$\mathbb{Z}_2$ gauge theory

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Abstract

The quantum $\mathbb{Z}_2$ gauge theory on the square lattice studied. This exhibits a phase with topological order, and one without, and a continuous phase transition between them.

I. INTRODUCTION

Wegner defined the $\mathbb{Z}_2$ gauge theory as a classical statistical mechanics partition function on the cubic lattice. We consider the partition function [1]

$$\begin{aligned}
\widetilde{\mathcal{Z}}_{\mathbb{Z}_2} &= \sum_{\{\sigma_{ij}\}=\pm 1} \exp\left(-\widetilde{\mathcal{H}}_{\mathbb{Z}_2}/T\right) \\
\widetilde{\mathcal{H}}_{\mathbb{Z}_2} &= -K \sum_{\square} \prod_{(ij)\in\square} \sigma_{ij} \,,
\end{aligned} \tag{1}$$

The degrees of freedom in this partition function are the binary variables $\sigma_{ij} = \pm 1$ on the links $\ell \equiv (ij)$ of the cubic lattice. The $\square$ indicates the elementary plaquettes of the cubic lattice.

We will present our discussion in this section entirely in terms of the corresponding quantum model on the square lattice. This degrees of freedom of the quantum model are qubits on the links, ℓ, of a square lattice. The Pauli operators σ_ℓ^α ($\alpha = x, y, z$) act on these qubits, and σ_{ij} variables in Eq. (1) are promoted to the operators σ_ℓ^z on the spatial links. We set $\sigma_{ij} = 1$ on the temporal links as a gauge choice. The Hamiltonian of the quantum $\mathbb{Z}_2$ gauge theory is [1, 2]

$$\mathcal{H}_{\mathbb{Z}_2} = -K \sum_{\square} \prod_{\ell\in\square} \sigma_\ell^z - g \sum_\ell \sigma_\ell^x \,, \tag{2}$$

where $\square$ indicates the elementary plaquettes on the square lattice, as indicated in Fig. 1a.

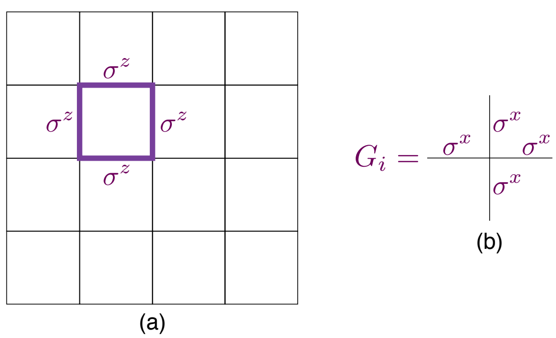


FIG. 1. (a) The plaquette term of the $\mathbb{Z}_2$ lattice gauge theory. (b) The operators G_i which generate $\mathbb{Z}_2$ gauge transformations.

On the infinite square lattice, we can define operators on each site, i, of the lattice which commute with $\mathcal{H}_{\mathbb{Z}_2}$ (see Fig. 1b)

$$G_i = \prod_{\ell\in +} \sigma_\ell^x \,, \tag{3}$$

which clearly obey $G_i^2 = 1$. We have $G_i \sigma^z_\ell G_i = \varrho_i \sigma^z_\ell$, where $\varrho_i = -1$ only if the site i is at the end of link ℓ, and $\varrho_i = 1$ otherwise: the G_i generates a space-dependent $\mathbb{Z}_2$ gauge transformation on the site i. There are an even number of σ^z_ℓ emanating from each site in the K term in $\mathcal{H}_{\mathbb{Z}_2}$, and so

$$[\mathcal{H}_{\mathbb{Z}_2}, G_i] = 0\,. \tag{4}$$

The spectrum of $\mathcal{H}_{\mathbb{Z}_2}$ depends upon the values of the conserved G_i, and here we will take

$$G_i = 1\,; \tag{5}$$

this corresponds to a 'pure' $\mathbb{Z}_2$ gauge theory with no matter fields. We will consider matter fields later.

