3

Classical phase transitions

Given the motivation outlined in Chapter 2, we will begin by discussing phase transitions in the context of classical statistical mechanics. This is a vast subject, and the reader can find many other books which explore many subtle issues. Here, our purpose will be to summarize the main ideas which will transfer easily to our subsequent discussions of quantum phase transitions.

We will consider the most important models of classical phase transitions: ferromagnets with N component spins residing on the sites, i, of a hypercubic lattice. For $N = 1$, this is the familiar Ising model with partition function we already met in (2.2):

$$\mathcal{Z} = \sum_{\{\sigma_i^z = \pm 1\}} \exp(-H)$$
$$H = -K \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z, \tag{3.1}$$

Note that we have refrained from inserting an explicit factor of temperature above, as the symbol T will be reserved for the temperature of the quantum systems we consider later. As is discussed in introductory statistical mechanics texts, this Ising model describes the vicinity of the liquid-gas critical point, with the average value $\langle \sigma_i^z \rangle$ measuring the density in the vicinity of site i. It can also model loss of ferromagnetism with increasing temperature in magnets in which the electronic spins preferentially align along a particular crystalline axis: this 'easy-axis' behavior can be induced by the spin-orbit interaction. For $N > 1$, we generalize (3.1) to the model of (2.3)

$$\mathcal{Z} = \prod_i \int D\mathbf{n}_i \delta\left(\mathbf{n}_i^2 - 1\right) \exp\left(K \sum_{\langle i,j \rangle} \mathbf{n}_i \cdot \mathbf{n}_j\right). \tag{3.2}$$

The $N = 3$ case (known as the 'Heisenberg' model) describes ferromagnetism in materials with sufficiently weak small spin-orbit couplings, so that the spins can freely orient along any direction. The $N = 2$ case (known as the 'XY' model) describes the superfluid-normal transition in liquid helium and other superfluids, as we will see in Section 8.3.

The observables, $\mathcal{O}$, of the classical models will be arbitrary functions of the $\mathbf{n}_i$, and we will be interested in their expectation values defined by

$$\langle \mathcal{O} \rangle \equiv \frac{1}{\mathcal{Z}} \sum_{\{\sigma_i^z = \pm 1\}} \exp(-H) \mathcal{O} \tag{3.3}$$

for $N = 1$, and similarly for $N \geq 1$.

As discussed in Chapter 2, the models in Eqs. (3.1) and (3.2) undergo phase transitions at some critical $K = K_c$. We are interested in describing the nature of the spin correlations in the vicinity of this critical point, and especially their universal aspects. We will begin in Section 3.1 by describing this transition using a variational method which leads to a 'mean field' theory. A more general formulation of the mean field results appears in the framework of Landau theory in Section 3.2, which allows easy treatment of spatial variations. Finally, corrections to Landau theory will be considered in Section 3.3.

3.1 Mean field theory

First, we give a heuristic derivation of mean field theory. Here, and below, for notational simplicity we will focus on the $N = 1$ Ising case, although the generalization to $N > 1$ is not difficult.

We focus on the fluctuations on a particular spin σ_i^z. This spin feels the local Hamiltonian

$$\begin{aligned} & -K\sigma_i^z \left(\sum_{j \text{ neighbor of } i} \sigma_j^z \right) \\ \approx & -K\sigma_i^z \left(\sum_{j \text{ neighbor of } i} \left\langle \sigma_j^z \right\rangle \right) \\ = & -2DKN_0\sigma_i^z. \end{aligned} \tag{3.4}$$

The mean field approximation is on the second line, where we replace all the neighboring spins by their average value. Here N_0 is the ferromagnetic moment, defined by

$$N_0 \equiv \langle \sigma_i^z \rangle \tag{3.5}$$

in the full theory, where the translational invariance of H guarantees the independence of N_0 on the site i. Given the simple effective Hamiltonian for site i in Eq. (3.4), we can now evaluate

$$\langle\sigma_i^z\rangle = \frac{\sum_{\sigma_i^z=\pm1}\sigma_i^z \exp\left(2DKN_0\sigma_i^z\right)}{\sum_{\sigma_i^z=\pm1}\exp\left(2DKN_0\sigma_i^z\right)} = \tanh(2DKN_0) \tag{3.6}$$

Combining Eqs. (3.5) and (3.6), we have our central mean field equation

$$N_0 = \tanh(2DKN_0) \tag{3.7}$$

for the value of N_0. We will discuss the nature of its solutions shortly.

