Quality of $Z_2$ gauge theory
and quantum Ising model

Ising gauge theory

\[ H = -K \sum_{D} \prod_{l \in D} b_{z} - g \sum_{l} b_{z} \]

Define operators on the dual lattice sites

\[ \mu^{x_{i}} \text{ and } \mu^{z_{i}} \]
From this definition it is easy to see that
\[ \mu_l^x \mu_l^x = -\mu_l^x \mu_l^x \]
and \[ \mu_l^z \mu_l'^z = \mu_l^z \mu_l'^z \quad \text{for} \quad l \neq l' \]
and \[ \mu_l^x \mu_l^z = \mu_l^z \mu_l^x = 1. \]

So \[ \mu_l^x \] and \[ \mu_l^z \] are a dual set of qubits.

Now note that
\[ o^x = \mu_1^2 \mu_2^2. \]
\( \sigma^x = \mu_1 \mu_2 \frac{G_a G_b G_c G_d \cdots}{1} \)

So

\[
H = -g \sum_{\langle ij \rangle} \mu_i \mu_j - k \sum_i \mu_i
\]

This is the square lattice Ising model in a transverse field.
Mapping between quantum Ising model in a transverse field in $d$ dim.
and classical Ising model in $d+1$ dim.

First consider $d=0$, i.e. a single quantum spin

$$H = -g \sigma^x$$

$$Z = \text{Tr} e^{-\beta H} = 2 \cosh (\beta g)$$

$$= \underbrace{\text{Tr} e^{-\Delta t H} e^{-\Delta t H} e^{-\Delta t H}}_{N \text{ times}}$$

$$N \Delta t = \beta.$$ 

Insert complete set of states between each exponential.

Like eigenstates of $\sigma^z$:

$$|s\rangle = \pm |\pm \rangle$$

$s = \pm 1$. 

\[ \langle s_1 | e^{\Delta t g s^x} | s_2 \rangle \]
\[ = \langle s_1 | \cosh (\Delta t g) + e^{\Delta t g} \sinh (\Delta t) | s_2 \rangle \]
\[ = A \exp (B s_1 s_2) . \]

\[ s_1 = s_2 = 1 \]
\[ \cosh (\Delta t g) = A e^B \]
\[ s_1 = -s_2 = 1 \quad + \sinh (\Delta t g) = A e^{-B} \]
\[ \Rightarrow A^2 = \cosh (\Delta t g) \sinh (\Delta t g) \]
\[ e^{-2B} = \tanh (\Delta t g) . \]

\[ Z = A^n \sum_{s_i = \pm 1} \exp (B s_i s_{i+1}) \]
\[ s_1 = s_{n+1} \]

\[ \Rightarrow \text{Classical Ising chain in 1 dimension} \]
\[ \text{(with periodic boundary conditions)} \]
Now consider $d > 1$

$$H = -J \sum_{<ij>} \sigma_i^z \sigma_j^z - g \sum_i \sigma_i^x$$

$$Z = T_n e^{-\beta H} = T_n \left( e^{-\Delta t H} e^{-\Delta t H} \cdots e^{-\Delta t H} \right)_{N \ \text{times}}$$

Now use $\Delta t \to 0$, $N \to \infty$, $\Delta t N = \beta$ fixed.

$$e^{\Delta t J \sum_{<ij>} \sigma_i^z \sigma_j^z + \Delta t g \sum_i \sigma_i^x}$$

$$\approx \exp \left( \Delta t J \sum_{<ij>} \sigma_i^z \sigma_j^z \right) \exp \left( \Delta t g \sum_i \sigma_i^x \right) + O((\Delta t)^2)$$

Inserting complete set of states as in $d=0$ we obtain a $(d+1)$-dim. Ising model.
$H = \sum \prod \sigma_z \sigma_z \sigma_z \sigma_z$

where $\prod$ includes spacetime plaquette