XY model: particle-vortex duality

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Abstract

We consider the classical XY model in $D = 1, 2, 3$ dimensions. In $D = 2$, we perform a duality transform to a theory of vortices and discuss confinement-deconfinement theory of vortices, as in the Kosterlitz-Thouless theory. In $D = 3$, we obtain a U(1) gauge theory of line vortices, as in the Dasgupta-Halperin theory.

I. XY MODEL IN $D = 1$

First we consider the simpler case of one dimension, when there is no ordered phase at any T. The lattice partition function is

$$\mathcal{Z} = \prod_i \int_0^{2\pi} \exp\left(K \sum_i \cos(\theta_i - \theta_{i+1})\right) \tag{1}$$

where

$$K \equiv \frac{J}{T} \tag{2}$$

and we also define the complex order parameter

$$\psi = \phi_1 + i\phi_2 = e^{i\theta} \tag{3}$$

Anticipating that there is no ordered phase at any T, let us work in the low T limit, $T \ll J$. Then, we expect that θ_i varies slowly with i, and we can take the continuum limit to write

$$\mathcal{Z} = \int \mathcal{D}\theta(x) \exp\left(-\frac{K}{2}\int dx \left(\frac{d\theta}{dx}\right)^2\right) \tag{4}$$

As this is a Gaussian action, we can easily evaluate the correlation functions of the order parameter

$$\begin{aligned} \langle \psi(x)\psi^*(0)\rangle &= \exp\left(-\frac{1}{K}\int \frac{dk}{2\pi}\frac{(1-\cos(kx))}{k^2}\right) \\ &= \exp\left(-\frac{|x|}{2K}\right) \end{aligned} \tag{5}$$

So the correlation length is

$$\xi = 2K = \frac{2J}{T} \tag{6}$$

which diverges only at $T = 0$. We also expect exponential decay of correlations at very high T, and so the low T and high T limits are smoothly connected without an intervening phase transition.

A. Quantum interpretation

We can also interpret Eq. (4) as the Feynman path integral of a particle with mass K and co-ordinate θ, moving on a unit circle with imaginary time x. The Hamiltonian of this quantum particle is ($\hbar = 1$).

$$H = -\frac{1}{2K}\frac{d^2}{d\theta^2} \tag{7}$$

So the eigenenergies are

$$E_n = \frac{n^2}{2K} \quad , \quad n = 0, \pm 1, \pm 2, \ldots \tag{8}$$

In terms of these eigenstates $|n\rangle$, the correlation function in Eq. (5) can we written as (for $x > 0$)

$$\begin{aligned}
\langle \psi(x)\psi^*(0)\rangle &= \langle 0|\, \psi \exp(-Hx)\psi^* \,|0\rangle \\
&= \langle -1|\exp(-Hx)\,|-1\rangle \\
&= \exp(-E_{-1}x)
\end{aligned} \tag{9}$$

which agrees with Eq. (5).

II. VORTICES IN THE XY MODEL IN $D = 2$

First, let us just carry out precisely the same analysis as that carried out above in $D = 1$. We will ultimately show that such an analysis yields the correct results as $T \to 0$ even in $D = 2$. However, unlike $D = 1$, the results only below a critical temperature T_{KT}.

The continuum theory analog of Eq. (4) is

$$\mathcal{Z} = \int \mathcal{D}\theta(x) \exp\left(-\frac{K}{2\pi}\int d^2x\, (\nabla_x\theta)^2\right) \tag{10}$$

where now

$$K = \frac{\pi J}{T}\,. \tag{11}$$

The correlator in Eq. (5) now maps to

$$\begin{aligned}
\langle \psi(x)\psi^*(0)\rangle &= \exp\left(-\frac{\pi}{K}\int^{\Lambda} \frac{d^2k}{4\pi^2}\frac{(1-\cos(kx))}{k^2}\right) \\
&\approx \exp\left(-\frac{1}{2K}\ln(\Lambda|x|)\right) \quad , \quad |x| \to \infty \\
&= 1/(\Lambda x)^{1/(2K)}
\end{aligned} \tag{12}$$

So at low T, the two-point correlator decays only as a power-law. On the other hand, we know from the high temperature expansion that at sufficiently high T, the two-point correlator must decay exponentially. These differences are resolved by a vortex unbinding transition at T_{KT}.

See the attached notes from the book by Plischke and Bergersen

A. Duality transform in $D = 2$

We recall the partition function

$$\mathcal{Z} = \prod_i \int_0^{2\pi} d\theta_i e^{-\mathcal{S}} \tag{13}$$

where the action is the 'XY' model

$$\mathcal{S} = -\frac{K}{\pi} \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j). \tag{14}$$

Our treatment below follows classic JKKN paper [1].