Let us now give a more formal derivation of Eq. (3.7) using the variational method. This method relies on the choice of an arbitrary mean-field Hamiltonian H_{MF}. Naturally, we want to choose H_{MF} so that we are able to easily evaluate its partition function $\mathcal{Z}_{MF}$, and the expectation values of all the observables, which we denote $\langle\mathcal{O}\rangle_{MF}$ after evaluation as in (3.3) but with H replaced by H_{MF}. We now want to optimize the choice of H_{MF} by a variational principle which bounds the exact free energy $\mathcal{F} = -\ln\mathcal{Z}$. Of course, the best possible choice is $H_{MF} = H$, but this does not allow easy evaluation of correlations. The variational principle descends from the theorem

$$\mathcal{F} \leq \mathcal{F}_{MF} + \langle H - H_{MF}\rangle_{MF}\,. \tag{3.8}$$

The proof of the theorem proceeds as follows (here, and below, we use the symbol "Tr" to denote the sum over all the σ_i^z):

$$\begin{aligned} e^{-\mathcal{F}} &= \mathrm{Tr} e^{-H} \\ &= \mathrm{Tr} e^{-(H-H_{MF})-H_{MF}} \\ &= e^{-\mathcal{F}_{MF}}\left\langle e^{-(H-H_{MF})}\right\rangle_{MF} \end{aligned} \tag{3.9}$$

We now use the statement of the convexity of the exponential function, which is

$$\left\langle e^{-\mathcal{O}}\right\rangle \geq e^{-\langle\mathcal{O}\rangle}. \tag{3.10}$$

Taking logarithms of both sides of Eq. (3.9), we finally obtain Eq. (3.8).

Returning to the Ising model, we choose the simplest H_{MF} consisting of a set of decoupled spins in a 'mean field' h_{MF}:

$$H_{MF} = -h_{MF}\sum_i \sigma_i^z \tag{3.11}$$

Then, we see immediately that

$$\mathcal{F}_{MF} = -M\ln(2\cosh h_{MF}), \tag{3.12}$$

where M is the total number of sites on the lattice, and

$$N_0 = \langle \sigma_i^z \rangle_{MF} = -\frac{1}{M}\frac{\partial \mathcal{F}_{MF}}{\partial h_{MF}} = \tanh(h_{MF}) \tag{3.13}$$

Using Eq. (3.8) to bound the free energy, we have

$$\begin{aligned} \mathcal{F} &\leq \mathcal{F}_{MF} - K\sum_{\langle ij \rangle} \langle \sigma_i^z \sigma_j^z \rangle + h_{MF}\sum_i \langle \sigma_i^z \rangle \\ &\leq \mathcal{F}_{MF} - MKDN_0^2 + Mh_{MF}N_0 \end{aligned} \tag{3.14}$$

We now have upper bounds for the free energy for every value of the, so far, undetermined parameter h_{MF}. Clearly we want to choose h_{MF} to minimize the right hand side of Eq. (3.14); we will declare the resulting upper bound as our approximate result for $\mathcal{F}$ — this is the mean field approximation. Actually, it is helpful to trade the variational parameter h_{MF} with the value of the ferromagnetic moment N_0, which are related to each other by Eq. (3.13). So our variational parameter is now N_0, and our mean field free energy is a function of $\mathcal{N}_0$ given by

$$\mathcal{F}(N_0)/M = \mathcal{F}_{MF}(N_0)/M - KDN_0^2 + N_0 h_{MF}(N_0) \tag{3.15}$$

where the functions $\mathcal{F}_{MF}(N_0)$ and $h_{MF}(N_0)$ are defined by Eqs. (3.12) and (3.13). Using these expressions, we obtain the explicit expression

$$\frac{\mathcal{F}(N_0)}{M} = -KDN_0^2 + \frac{1+N_0}{2}\ln\frac{1+N_0}{2} + \frac{1-N_0}{2}\ln\frac{1-N_0}{2}, \tag{3.16}$$

which can be interpreted as the sum of estimates of the internal energy and entropy of the Ising spins (See exercise 3.3.1). Our task is now to minimize Eq. (3.16) over values of N_0 for each K. Before examining the results of this, let us examine the nature of the stationarity condition by taking the derivative of Eq. (3.15):

$$\frac{1}{M}\frac{\partial \mathcal{F}}{\partial N_0} = \frac{1}{M}\frac{\partial \mathcal{F}}{\partial h_{MF}}\frac{\partial h_{MF}}{\partial N_0} - 2KDN_0 + h_{MF} + N_0\frac{\partial h_{MF}}{\partial N_0} \tag{3.17}$$

Using Eq. (3.13) we observe that the first and last terms cancel, and so the stationarity condition is simply $h_{MF} = 2KDN_0$, which is finally equivalent to our earlier heuristic result in Eq. (3.7).

Rather than solving Eq. (3.7), it is more instructive to examine the solution by plotting Eq. (3.16) as a function of N_0 for different K. This is shown in Fig. 3.1. We notice a qualitative change in the nature of the minimization at $K = K_c = 1/(2D)$. For $K < K_c$ (high temperatures) the free energy is minimized by $N_0 = 0$: this corresponds to the high temperature 'paramagnetic' phase. However, for $K > K_c$ we have two degenerate minima at