Wegner [1] showed that there were two gapped phases in the theory, which are necessarily separated by a phase transition. Remarkably, unlike all previously known cases, this phase transition was not required by the presence of a broken symmetry in one of the phases: there was no local order parameter characterizing the phase transition. Instead, Wegner argued for the presence of a phase transition using the behavior of the Wegner-Wilson loop operator $W_{\mathcal{C}}$, which is the product of σ^z on the links of any closed contour $\mathcal{C}$ on the direct square lattice, as illustrated in Fig. 2. ($W_{\mathcal{C}}$ is usually, and improperly, referred to just as a Wilson loop.) The two phases are:

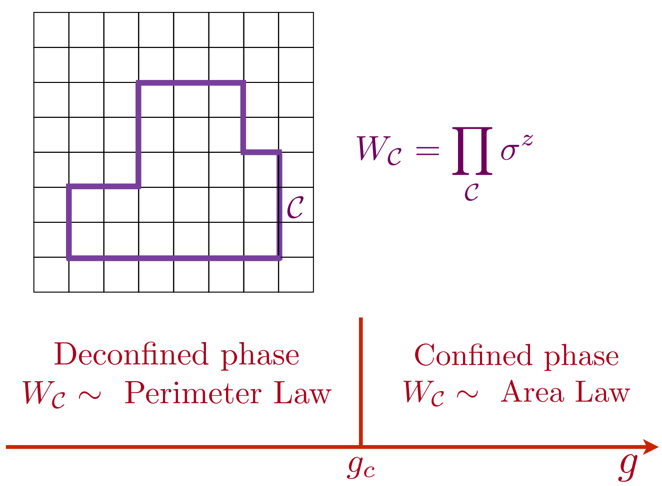


FIG. 2. The Wegner-Wilson loop operator $W_{\mathcal{C}}$ on the closed loop $\mathcal{C}$. Shown below is a schematic ground state phase diagram of $\mathcal{H}_{\mathbb{Z}_2}$, with the distinct behaviors of $W_{\mathcal{C}}$ in the deconfined and confined phases.

(i) At $g \gg K$ we have the 'confining' phase. In this phase $W_{\mathcal{C}}$ obeys the area law: $\langle W_{\mathcal{C}} \rangle \sim$

$\exp(-\alpha A_{\mathcal{C}})$ for large contours $\mathcal{C}$, where $A_{\mathcal{C}}$ is the area enclosed by the contour $\mathcal{C}$ and α is a constant. This behavior can easily be seen by a small K expansion of $\langle W_{\mathcal{C}} \rangle$: one power of K is needed for every plaquette enclosed by $\mathcal{C}$ for the first non-vanishing contribution to $\mathcal{W}_C$. The rapid decay of $\langle W_{\mathcal{C}} \rangle$ is a consequence of the large fluctuations in the $\mathbb{Z}_2$ flux, $\prod_{\ell \in \square} \sigma^z_\ell$, through each plaquette (*ii*) At $K \gg g$ we have the 'deconfined' phase. In this phase, the $\mathbb{Z}_2$ flux is expelled, and $\prod_{\ell \in \square} \sigma^z_\ell$ usually equals $+1$ in all plaquettes. We will see later that the flux expulsion is analogous to the Meissner effect in superconductors. The small residual fluctuations of the flux lead to a perimeter law decay, $\langle W_{\mathcal{C}} \rangle \sim \exp(-\alpha' P_{\mathcal{C}})$ for large contours $\mathcal{C}$, where $P_{\mathcal{C}}$ is the perimeter of the contour $\mathcal{C}$ and α' is a constant.

Along with establishing the existence of a phase transition using the distinct behaviors of the Wegner-Wilson loop, Wegner also determined the critical properties of the transition. He performed a Kramers-Wannier duality transformation, and showed that the $\mathbb{Z}_2$ gauge theory was equivalent to the classical Ising model. This establishes that the confinement-deconfinement transition is in the universality class of the the Ising Wilson-Fisher [3] conformal field theory in 3 spacetime dimensions (a CFT3). The phase with the dual Ising order is the confining phase, and the phase with Ising 'disorder' is the deconfined phase. We will derive the this Ising criticality in Section **??** by a different method. For now, we note that the critical theory is not precisely the Wilson-Fisher Ising CFT, but what we call the Ising* theory. In the Ising* theory, the only allowed operators are those which are invariant under $\phi \to -\phi$, where ϕ is the Ising primary field [4, 5].