It is convenient to write this is a lattice gauge theory notation

$$\mathcal{S} = -\frac{K}{\pi} \sum_{i,\mu} \cos(\Delta_\mu \theta_i), \tag{15}$$

where μ extends over x, τ, the two directions of spacetime. Here Δ_μ defines a discrete lattice derivative with $\Delta_\mu f(x_i) \equiv f(x_i + \hat{\mu}) - f(x_i)$, with $\hat{\mu}$ a vector of unit length.

Now we introduce the Villain representation

$$e^{-K(1-\cos(\theta))/\pi} \approx \sum_{n=-\infty}^{\infty} e^{-K(\theta - 2\pi n)^2/(2\pi)} \tag{16}$$

which is clearly valid for large K. We will use it for all values of K: this is OK because the right-hand-side preserves an essential feature for all K—periodicity in θ. Then we can write the partition function as

$$\mathcal{Z} = \sum_{m_{i\mu}} \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} e^{-\mathcal{S}} \tag{17}$$

with

$$\mathcal{S} = \frac{K}{2\pi} \sum_{i,\mu} (\Delta_\mu \theta_i - 2\pi m_{i\mu})^2, \tag{18}$$

where the $m_{i\mu}$ are independent integers on all the links of the square lattice. Now we need the exact Fourier series representation of a periodic function of θ

$$\sum_{n=-\infty}^{\infty} e^{-K(\theta-2\pi n)^2/(2\pi)} = \frac{1}{\sqrt{2K}} \sum_{p=-\infty}^{\infty} e^{-\pi p^2/(2K) - ip\theta}. \tag{19}$$

Note that both sides of the equation are invariant under $\theta \to \theta + 2\pi$. Then (17) can be rewritten as (ignoring overall normalization constants)

$$\mathcal{Z} = \sum_{p_{i\mu}} \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} e^{-\mathcal{S}} \tag{20}$$

with

$$\mathcal{S} = \frac{\pi}{2K} \sum_{i,\mu} p_{i\mu}^2 + i p_{i\mu} \Delta_\mu \theta_i. \tag{21}$$

Again, the $p_{i\mu}$ are an independent set of integers on the links of the square lattice. Now the advantage of (21) is that all the integrals over the θ_i factorize, and each θ_i integral can be performed exactly. Each integral leads to a divergence-free constraint on the $p_{i\mu}$ integers

$$\Delta_\mu p_{i\mu} = 0. \tag{22}$$

We can view $p_{i\mu}$ as the number current of a boson (particle) moving in $D = 2$ lattice spacetime: the divergence-free constraint shows that the number of these particles is conserved. We can solve this constraint by writing $p_{i\mu}$ as the 'curl' of another integer valued field, $h_\jmath$, which resides on the sites, $\jmath$, of the dual lattice

$$p_{i\mu} = \epsilon_{\mu\nu} \Delta_\nu h_\jmath. \tag{23}$$

Then the partition function becomes that of a 'height' or 'solid-on-solid' (SOS) model

$$\mathcal{Z} = \sum_{h_\jmath} e^{-\mathcal{S}} \tag{24}$$

with

$$\mathcal{S} = \frac{\pi}{2K} \sum_{\jmath,\mu} \left(\Delta_\mu h_\jmath\right)^2 \tag{25}$$

This can also describe the statistical mechanics of a two-dimensional surface upon which atoms are being added discretely on the sites $\jmath$, and $h_\jmath$ is the 'height' of the atomic surface.

We are now almost at the final, dual, form of the original XY model in (15). We simply have to approximate the discrete height field $h_\jmath$ by a continuous field $\phi_\jmath$. Formally, we can do this by writing, for any function $f(h)$

$$\begin{aligned}
\sum_{h=-\infty}^{\infty} f(h) &= \int_{-\infty}^{\infty} d\phi f(\phi/\pi) \sum_{h=-\infty}^{\infty} \delta(\phi - \pi h) \\
&= \frac{1}{\pi} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi f(\phi/\pi) e^{2ip\phi} \\
&= \frac{1}{\pi} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi f(\phi/\pi) e^{(\ln y)p^2 + 2ip\phi} \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} d\phi f(\phi/\pi) \left[1 + 2 \sum_{p=1}^{\infty} y^{p^2} \cos(2p\phi)\right]
\end{aligned} \tag{26}$$

for $y = 1$. We now substitute (26) into (25), and then examine the situation for $y \to 0$: this is the only "unjustified" approximation we make here. It is justified more carefully by JKKN using renormalization group arguments, who show that all of the basic physics is apparent already at small y. In such a limit we can write the exact result (26) as approximately

$$\sum_{h=-\infty}^{\infty} f(h) \approx \frac{1}{\pi} \int_{-\infty}^{\infty} d\phi f(\phi/\pi) \exp\left(2y\cos(2\phi)\right) \tag{27}$$