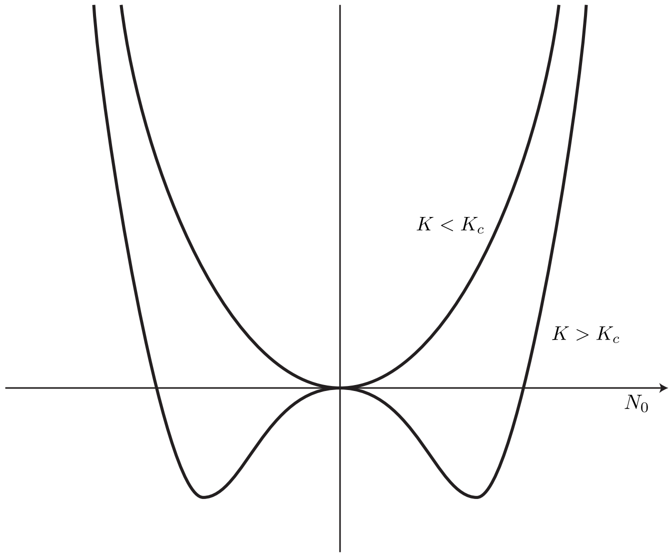

Figure 3.1 Plot of (3.16) as function of N_0 for different K.

non-zero values of N_0 which have the same magnitude but opposite signs. The system will 'spontaneously' choose one of these equivalent minima, leading to ferromagnetic order. Note that this choice is not invariant under the spin-flip symmetry of the underlying H, and so this is a simple illustration of the phenomenon of 'spontaneous symmetry breaking'. The critical point $K = K_c$ is the position of the phase transition between the paramagnetic and ferromagnetic phases.

3.2 Landau theory

The main idea of mean field theory has been to represent the 2^M degrees of freedom in the Ising model by a single mean magnetization, N_0. The free energy is determined as a function N_0, and then minimized to obtain the optimal equilibrium state.

Landau theory retains the idea of free energy optimization, but generalizes N_0 to a continuum field $\phi_\alpha(x)$. Here $\alpha = 1 \ldots N$, and x is a D-dimensional co-ordinate associated with a hypercubic lattice of spacing a. We have now returned to a consideration of the theory for general N. The central actor in Landau theory will be a free energy *functional*, $\mathcal{F}[\phi_\alpha(x)]$, which has to be minimized with respect to variations in $\phi_\alpha(x)$.

For now, we will keep the definition of $\phi_\alpha(x)$ somewhat imprecise. Physically, $\phi_\alpha(x)$ represents a coarse-grained average of the local magnetization

$n_{i\alpha}$ in the vicinity of $x = x_i$, as we discussed in (2.10) for the corresponding quantum model

$$\phi_\alpha(x) \sim \sum_{i \in \mathcal{N}(x)} n_{i\alpha} \tag{3.18}$$

As discussed below (2.10), we view ϕ_α as a "soft" spin whose magnitude can vary freely over all positive values.

The Landau free energy functional $\mathcal{F}[\phi_\alpha(x)]$ is now derived from a few basic principles:

- The Hamiltonian is invariant under a common O(N) rotation, $n_{i\alpha} \to R_{\alpha\beta} n_{i\beta}$, where $R_{\alpha\beta}$ is a rotation matrix applied to all sites i. So the free energy should also be invariant under global rotations of the ϕ_α. We saw an example of this for $N = 1$: then the only symmetry is $\sigma_i^z \to -\sigma_i^z$, and consequently, Eq. (3.16) is an even function of N_0.
- Near the critical point at $K = K_c$, the average value N_0 was smaller than the natural value $|\sigma_i^z| = 1$. We expect this to also hold for $N > 1$. After the coarse-graining in Eq. (3.18), we expect that ϕ_α is small in a similar sense. Our main interest will be in the vicinity of K_c, and therefore the Landau functional will be expanded in powers of ϕ_α.
- A key step by which Landau theory improves mean field theory is that it allows for spatial variations in the local magnetic order. We will assume here that the important spatial variations occur on a scale which is much larger than a lattice spacing. This assumption will be seen to be valid later, provided we are close to the critical point $K = K_c$. With this assumption, we will be able to expand the free energy functional in gradients of ϕ_α.

We are now prepared to write down the important terms in $\mathcal{F}[\phi_\alpha(x)]$. Expanding in powers and gradients of $\phi_\alpha(x)$ we have

$$\mathcal{F} = \int d^D x \left\{ \frac{1}{2} \left[\mathcal{K} (\nabla_x \phi_\alpha)^2 + r \phi_\alpha^2(x) \right] + \frac{u}{4!} \left(\phi_\alpha^2(x) \right)^2 \right\}, \tag{3.19}$$

expressed in terms of the parameters $\mathcal{K}$, r, u. We may regard these as unknown phenomenological parameters that have to be determined by fitting to experimental or numerical data. However, we can obtain initial estimates by matching to the results on mean field theory in Section 3.1. First, we set the overall normalization of ϕ_α by setting its average value equal to that of the $n_{i\alpha}$:

$$\langle \phi_\alpha(x_i) \rangle = \langle n_{i\alpha} \rangle . \tag{3.20}$$

Then comparing Eq. (3.19) with the expansion of Eq. (3.16) to quartic order, we obtain for $N = 1$

$$r = a^{-D}(1 - 2DK) \quad , \quad u = 2a^{-D}. \tag{3.21}$$

Of course, this method does not yield the value of $\mathcal{K}$, because mean field theory does not have any spatial variations. To estimate $\mathcal{K}$ we may examine the energy of a domain wall at low temperatures between two oppositely oriented ferromagnetic domains with $\sigma_i^z = \pm 1$. Such a domain wall has energy $2K$ per unit length; computing the domain wall energy using Eq. (3.19), we obtain the estimate $\mathcal{K} \approx 2Ka^{2-D}$.