II. TOPOLOGICAL ORDER

While Wegner's analysis yields a satisfactory description of the pure $\mathbb{Z}_2$ gauge theory, the Wegner-Wilson loop is, in general, not a useful diagnostic for the existence of a phase transition. Once we add dynamical matter fields (as we will do below), $W_{\mathcal{C}}$ invariably has a perimeter law decay, although the confinement-deconfinement phase transition can persist.

The modern interpretation of the existence of the phase transition in the $\mathbb{Z}_2$ lattice gauge theory is that it is present because the deconfined phase has $\mathbb{Z}_2$ 'topological' order [6–12], while the confined phase is 'trivial'. We now describe two characteristics of this topological order: both characteristics can survive the introduction of additional degrees of freedom; but we will see that the first is more robust, and is present even in cases with gapless excitations carrying $\mathbb{Z}_2$ charges.

The first characteristic is that there are stable low-lying excitations of the topological phase in the infinite lattice model which cannot be created by the action of any local operator on the ground state (*i.e.* there are 'superselection' sectors [10]). This excitation is a particle, often called a 'vison', which carries $\mathbb{Z}_2$ flux of -1 [13–15]. Recall that the ground state of the deconfined phase expelled the $\mathbb{Z}_2$ flux: at $g = 0$ the state with all spins up, $|\Uparrow\rangle$, (*i.e.* eigenstates of σ^z_ℓ with eigenvalue $+1$) is a ground state, and this has no $\mathbb{Z}_2$ flux. This state is not an eigenstate of the G_i, but this is easily remedied by a gauge transformation:

$$|0\rangle = \prod_i (1 + G_i)\, |\Uparrow\rangle \tag{6}$$

is an eigenstate of all the G_i. Now we apply the σ^x_ℓ operator on a link ℓ, the neighboring plaquettes acquire $\mathbb{Z}_2$ flux of -1. We need a non-local 'string' of σ^x operators to separate these $\mathbb{Z}_2$ fluxes so that we obtain 2 well separated vison excitations; see Fig. 3. Each vison is stable in its own

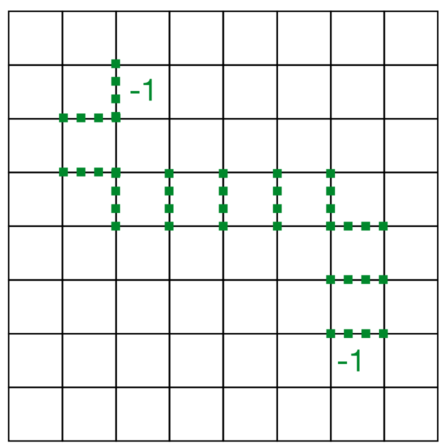
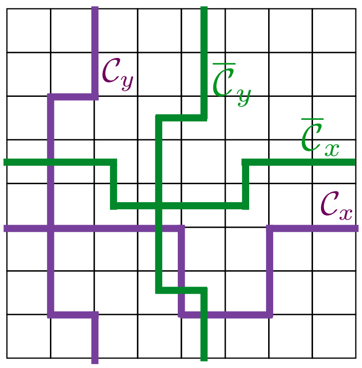


FIG. 3. Two visons (indicated by the -1's in the plaquettes) connected by an invisible string. The dashed lines indicate the links, ℓ, on which the σ^x_ℓ operators acted on $|0\rangle$ to create a pair of separated visons. The plaquettes with an even number of dashed lines on their edges carry no $\mathbb{Z}_2$ fluxes, and so are 'invisible'.

region, and it can only be annihilated when it encounters another vison. Such a vison particle is present only in the deconfined phase: all excitations in the confined phase can be created by local operators, as is easily verified in a small K expansion.