Substituting (27) into (25), we find that our final dual theory of the XY model is the sine-Gordon theory!

$$\mathcal{Z} = \prod_{\jmath} \int d\phi_{\jmath}\, e^{-\mathcal{S}_{sg}} \tag{28}$$

with

$$\mathcal{S}_{sg} = \frac{1}{2\pi K} \sum_{\jmath,\mu} \left(\Delta_\mu \phi_{\jmath}\right)^2 - 2y \sum_{\jmath} \cos(2\phi_{\jmath}) \tag{29}$$

If we now expand in powers of y, and integrate out ϕ, we obtain the partition function of a plasma of vortices and anti-vortices:

$$\mathcal{Z} = \sum_{N=0}^{\infty} \frac{y^{2N}}{(N!)^2} \prod_{i=1}^{N} \int dx_{+i} dx_{-i} \exp\left(K \sum_{j \neq k=1}^{2N} p_j p_k \ln|x_j - x_k| \right) \tag{30}$$

where $x_j = x_{+j}$ and $p_j = 1$ for $j = 1 \ldots N$ (representing the vortices) and $x_j = x_{-,j-N}$ and $p_j = -1$ for $j = N+1 \ldots 2N$ (representing the anti-vortices). So we can identify

$$V_{\pm} = e^{\pm 2i\phi} \tag{31}$$

as the vortex/anti-vortex operators.

B. Mappings of observables

We begin with the boson current J_μ. This can obtained by coupling the XY model to a fixed external gauge field A_μ by replacing (15) by

$$\mathcal{S} = -\frac{K}{\pi} \sum_{i,\mu} \cos(\Delta_\mu \theta_i - A_{i\mu}), \tag{32}$$

and defining the current

$$J_{i\mu} = \frac{\delta \mathcal{S}}{\delta A_{i\mu}}. \tag{33}$$

Note that because the chemical potential of the bosons is i times the time-component of the vector potential (in Euclidean time), the boson density operator is i times J_τ (this factor of i must be kept in mind in all Euclidean path integrals). Then carrying through the mappings above we find that the sine-Gordon action is replaced by

$$\mathcal{S}_{sg} = \frac{1}{2\pi K} \sum_{\jmath,\mu} \left(\Delta_\mu \phi_{\jmath}\right)^2 - 2y \sum_{\jmath} \cos(2\phi_{\jmath}) + \frac{i}{\pi} A_\mu \epsilon_{\mu\nu} \Delta_\nu \phi_{\jmath} \tag{34}$$

So now we can identify the current operator

$$J_{i\mu} = \frac{i}{\pi}\epsilon_{\mu\nu}\Delta_\nu \phi_\jmath \,. \tag{35}$$

Note that this current is automatically conserved: *i.e.* $\Delta_\mu J_\mu = 0$.

Another useful observable is the *vorticity*. A little thought using the action (18) shows that we can identify

$$v_\jmath = \epsilon_{\mu\nu}\Delta_\mu m_{i\nu} \tag{36}$$

as the integer vorticity on site $\jmath$. So we extend the action (18) to

$$\mathcal{S} = \frac{K}{2\pi}\sum_{i,\mu}(\Delta_\mu \theta_i - 2\pi m_{i\mu})^2 + i2\pi\lambda_\jmath \epsilon_{\mu\nu}\Delta_\mu m_{i\nu} \tag{37}$$

Now every vortex/antivortex in the partition function at site $\jmath$ appears with a factor of $e^{\pm i2\pi\lambda_\jmath}$. For the duality mapping we need an extended version of the identify (19)

$$\sum_{n=-\infty}^{\infty} e^{-K(\theta-2\pi n)^2/(2\pi)+i2\pi n\lambda} = \frac{1}{\sqrt{2K}}\sum_{p=-\infty}^{\infty} e^{-\pi(p-\lambda)^2/(2K)-i(p-\lambda)\theta}, \tag{38}$$

which holds for any real λ, θ, and K. Then we find that (29) maps to

$$\mathcal{S}_{sg} = \frac{1}{2\pi K}\sum_{\jmath,\mu}(\Delta_\mu \phi_\jmath)^2 - 2y\sum_\jmath \cos(2\phi_\jmath - 2\pi\lambda_\jmath) \tag{39}$$

We therefore observe that each factor $e^{\pm i2\pi\lambda_\jmath}$ appears with a factor of $ye^{\pm 2i\phi}$. So we identify y with the vortex 'fugacity', and confirm that $V_\pm = e^{\pm 2i\phi}$ is the vortex/anti-vortex operator.