An interesting feature of Eq. (3.21) is that $r = 0$ at $K = K_c$. In fact, we expect quite generally that $r \sim K_c - K$, as a simple argument now shows. The optimum value of $\phi_\alpha(x)$ under Eq. (3.19) is clearly given by a space-independent solution (provided $\mathcal{K} > 0$). For $r > 0$ (and assuming that $u > 0$ generally), the minimum of $\mathcal{F}$ is obtained by $\phi_\alpha(x) = 0$. This is clearly the paramagnetic phase. In contrast, for $r < 0$, it will pay to choose a space-independent but non-zero ϕ_α. The O(N) invariance of $\mathcal{F}$ guarantees that there are a degenerate set of minima that map onto each other under O(N) rotations. So let us orient ϕ_α along the $\alpha = 1$ axis, and write

$$\phi_\alpha = \delta_{\alpha,1} N_0 \tag{3.22}$$

Inserting this into Eq. (3.19) and minimizing for $r < 0$ we obtain

$$N_0 = \sqrt{\frac{-6r}{u}} \tag{3.23}$$

This shows that N_0 vanishes as $r \nearrow 0$, and the approach to the critical point allows us to introduce the *critical exponent* β defined by

$$N_0 \sim (-r)^\beta \tag{3.24}$$

Both mean field and Landau theory predict that $\beta = 1/2$, as can also be verified by a minimization of the full expression in Eq. (3.16). Related analyses of other observables allow us to obtain other critical exponents, as we explore in Exercise 3.3.4.

3.3 Fluctuations and perturbation theory

We now want to proceed beyond the mean field treatment of the phase transition at $K = K_c$. We expect that the value of K_c will have corrections to its mean field value of $1/(2D)$: we will not focus on these here because they are *non universal*, *i.e.* dependent upon specific details of the microscopic

Hamiltonian. Rather, our focus will be on universal quantities like the critical exponent β. The structure of Landau theory already suggests reasons why β may be universal: notice that the value of β depended only on the quartic polynomial structure of the free energy, which in turn followed from symmetry considerations. Modifying the form of H, *e.g.* by adding second-neighbor ferromagnetic couplings, would not change the arguments leading to the Landau free energy functional, and we would still obtain $\beta = 1/2$ although K_c would change.

Indeed, the universality suggested by Landau theory is too strong. It indicates that any ferromagnet with O(N) symmetry always has $\beta = 1/2$. We will now see that there are fluctuation corrections to Landau theory, and that the universal quantities depend not only on symmetry, but also on the dimensionality, D. The Landau theory predictions are correct for $D > 4$, while there are corrections for $D \leq 4$. We will need the renormalization group approach to compute universal quantities for $D < 4$, and this will be described in Chapter 4.

One way to address fluctuation corrections is to return to the underlying partition function in Eq. (3.1), and expand it as a power series in K or in $1/K$. Such series expansions have been carried out to very high orders, and they are efficient and accurate methods for describing the behavior at high and low temperatures. However, they are not directly suited for addressing the vicinity of the critical point $K = K_c$. Instead, we would like to use a method which builds on the success of Landau theory, and yields its results at leading order. The coarse-graining arguments associated with Eq. (3.18) suggest a route to achieving this: rather than summing over all the individual spins in Eq. (3.1), we should integrate over all values of the collective field variable $\phi_\alpha(x)$. In other words, we should regard the expression in Eq. (3.19) not as the free energy functional, but as the Hamiltonian (or 'action') of a classical statistical mechanics problem in which the degrees of freedom are represented by the field $\phi_\alpha(x)$. The partition function is therefore represented by the functional integral

$$\mathcal{Z} = \int \mathcal{D}\phi_\alpha(x) \exp(-\mathcal{S}_\phi),$$

$$\mathcal{S}_\phi = \int d^D x \left\{ \frac{1}{2} \left[(\nabla_x \phi_\alpha)^2 + r\phi_\alpha^2(x) \right] + \frac{u}{4!} \left(\phi_\alpha^2(x) \right)^2 \right\}. \qquad (3.25)$$

Here the symbol $\int \mathcal{D}\phi_\alpha(x)$ represents an infinite dimensional integral over the values of the field $\phi_\alpha(x)$ at every spatial point x. Whenever in doubt, we will interpret this somewhat vague mathematical definition by discretizing

x to a set of lattice points of small spacing $\sim 1/\Lambda$. Equivalently, we will Fourier transform $\phi_\alpha(x)$ to $\phi_\alpha(k)$, and impose a cutoff $|k| < \Lambda$ in the set of allowed wavevectors.

We have set the co-efficient of the gradient term $\mathcal{K}$ equal to unity in (3.25). This is to avoid clutter of notation, and is easily accomplished by an appropriate rescaling of the field ϕ_α and the spatial co-ordinates.

An immediate advantage of the representation in Eq. (3.25) is that Landau theory is obtained simply by making the saddle-point approximation to the functional integral. We can also see that, as will be described in more detail below, systematic corrections to Landau theory appear in an expansion in powers of the quartic coupling u. The remainder of this chapter is devoted to explaining how to compute the terms in the u expansion. Each term has an efficient representation in terms of "Feynman diagrams", from which an analytic expression can also be obtained.