The second topological characteristic emerges upon considering the low-lying states of $\mathcal{H}_{\mathbb{Z}_2}$ on a topologically non-trivial geometry, like the torus. A key observation in such geometries is that the G_i (and their products) do not exhaust the set of operators which commute with $\mathcal{H}_{\mathbb{Z}_2}$. On a torus, there are 2 additional independent operators which commute with $\mathcal{H}_{\mathbb{Z}_2}$: these operators, V_x, V_y, are illustrated in Fig. 4 (these are analogs of 'tHooft loops). The operators are defined on contours,



$$V_x = \prod_{\overline{\mathcal{C}}_x} \sigma^x \quad , \quad V_y = \prod_{\overline{\mathcal{C}}_y} \sigma^x$$

$$W_x = \prod_{\mathcal{C}_x} \sigma^z \quad , \quad W_y = \prod_{\mathcal{C}_y} \sigma^z$$

$$V_x W_y = -W_y V_x \quad , \quad V_y W_x = -W_x V_y$$

and all other pairs commute.

$$[H, V_x] = [H, V_y] = 0$$

FIG. 4. Operators in a torus geometry: periodic boundary conditions are implied on the lattice.

$\overline{\mathcal{C}}_{x,y}$ which reside on the dual square lattice, and encircle the two independent cycles of the torus. The specific contours do not matter, because we can deform the contours locally by multiplying them with the G_i. It is also useful to define Wegner-Wilson loop operators $W_{x,y}$ on direct lattice contours $\mathcal{C}_{x,y}$ which encircle the cycles of the torus; note that the $W_{x,y}$ do not commute with $\mathcal{H}_{\mathbb{Z}_2}$, while the $V_{x,y}$ do commute. Because the contour $\mathcal{C}_x$ intersects the contour $\overline{\mathcal{C}}_y$ an odd number of times (and similarly with $\mathcal{C}_y$ and $\overline{\mathcal{C}}_x$) we obtain the anti-commutation relations

$$W_x V_y = -V_y W_x \quad , \quad W_y V_x = -V_x W_y \,, \tag{7}$$

while all other pairs commute.

With this algebra of topologically non-trivial operators at hand, we can now identify the distinct signatures of the phases without and with topological order. All eigenstates of $\mathcal{H}_{\mathbb{Z}_2}$ must also be eigenstates of V_x and V_y. First, consider the non-topological confining phase at large g. At $g = \infty$, the ground state, $|\Rightarrow\rangle$, has all spins pointing to the right (*i.e.* all qubits are eigenstates of σ^x_ℓ with eigenvalue $+1$). This state clearly has eigenvalues $V_x = V_y = +1$. States with $V_x = -1$ or $V_y = -1$ must have at least one spin pointing to the left, and so cost a large energy g: such states cannot be degenerate with the ground state, even in the limit of an infinite volume for the torus.

Next, consider the topological deconfined phase at small g. The ground state $|0\rangle$ is not an eigenstate of $V_{x,y}$, but is instead an eigenstate of $W_{x,y}$ with $W_x = W_y = 1$. The state $V_x |0\rangle$ is easily seen to be an eigenstate of $W_{x,y}$ with $W_x = 1$ and $W_y = -1$: so this state has $\mathbb{Z}_2$ flux of

-1 through one of the holes of the torus. At $g = 0$, the state $V_x |0\rangle$ is also a ground state of $\mathcal{H}_{\mathbb{Z}_2}$, degenerate with $|0\rangle$: see Fig. 5. Similarly, we can create two other ground states, $V_y |0\rangle$ and

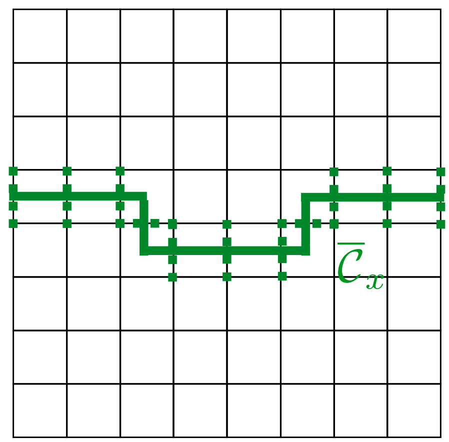


FIG. 5. The state $V_x |0\rangle$ (the dashed lines indicate σ^x operators on the ground state: periodic boundary conditions are implied on the lattice. Notice that every plaquette has $\mathbb{Z}_2$ flux $+1$, and so this is a ground state at $g = 0$. This state has $W_x = 1$ and $W_y = -1$. At small non-zero g, there is a non-zero tunnelling amplitude between $|0\rangle$ and $V_x |0\rangle$ of order g^{L_x}, where L_x is the length of $\overline{\mathcal{C}}_x$.