C. Renormalization group analysis

We will discuss some important properties of the sine–Gordon field theory $\mathcal{S}_{sg}$ in Eq. (39) as a function of the dimensionless coupling K and the vortex fugacity y.

First, we note that the theory $\mathcal{S}_{sg}$ at $y = 0$ is precisely the dual of the low temperature theory in Eq. (10). We can compute correlators of the operators $e^{ip\theta}$ and $e^{ip\phi}$ in such a phase as before and obtain

$$\left\langle e^{ip\theta(x)}e^{-ip'\theta(0)}\right\rangle \sim \delta_{pp'}/x^{p^2/2K}, \qquad \left\langle e^{ip\phi(x)}e^{-ip'\phi(0)}\right\rangle \sim \delta_{pp'}/x^{p^2K/2}; \tag{40}$$

Note that the correlator of the vortex operator $e^{2i\phi}$ is $\sim \exp(-2K\ln(|x|))$ which is precisely the exponential of the vortex/anti-vortex interaction energy.

Note that these correlators are both power laws, indicating that the theory is scale invariant along the line $y = 0$ (indeed it is conformally invariant). From Eq. (40) we see that this is a line of critical points along which the exponents vary continuously as a function of the dimensionless parameter K. The technology of renormalization group scale transformations can therefore be applied freely at any point along this line. We can talk of scaling dimensions of operators, and the results (40) show that

$$\dim[e^{ip\theta}] = \frac{p^2}{4K}, \qquad \dim[e^{ip\phi}] = \frac{p^2 K}{4}. \tag{41}$$

Using this, and the scaling dimensions (41) for $p = 2$, we immediately obtain the scaling dimension $\dim[y] = 2 - K$ along the $v = 0$ line. This can be written as a renormalization group flow equation under the rescaling $\Lambda \to \Lambda e^{\ell}$:

$$\frac{dy}{d\ell} = (2 - K)v. \tag{42}$$

So the critical fixed line $y = 0$ is stable for $K < 2$. However, this flow equation is not the complete story, especially when K approaches 2. For $|K - 2| \sim |y|$ we see that the term on the right-hand side is not linear in the small parameter y, but quadratic. To be consistent, then, we also have to consider other terms of order y^2 that might arise in the flow equations. As we will see below, there is a renormalization of K that appears at this order.

The flow equations at order y^2 are generated by decomposing the field $\phi(x)$ into a background slowly varying component $\phi_<(x)$ and a rapidly varying component $\phi_>(x)$, which will be integrated out to order y^2:

$$\phi(x) = \phi_<(x) + \phi_>(x), \tag{43}$$

where $\phi_<$ has spatial Fourier components at momenta smaller than $\Lambda e^{-\ell}$, while $\phi_>$ has components between $\Lambda e^{-\ell}$ and Λ. Inserting (43) into (39), to linear order in y we generate the following effective coupling for $\phi_<$:

$$\begin{aligned} & y \int d^2x \, \langle \cos(2\phi_<(x) + 2\phi_>(x)) \rangle_0 \\ &\qquad = y \int d^2x \cos(2\phi_<(x)) \left\langle e^{i2\phi_>(x)} \right\rangle_0 \\ &\qquad = y \int d^2x \cos(2\phi_<(x)) e^{-2\langle \phi_>^2 \rangle_0} \\ &\qquad \approx y \left(1 - K \frac{d\Lambda}{\Lambda}\right) \int d^2x \cos(2\phi_<(x)), \end{aligned} \tag{44}$$

where the subscript 0 indicates an average with respect to the free $y = 0$ Gaussian action of $\phi_>$, and $d\Lambda = \Lambda(1 - e^{-\ell})$. When combined with a rescaling of coordinates $x \to xe^{-\ell}$ to restore the cutoff to its original value, it is clear that (44) leads to the flow equation (42). The same procedure applies to quadratic order in v. As the algebra is a bit cumbersome, we will only schematically indicate the steps. We generate terms such as

$$\begin{aligned} & y^2 \int d^2x_1 d^2x_2 \cos(2\phi_<(x_1) \pm 2\phi_<(x_2)) \exp\left(\mp 4 \langle \phi_>(x_1)\phi_>(x_2) \rangle_0\right) \\ &\qquad = y^2 \int d^2x_1 d^2x_2 \cos(2\phi_<(x_1) \pm 2\phi_<(x_2)) \exp\left(\mp f(x_1 - x_2) d\Lambda\right), \end{aligned} \tag{45}$$