3.3.1 Gaussian integrals

We introduce the technology of Feynman diagrams in the simplest possible setting. Let us discretize space, and write the $\phi_\alpha(x_i)$ variables as y_i; we drop the α label to avoid clutter of indices. Then consider the multidimensional integral

$$\mathcal{Z}(u) = \int \mathcal{D}y \exp\left(-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j - \frac{u}{24}\sum_i y_i^4\right), \tag{3.26}$$

where the off-diagonal terms in the matrix A arise from the spatial gradient terms in $\mathcal{S}_\phi$. In this section, we will consider A to be an arbitrary positive definite, symmetric matrix. The positive definiteness requires that $r > 0$ *i.e.* $K < K_c$. Also, we have defined

$$\int \mathcal{D}y = \prod_i \int_{-\infty}^{\infty} \frac{dy_i}{\sqrt{2\pi}}, \tag{3.27}$$

and are interested in the expansion of $\mathcal{Z}(u)$ in powers of u. Thinking of Eq. (3.26) as a statistical mechanics ensemble, we will also be interested in the power series expansion of correlators like

$$C_{ij}(u) \equiv \langle y_i y_j \rangle \equiv \frac{1}{\mathcal{Z}(u)} \int \mathcal{D}y\, y_i y_j \exp\left(-\frac{1}{2}\sum_{k\ell} y_k A_{k\ell} y_\ell - \frac{u}{24}\sum_k y_k^4\right). \tag{3.28}$$

First, we note the exact expressions for these quantities at $u = 0$. The

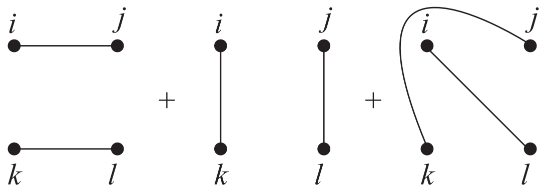

Figure 3.2 Diagrammatic representation of the 4-point correlator in (3.33). Each line is factor of the propagator in (3.31).

partition function is

$$\mathcal{Z}(0) = (\det A)^{-1/2} \tag{3.29}$$

This result is most easily obtained by performing an orthogonal rotation of the y_i to a basis which diagonalizes the matrix A_{ij} before performing the integral (exercise 3.3.5). Also useful for the u expansion is the identity

$$\int \mathcal{D}y \exp\left(-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j - \sum_i J_i y_i\right) = (\det A)^{-1/2} \exp\left(\frac{1}{2}\sum_{ij} J_i A^{-1}_{ij} J_j\right), \tag{3.30}$$

which is obtained by shifting the y_i variables to complete the square in the argument of the exponential. By taking derivatives of this identity with respect to the J_i, and then setting $J_i = 0$, we can generate expressions of all the correlators at $u = 0$. In particular, the two-point correlator is

$$C_{ij}(0) = A^{-1}_{ij}. \tag{3.31}$$

For the $2n$-point correlator, we have an expression known as Wick's theorem:

$$\langle y_1 y_2 \ldots y_{2n} \rangle = \sum_P \langle y_{P1} y_{P2} \rangle \ldots \left\langle y_{P(2n-1)} y_{P2n} \right\rangle \tag{3.32}$$

where the summation over P represents the sum over all possible products of pairs, and we reiterate that both sides of the equation are evaluated at $u = 0$. Thus for the 4-point correlator, we have

$$\langle y_i y_j y_k y_\ell \rangle = \langle y_i y_j \rangle \langle y_k y_\ell \rangle + \langle y_i y_k \rangle \langle y_j y_\ell \rangle + \langle y_i y_\ell \rangle \langle y_j y_k \rangle \tag{3.33}$$

There is a natural diagrammatic representation of the right hand side of Eq. (3.33): we represent each distinct field y_i by a dot, and then draw a line between dots i and j to represent each factor of $C_{ij}(0)$. See Fig 3.2.

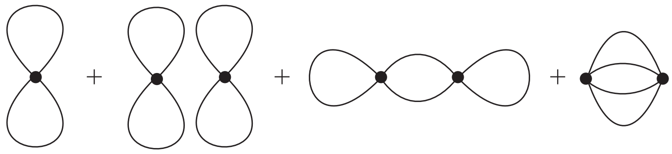

Figure 3.3 Diagrams for the partition function to order u^2.

We can now generate the needed expansions of $\mathcal{Z}(u)$ and $C_{ij}(u)$ simply by expanding the integrands in powers of u, and evaluating the resulting series term by term using Wick's theorem. What follows is simply a set of very useful diagrammatic rules for efficiently obtaining the answer at each order. However, whenever in doubt on the value of a diagram, it is often easiest to go back to this primary definition.