$V_y V_x |0\rangle$, which are also eigenstates of $W_{x,y}$ with distinct eigenvalues. So at $g = 0$, we have a 4-fold degeneracy in the ground state, and all other states are separated by an energy gap. When we turn on a non-zero g, the ground states will no longer be eigenstates of $W_{x,y}$ because these operators do not commute with $\mathcal{H}_{\mathbb{Z}_2}$. Instead the ground states will become eigenstates of $V_{x,y}$; at $g = 0$ we can take the linear combinations $(1 \pm V_x)(1 \pm V_y) |0\rangle$ to obtain degenerate states with eigenvalues $V_x = \pm 1$ and $V_y = \pm 1$. At non-zero g, these 4 states will no longer be degenerate, but will acquire an exponentially small splitting of order $g(g/K)^L$, where L is a linear dimension of the torus: this is due to a non-zero tunneling amplitude between states with distinct $\mathbb{Z}_2$ fluxes through the holes of the torus.

The presence of these 4 lowest energy states, which are separated by an energy splitting which vanishes exponentially with the linear size of the torus, is one of the defining characteristics of $\mathbb{Z}_2$ topological order. We can take linear combinations of these 4 states to obtain distinct states with eigenvalues $W_x = \pm 1$, $W_y = \pm 1$ of the $\mathbb{Z}_2$ flux through the holes of the torus; or we can take energy eigenvalues, which are also eigenstates of $V_{x,y}$ with $V_x = \pm 1$, $V_y = \pm 1$. These feature are present throughout the entire deconfined phase, while the confining state has a unique ground state with

$V_x = V_y = 1$. See Fig. 6.

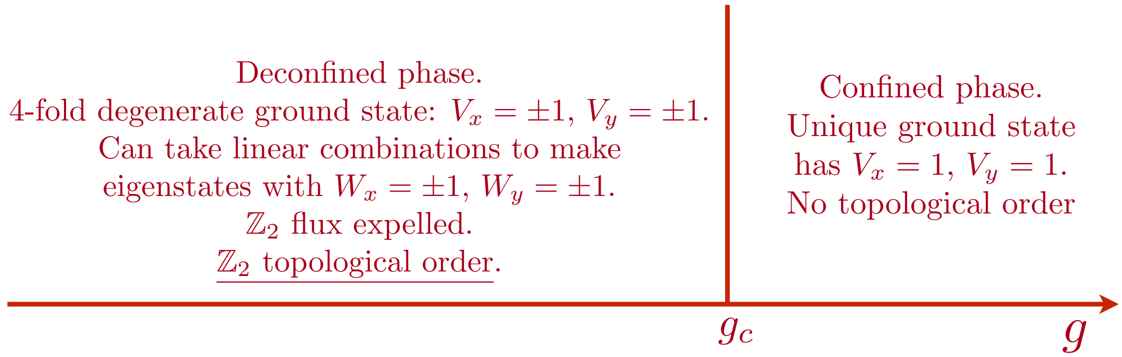


FIG. 6. An updated version of the phase diagram of $\mathcal{H}_{\mathbb{Z}_2}$ in Fig. 2. The confinement-deconfinement phase transition is described by the Ising* Wilson-Fisher CFT.

We close this section by noting that the $\mathbb{Z}_2$ topological order described above can also be realized in a U(1)×U(1) Chern-Simons gauge theory [9, 11]. This is the theory with the 2+1 dimensional Lagrangian

$$\mathcal{L}_{\rm cs} = \frac{i}{\pi} \int d^3x \, \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu b_\lambda \,, \tag{8}$$

where a_μ and b_μ are the 2 U(1) gauge fields. The Wilson loop operators of these gauge fields

$$W_i = \exp\left(i \int_{\mathcal{C}_i} a_\mu dx_\mu \right) \quad , \quad V_i = \exp\left(i \int_{\overline{\mathcal{C}}_i} b_\mu dx_\mu \right) \,, \tag{9}$$

are precisely the operators $W_{x,y}$ and $V_{x,y}$ when the contours $\mathcal{C}_i$ and $\overline{\mathcal{C}}_i$ encircle the cycles of the torus. This can be verified by reproducing the commutation relations in Eq. (7) from Eq. (8). We will present an explicit derivation of $\mathcal{L}_{\rm cs}$ later in the course.

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