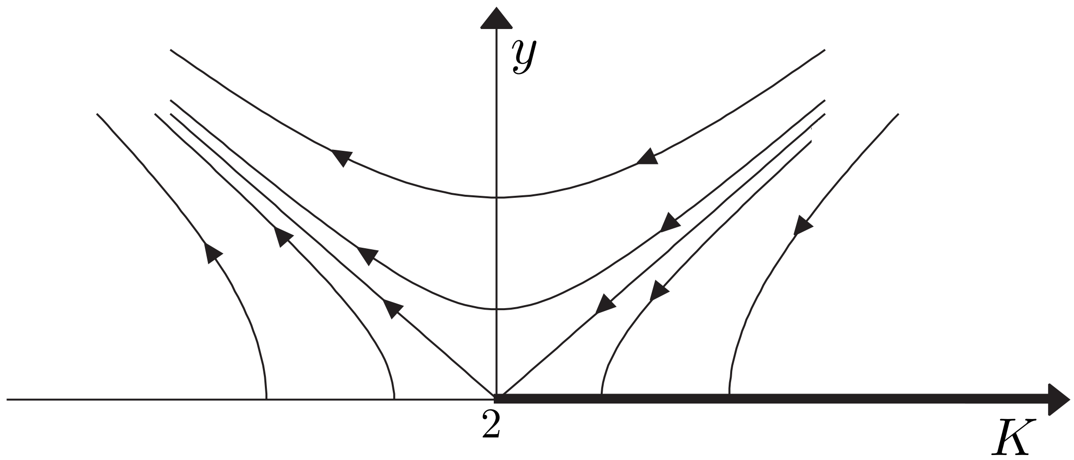

FIG. 1. RG flows of the sine-Gordon theory $\mathcal{S}_{sg}$. The thick fixed line corresponds to the low temperature phase of the XY model, $T < T_{KT}$.

where $f(x_1 - x_2)$ is some regularization-dependent function that decays on spatial scale $\sim\Lambda^{-1}$. For this last reason we may expand the other terms in (45) in powers of $x_1 - x_2$. The terms with the $+$ sign then generate a $\cos(4\phi)$ interaction; we will ignore this term as the analog of the arguments used to obtain (42) show that this term is strongly irrelevant for $K \sim 2$. The terms with the $-$ sign generate gradients on $\phi_<$ and therefore lead to a renormalization of K. In this manner we obtain the flow equation

$$\frac{dK}{d\ell} = -\delta v^2, \tag{46}$$

where δ is a positive, regularization-dependent constant (it also depends upon K, but we can ignore this by setting $K = 2$ in δ at this order).

A fairly complete understanding of the properties of $\mathcal{S}_{sg}$ follows from an analysis of Eqns. (42) and (46). The flow trajectories are shown in Fig. 1: They lie along the hyperbolae $4\delta v^2 - (2-K)^2 =$ constant.

To facilitate the integration of the flow equations (42) and (46) we change variables to

$$y_{1,2} = \sqrt{\delta} v \mp (K/2 - 1). \tag{47}$$

Then Eqns. (42) and (46) become

$$\begin{aligned} \frac{dy_1}{d\ell} &= y_1(y_1 + y_2), \\ \frac{dy_2}{d\ell} &= -y_2(y_1 + y_2). \end{aligned}$$

It is clear from these equations that one integral is simply $y_1 y_2 = C$, where C is a constant determined by the initial conditions; the first equation is then easily integrated to give

$$\tan^{-1}\frac{y_1(\ell)}{\sqrt{C}} - \tan^{-1}\frac{y_1(0)}{\sqrt{C}} = \sqrt{C}\ell. \tag{48}$$

By the usual scaling argument, the characteristic inverse correlation length, ξ^{-1}, in the disordered phase is of order $e^{-\ell^*}$, where ℓ^* is the value of ℓ over which y_1 grows from an initial value of order

$$\epsilon \sim \frac{T - T_{KT}}{T_{KT}} \ll 1 \tag{49}$$

to a value of order unity. From the initial conditions, we expect the constant C to also be of order ϵ, and so let us choose $C = \epsilon$; then a straightforward analysis of (48) gives us

$$\xi^{-1} \sim \exp\left(-\frac{\pi}{2\sqrt{\epsilon}}\right). \tag{50}$$

This singularity, and the flow analysis above, are characteristic of a "Kosterlitz–Thouless" transition.