For $\mathcal{Z}(u)$, expanding to order u^2, we obtain the diagrams shown in Fig 3.3 which evaluate to the expression

$$\frac{\mathcal{Z}(u)}{\mathcal{Z}(0)} = 1 - \frac{u}{8}\sum_i (A^{-1}_{ii})^2 + \frac{1}{2}\left(\frac{u}{8}\sum_i (A^{-1}_{ii})^2\right)^2 + \frac{u^2}{16}\sum_{i,j} A^{-1}_{ii}A^{-1}_{jj}\left(A^{-1}_{ij}\right)^2 + \frac{u^2}{48}\sum_{i,j}\left(A^{-1}_{ij}\right)^4 + \mathcal{O}(u^3) \tag{3.34}$$

We are usually interested in the free energy, which is obtained by taking the logarithm of the above expression, yielding

$$\ln\frac{\mathcal{Z}(u)}{\mathcal{Z}(0)} = -\frac{u}{8}\sum_i (A^{-1}_{ii})^2 + \frac{u^2}{16}\sum_{i,j} A^{-1}_{ii}A^{-1}_{jj}\left(A^{-1}_{ij}\right)^2 + \frac{u^2}{48}\sum_{i,j}\left(A^{-1}_{ij}\right)^4 + \mathcal{O}(u^3) \tag{3.35}$$

Now, notice an important feature of Eq. (3.35): the terms here correspond precisely to the subset of the terms in Fig. 3.3 associated with the *connected* diagrams. These are diagrams in which all points are connected to each other by at least one line, and this result is an example of the "linked cluster theorem". We will not prove this very useful result here: at all orders in u, we can obtain the perturbation theory for the free energy by keeping only the connected diagrams in the expansion of the partition function.

Now let us consider the u expansion of the two-point correlator, $C_{ij}(u)$, in Eq. (3.28). Here, we have to expand the numerator and denominator in Eq. (3.28) in powers of u, evaluate each term using Wick's theorem, and then divide the result series. Fortunately, the linked cluster theorem simplifies things a great deal here too. The result of the division is simply to cancel all the disconnected diagrams. Thus, we need only expand the numerator,

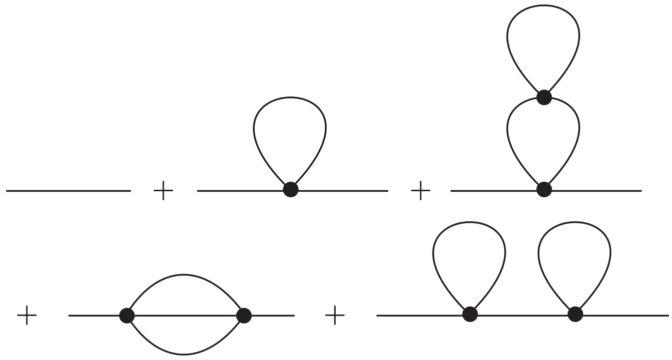

Figure 3.4 Diagrams for the two-point correlation function to order u^2.

and keep only connected diagrams. The diagrams are shown in Fig. 3.4 to order u^2, and they evaluate to

$$C_{ij}(u) = A^{-1}_{ij} - \frac{u}{2}\sum_k A^{-1}_{ik}A^{-1}_{kk}A^{-1}_{kj} + \frac{u^2}{4}\sum_{k,\ell} A^{-1}_{ik}A^{-1}_{kk}A^{-1}_{k\ell}A^{-1}_{\ell\ell}A^{-1}_{\ell j}$$
$$+ \frac{u^2}{4}\sum_{k,\ell} A^{-1}_{ik}\left(A^{-1}_{k\ell}\right)^2 A^{-1}_{\ell\ell}A^{-1}_{kj} + \frac{u^2}{6}\sum_{k,\ell} A^{-1}_{ik}\left(A^{-1}_{k\ell}\right)^3 A^{-1}_{\ell j}. \tag{3.36}$$

We now state the useful *Dyson's theorem*. For this, it is useful to consider the expansion of the inverse of the C_{ij} matrix, and write it as

$$C^{-1}_{ij} = A_{ij} - \Sigma_{ij} \tag{3.37}$$

where the matrix Σ_{ij} is called the "self energy", for historical reasons not appropriate here. Using Eq. (3.36), some algebra shows that to order u^2

$$\Sigma_{ij}(u) = -\delta_{ij}\frac{u}{2}A^{-1}_{ii} + \delta_{ij}\frac{u^2}{4}\sum_k \left(A^{-1}_{ik}\right)^2 A^{-1}_{kk} + \frac{u^2}{6}\left(A^{-1}_{ij}\right)^3, \tag{3.38}$$

and these are shown graphically in Fig. 3.5. Dyson's theorem states that we can obtain the expression for the Σ_{ij} directly from the graphs for C_{ij} in Fig. 3.4 by two modifications: (i) drop the factors of A^{-1} associated with external lines, and (ii) keep only the graphs which are one-particle irreducible (1PI). The latter are graphs which do not break into disconnected pieces when one internal line is cut; the last graph in Fig. 3.4 is one-particle reducible, and so does not appear in Eq. (3.38) and Fig. 3.5.

The expression in Eq. (3.38) will be the basis for much of the analysis in Part II.