III. PARTICLE-VORTEX DUALITY IN $D = 3$

The initial analysis in $D = 3$ tracks that in $D = 2$: everything in Section II A until Eq. (22) also applies in $D = 3$. However, the solution of the divergence-free condition on the boson number current, $p_{i\mu}$ now takes the form

$$p_{i\mu} = \epsilon_{\mu\nu\lambda}\Delta_\nu h_{\jmath\lambda} \tag{51}$$

where $h_{\jmath\mu}$ is now an integer-valued field on the links of the dual lattice. Then, promoting $h_{\jmath\mu}$ to a continuous field $a_{\jmath\mu}/(2\pi)$ (which replaces ϕ/π in (27)), the sine-Gordon theory in (28) and (29) is replaced by

$$\mathcal{Z} = \prod_{\jmath,\mu}\int da_{\jmath\mu}\, e^{-\mathcal{S}} \tag{52}$$

with

$$\mathcal{S} = \frac{1}{2K}\sum_{\jmath,\mu}\left(\epsilon_{\mu\nu\lambda}\Delta_\nu a_{\jmath\lambda}\right)^2 - 2y\sum_{\jmath}\cos(a_{\jmath\mu})) \tag{53}$$

Notice that the first terms has the form of Maxwell term in electrodynamics, and is invariant under gauge transformations. We can also make the second term gauge invariant by introducing an angular scalar field $\vartheta_\jmath$ on the links of the dual lattice. Then we obtain the action of scalar electrodynamics on a lattice, which is the the final form of the lattice dual theory:

$$\mathcal{Z} = \prod_{\jmath}\int d\vartheta_j \prod_{\jmath,\mu}\int da_{\jmath\mu}\, e^{-\mathcal{S}_{\rm qed}} \tag{54}$$

with

$$\mathcal{S}_{\rm qed} = \frac{1}{2K} \sum_{\jmath,\mu} \left(\epsilon_{\mu\nu\lambda} \Delta_\nu a_{\jmath\lambda}\right)^2 - 2y \sum_{\jmath} \cos(\Delta_\mu \vartheta_\jmath - a_{\jmath\mu}). \tag{55}$$

Clearly, we have not changed anything, apart from an overall constant in the path integral: we can absorb the ϑ by a gauge transformation of $a_{\jmath\mu}$, and then the $\vartheta_\jmath$ integrals just yield a constant prefactor.

A. Mapping of observables

The mappings in Section II B have direct generalizations to $D = 3$.

By coupling the XY model to an external vector potential A_μ, we find that the boson current operator is given by (replacing (35))

$$J_{i\mu} = \frac{i}{2\pi} \epsilon_{\mu\nu\lambda} \Delta_\nu a_{\jmath\lambda} \tag{56}$$

So the boson current maps to the electromagnetic flux of the dual scalar QED theory.

Similarly, we can now define the vorticity current by (replacing (36))

$$v_{\jmath\mu} = \epsilon_{\mu\nu\lambda} \Delta_\nu m_{i\lambda}, \tag{57}$$

and then the analog of (39) in the presence of a source $2\pi\lambda_\mu$ coupling to the vorticity current is

$$\mathcal{S}_{\rm qed} = \frac{1}{2K} \sum_{\jmath,\mu} \left(\epsilon_{\mu\nu\lambda} \Delta_\nu a_{\jmath\lambda}\right)^2 - 2y \sum_{\jmath} \cos(\Delta_\mu \vartheta_\jmath - a_{\jmath\mu} - 2\pi\lambda_{\jmath\mu}). \tag{58}$$

This tells us that the gauge-invariant current of the dual scalar field $e^{i\vartheta}$ is precisely equal to the vorticity current of the original boson theory.

B. Universal continuum theory

The connection described so far may appear specialized to particular lattice XY models. However, it is possible to state the particle-vortex mapping in rather precise and universal times as an exact correspondence between two different field theories, as was first argued by Dasgupta and Halperin [2].

In direct boson perspective, we have already seen that the vicinity of the superfluid-insulator transition is described by a field theory for the complex field $\psi \sim e^{i\theta}$

$$\begin{aligned} \mathcal{Z}_\psi &= \int \mathcal{D}\psi \, e^{-\mathcal{S}_\psi} \\ \mathcal{S}_\psi &= \int d^3x \left[|\partial_\mu \psi|^2 + r|\psi|^2 + u|\psi|^4 \right]. \end{aligned} \tag{59}$$

In the dual vortex formulation, we can deduce a field theory from (58) for the complex field $\phi \sim e^{i\vartheta}$:

$$
\begin{aligned}
\mathcal{Z}_\phi &= \int \mathcal{D}\phi \mathcal{D}a_\mu \, e^{-\mathcal{S}_\phi} \\
\mathcal{S}_\phi &= \int d^3x \left[|(\partial_\mu - i a_\mu)\phi|^2 + s|\phi|^2 + v|\phi|^4 + \frac{1}{2K} \left(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda\right)^2 \right] .
\end{aligned} \tag{60}
$$

The precise claim is that the universal theory describing the phase transition in $\mathcal{S}_\psi$, as the parameter r is tuned across a symmetry-breaking transition at $r = r_c$, is identical to the theory describing the phase transition in $\mathcal{S}_\phi$ as a function of the tuning parameter s. A key observation is that the duality reverses the phases in which the fields are condensed; in particular the phases are

- **XY order**: In $\mathcal{S}_\psi$: $\langle \psi \rangle \neq 0$ and $r < r_c$. However, in $\mathcal{S}_\phi$: $\langle \phi \rangle = 0$ and $s > s_c$.
- **XY disorder**: In $\mathcal{S}_\psi$: $\langle \psi \rangle = 0$ and $r > r_c$. However, in $\mathcal{S}_\phi$: $\langle \phi \rangle \neq 0$ and $s < s_c$.

We can also extend this precise duality to include the presence of an arbitrary spacetime dependent external gauge field A_μ. In the particle theory, this couples minimally to ψ, as expected

$$
\begin{aligned}
\mathcal{Z}_\psi[A_\mu] &= \int \mathcal{D}\psi \, e^{-\mathcal{S}_\psi} \\
\mathcal{S}_\psi &= \int d^3x \left[|(\partial_\mu - i A_\mu)\psi|^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4 \right] ,
\end{aligned} \tag{61}
$$

while in the vortex theory, the coupling follows from (56):

$$
\begin{aligned}
\mathcal{Z}_\phi[A_\mu] &= \int \mathcal{D}\phi \mathcal{D}a_\mu \, e^{-\mathcal{S}_\phi} \\
\mathcal{S}_\phi &= \int d^3x \left[|(\partial_\mu - i a_\mu)\phi|^2 + s|\phi|^2 + \frac{v}{2}|\phi|^4 + \frac{1}{2K} \left(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda\right)^2 + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \right] .
\end{aligned} \tag{62}
$$

The last term is a 'mutual' Chern-Simons term. The equivalence between (61) and (62) is reflected in the equality of their partition functions as functionals of A_μ

$$
\mathcal{Z}_\psi[A_\mu] = \mathcal{Z}_\phi[A_\mu], \tag{63}
$$

after a suitable normalization, and mappings between renormalized couplings away from the quantum critical point. This is a powerful non-perturbative connection between two strongly interacting field theories, and holds even for large and spacetime-dependent A_μ. It maps arbitrary multi-point correlators of the particle current to associated correlators of the electromagnetic flux in the vortex theory. In particular, by taking one derivative of both theories with respect to A_μ, we have the operator identification

$$
\psi^* \partial_\mu \psi - \psi \partial_\mu \psi^* = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda \tag{64}
$$

between the particle and the vortex theories.

Let us now consider the nature of the excitations in the XY ordered and disordered phases in turn.

1. XY order

In the particle theory, $r < r_c$, the ψ field is condensed. So the only low-energy excitation is the Nambu-Goldstone mode associated with the phase of ψ. We write $\psi \sim e^{i\theta}$, and the effective theory for θ is

$$\mathcal{S} = \frac{\rho_s}{2} \int d^3x \, (\partial_\mu \theta - A_\mu)^2 \tag{65}$$

where ρ_s is the helicity modulus, and we have included the form of the coupling to the external field A_μ by gauge invariance.

In the vortex theory, $s > s_c$, and so ϕ is uncondensed and gapped. Let us ignore the ϕ field to begin with. Then the only gapless fluctuations are associated with photon a_μ, and its low energy effective action is

$$\mathcal{S} = \int d^3x \left[\frac{1}{8\pi^2 \rho_s} \left(\epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda \right)^2 + \frac{i}{2\pi} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \right]. \tag{66}$$

In 2+1 dimensions, there is only one polarization of a gapless photon, and this corresponds precisely to the gapless θ scalar in the boson theory. Indeed, it is not difficult to prove that (65) and (66) are exactly equivalent, using a Hubbard-Stratanovich transformation (see the analysis below (71)), which is essential a continuous version of the discrete angle-integer transforms we have used in our duality analysis so far.

Turning to gapped excitations, in the vortex theory we have ϕ particles and anti-particles. They interact via a long-range interaction mediated by the exchange of the gapless photon. For static vortices, this interaction has the form a Coulomb interaction $\sim \ln(r)$ in 2+1 dimensions. In the boson theory, this logarithmic interaction precisely computes the interaction between vortices in the XY ordered phase.