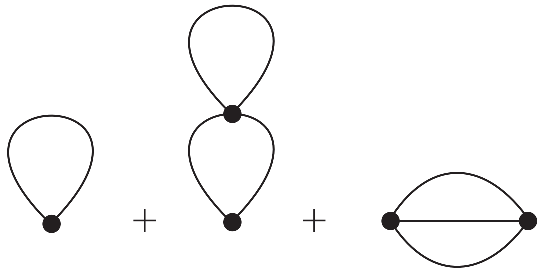

Figure 3.5 Diagrams for the self energy to order u^2.

3.3.2 Expansion for susceptibility

We now apply the results of Section 3.3.1 to our functional integral representation in Eq. (3.25) for the vicinity of the phase transition.

The problem defined by Eq. (3.25) has an important simplifying feature not shared by our general analysis of Eq. (3.26): translational invariance. This means that correlators only depend upon the differences of spatial coordinates, and that the analog of the matrix A can be diagonalized by a Fourier tranformation. So now we define the correlator

$$C_{\alpha\beta}(x-y) = \langle \phi_\alpha(x)\phi_\beta(y)\rangle - \langle \phi_\alpha(x)\rangle \langle \phi_\beta(y)\rangle \,, \tag{3.39}$$

where the subtraction allows generalization to the ferromagnetic phase; we will only consider the paramagnetic phase here.

The subtraction in Eq. (3.39) is also needed for the fluctuation-dissipation theorem. We will discuss the full version of this theorem in Section 7.1, but note a simpler version. We consider the susceptibility, $\chi_{\alpha\beta}$, the response of the system to an applied 'magnetic' field h_α under which the action changes as

$$\mathcal{S}_\phi \to \mathcal{S}_\phi - \int d^D x \, h_\alpha(x)\phi_\alpha(x) \tag{3.40}$$

Then

$$\chi_{\alpha\beta}(x-y) = \frac{\delta \langle \phi_\alpha(x)\rangle}{\delta h_\beta(y)} = C_{\alpha\beta}(x-y) \tag{3.41}$$

where the last equality follows from taking the derivative with respect to the field. Below we set $h_\alpha = 0$ after taking the derivative. The Fourier transform of the susceptibility $\chi_{\alpha\beta}$ is

$$\chi_{\alpha\beta}(k) = \int d^D x e^{-ikx} \chi_{\alpha\beta}(x) \tag{3.42}$$

In the paramagnetic phase $\chi_{\alpha\beta}(k) \equiv \delta_{\alpha\beta}\chi(k)$, and the susceptibility $\chi(k)$ will play a central role in our analysis.

We can also Fourier transform the field $\phi_\alpha(x)$ to $\phi_\alpha(k)$, and so obtain the following representation of the action from Eq. (3.25)

$$\mathcal{S}_\phi = \frac{1}{2}\int \frac{d^D k}{(2\pi)^D}|\phi_\alpha(k)|^2(k^2+r) \\ + \frac{u}{4!}\int \frac{d^D k}{(2\pi)^D}\frac{d^D q}{(2\pi)^D}\frac{d^D p}{(2\pi)^D}\phi_\alpha(k)\phi_\alpha(q)\phi_\alpha(p)\phi_\alpha(-k-p-q). \tag{3.43}$$

In this representation it is clear that the quadratic term in the action is diagonal, and so the inversion of the matrix A is immediate. In particular, from Eq. (3.31) we have the susceptibility at $u = 0$

$$\chi_0(k) = \frac{1}{k^2+r} \tag{3.44}$$

where we have defined $\chi_0(k)$ to be the value of $\chi(k)$ at $u = 0$. Dyson's theorem in Eq. (3.37) becomes a simple algebraic relation

$$\chi(k) = \frac{1}{1/\chi_0(k) - \Sigma(k)} = \frac{1}{k^2 + r - \Sigma(k)} \tag{3.45}$$

We will shortly obtain an explicit expression for $\Sigma(k)$.

Let us now explore some of the consequences of the $u = 0$ result in Eq. (3.44), which describes Gaussian fluctuations about mean field theory in the paramagnetic phase, $r > 0$. The zero momentum susceptibility, which we denote simply as $\chi \equiv \chi(k = 0) = 1/r$, diverges as we approach the phase transition at $K = K_c$ from the high temperature paramagnetic phase. This divergence is a key feature of the phase transition, and its nature is encoded in the critical exponent γ defined by

$$\chi \sim (K_c - K)^{-\gamma} \tag{3.46}$$

At this leading order in u we have $\gamma = 1$.

We can also examine the spatial correlations in the $u = 0$ theory. Performing the inverse Fourier transform to $C_{\alpha\beta}(x) = \delta_{\alpha\beta}C(x)$ we find

$$C(x) = \int \frac{d^D k}{(2\pi)^D}\frac{e^{ikx}}{(k^2+r)} = \frac{(2\pi)^{-D/2}}{(x\xi)^{(D-2)/2}}K_{(D-2)/2}(x/\xi), \tag{3.47}$$

where here K is the modified Bessel function, and we have introduced a characteristic length scale, ξ, defined by

$$\xi = 1/\sqrt{r}. \tag{3.48}$$

This is the correlation length, and is a measure of the distance over which fluctuations of ϕ_α (or the underlying spins σ_i^z) are correlated. This is evident from the limiting forms of Eq. (3.47) in various asymptotic regimes:

$$C(x) \sim \begin{cases} \dfrac{1}{x^{D-2}} & , \quad x \ll \xi \\ \dfrac{e^{-x/\xi}}{x^{(D-1)/2}\xi^{(D-3)/2}} & , \quad x \gg \xi \end{cases} . \tag{3.49}$$

As could be expected of a correlation length, the correlations decay exponentially to zero at distances larger than ξ.