2. XY disorder

This phase is simplest in the ψ theory: there are gapped particle and anti-particle excitations, quanta of ψ, which carry total A_μ charge $Q = 1$ and $Q = -1$ respectively. These excitations only have short-range interactions (associated with u).

The situation in the vortex theory is subtle, but ultimately yields the same set of excitations. The ϕ field is condensed. Consequently, by the Higgs mechanism, the a_μ gauge field has a non-zero "mass" and has a gap - so there is no gapless photon mode, as expected. But where are the excitations with quantized charges $Q = \pm 1$? These are '*vortices in vortices*'. In particular, $\mathcal{S}_\phi$ has solutions of its saddle point equations which are Abrikosov vortices. Because ϕ is condensed, any finite energy solution of ϕ must have the phase-winding of ϕ exactly match the line integral of a_μ. In particular, we can look for time-independent vortex saddle point solutions centered at the origin in which

$$\phi(x) = f(|x|)e^{i\vartheta(x)} \tag{67}$$

in which the angle ϑ winds by 2π upon encircling the origin. The saddle point equations show that $f(|x| \to 0) \sim |x|$, while

$$f(|x| \to \infty) = \sqrt{\frac{-s}{v}}. \tag{68}$$

Under these conditions, it is not difficult to show that finiteness of the energy requires

$$\oint dx_i \partial_i \vartheta = \oint dx_i a_i = \int d^2x \, \epsilon_{ij} \partial_i a_j \tag{69}$$

on any contour far from the center of the vortex. As the phase ϑ must be single-valued, we have from (64,69) that

$$Q = \frac{1}{2\pi} \int d^2x \epsilon_{ij} \partial_i a_j = \pm 1 \tag{70}$$

in the Abrikosov vortex/anti-vortex, as required. So the important conclusion is that the $Q = \pm 1$ particle and anti-particle excitations of the Mott insulator are the Abrikosov vortices and anti-vortices of the dual vortex theory.

There is another way to run through the above 'vortices in vortices' argument. Let us apply the mapping from (61) to (62) to the vortex theory. In other words, let us momentarily think of ϕ as the boson and a_μ as an external source field. Then the particle-to-vortex mapping from (61) to (62) applied to (62) yields a theory for a new dual scalar $\widetilde{\psi}$, and a new gauge field b_μ controlled by the action

$$\mathcal{S}_{\widetilde{\psi}} = \int d^3x \left[|(\partial_\mu - ib_\mu)\widetilde{\psi}|^2 + \widetilde{r}|\widetilde{\psi}|^2 + \frac{\widetilde{u}}{2}|\widetilde{\psi}|^4 + \frac{1}{2\widetilde{K}}\left(\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda\right)^2 + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu b_\lambda \right. \\ \left. \frac{1}{2K}\left(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda\right)^2 + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \right]. \tag{71}$$

Now we can exactly perform the Gaussian integral over a_μ: to keep issues of gauge invariance transparent, it is convenient to first decouple the Maxwell term using an auxilliary field P_μ

$$\mathcal{S}_{\widetilde{\psi}} = \int d^3x \left[|(\partial_\mu - ib_\mu)\widetilde{\psi}|^2 + \widetilde{r}|\widetilde{\psi}|^2 + \frac{\widetilde{u}}{2}|\widetilde{\psi}|^4 + \frac{1}{2\widetilde{K}}\left(\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda\right)^2 + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu b_\lambda \right. \\ \left. \frac{K}{2}P_\mu^2 - i\epsilon_{\mu\nu\lambda} P_\mu \partial_\nu a_\lambda + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \right]. \tag{72}$$

We can now perform the integral over a_μ, and obtain a delta function constraint which sets

$$P_\mu = b_\mu + A_\mu - \partial_\mu \alpha, \tag{73}$$

where α is an arbitrary scalar corresponding to a gauge choice; so we have

$$\mathcal{S}_{\widetilde{\psi}} = \int d^3x \left[|(\partial_\mu - ib_\mu)\widetilde{\psi}|^2 + \widetilde{r}|\widetilde{\psi}|^2 + \frac{\widetilde{u}}{2}|\widetilde{\psi}|^4 + \frac{1}{2\widetilde{K}}\left(\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda\right)^2 + \frac{K}{2}(b_\mu + A_\mu - \partial_\mu \alpha)^2 \right]. \tag{74}$$

The last term in (74) implies that the b_μ gauge field has been 'Higgsed' to the value $b_\mu = -A_\mu + \partial_\mu \alpha$. Setting b_μ to this value, we observe that α can be gauged away, and then $\mathcal{S}_{\widetilde{\psi}}$ reduces to the original particle theory in (61). So applying the particle-to-vortex duality to the vortex theory yields back the particle theory.

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