An important property of our expression in Eq. (3.48) for the correlation length is that it diverges upon the approach to the critical point. This divergence is also associated with a critical exponent, ν, defined by

$$\xi \sim (K_c - K)^{-\nu}, \tag{3.50}$$

and our present theory yields $\nu = 1/2$. In the vicinity of the phase transition, this large value of ξ provides an a posteriori justification of our taking a continuum perspective on the fluctuations. In other words, it supports our mapping from the lattice models in Eqs. (3.1) and (3.2) to the classical field theory in Eq. (3.25), where we replaced the lattice spin variables by the collective field ϕ_α using Eq. (3.18).

Let us now move beyond the $u = 0$ theory, and consider the corrections at order u. After mapping to Fourier space, the result in Eq. (3.38) for the self-energy yields

$$\Sigma(k) = -u\frac{(N+2)}{6}\int \frac{d^D p}{(2\pi)^D}\frac{1}{p^2 + r} \tag{3.51}$$

Here, and below, there is an implicit upper bound of $k < \Lambda$ needed to obtain finite answers for the wavevector integrals. The N dependence comes from keeping track of the spin index α along each line of the Feynman diagram, and allowing for the different possible contractions of such indices at each u interaction point. We then have from Eq. (3.45) our main result for the correction in the susceptibility

$$\frac{1}{\chi(k)} = k^2 + r + u\frac{(N+2)}{6}\int \frac{d^D p}{(2\pi)^D}\frac{1}{p^2 + r} + \mathcal{O}(u^2) \tag{3.52}$$

The first consequence of Eq. (3.52) is a shift in the position of the critical point. From Eq. (3.46), a natural way to define the position of the phase transition is by the zero of $1/\chi$. The order u correction in Eq. (3.52) shows that the critical point is no longer at $r = r_c = 0$, but at

$$r_c = -u\frac{(N+2)}{6}\int \frac{d^D p}{(2\pi)^D}\frac{1}{p^2} + \mathcal{O}(u^2) \tag{3.53}$$

Now, let us combine Eqs. (3.52) and (3.53) to determine the behavior of χ as $r \searrow r_c$. We introduce the coupling s defined by

$$s \equiv r - r_c, \tag{3.54}$$

which measures the deviation of the system from the critical point. Rewriting (3.52) in terms of s rather than r (we will always use s in favor or r in all subsequent analysis), we have

$$\frac{1}{\chi} = s + u\left(\frac{N+2}{6}\right)\int^{\Lambda}\frac{d^Dp}{(2\pi)^D}\left(\frac{1}{p^2+s}-\frac{1}{p^2}\right). \tag{3.55}$$

We are interested in the vicinity of the critical point, at which $s \to 0$.

A crucial point is that the nature of this limit depends sensitively on whether D is greater than or less than four. For $D > 4$, we can simply expand the integrand in (3.55) in powers of s and obtain

$$\frac{1}{\chi} = s(1 - c_1 u \Lambda^{D-4}), \tag{3.56}$$

where c_1 is a nonuniversal constant dependent upon the nature of the cutoff. Thus the effects of interactions appear to be relatively innocuous: The static susceptibility still diverges with the mean-field form $\chi(0) \sim 1/s$ as $s \to 0$, with the critical exponent $\gamma = 1$. This is in fact the generic behavior to all orders in u, and all the mean-field critical exponents apply for $D > 4$.

For $D < 4$, we notice that the integrand in (3.55) is convergent at high momenta, and so it is permissible to send $\Lambda \to \infty$. We then find that the correction to first order in u has a universal form

$$\frac{1}{\chi} = s\left[1 - \left(\frac{N+2}{6}\right)\frac{2\Gamma((4-D)/2)}{(D-2)(4\pi)^{D/2}}\frac{u}{s^{(4-D)/2}}\right]. \tag{3.57}$$

Notice that no matter how small u is, the correction term eventually becomes important for a sufficiently small s, and indeed it diverges as $s \to 0$. So for sufficiently large ξ, the mean field behavior cannot be correct, and a resummation of the perturbation expansion in u is necessary.

The situtation becomes worse at higher orders in u. As suggested by Eq. (3.57), the perturbation series for $1/(s\chi)$ is actually in powers of $u/s^{(4-D)/2}$, and so each successive term diverges more strongly as $s \to 0$. Thus the present perturbative analysis is unable to describe the vicinity of the critical point for $D < 4$. We will show this problem is cured by a renormalization group treatment in the following chapter.

Exercises

3.1 Consider an Ising model on a system of N sites. Let $N_\uparrow$ be the number of up spins. Calculate, $\Gamma(N_\uparrow, N)$, the total number of ways these $N_\uparrow$ spins could have been placed among the N sites. Obtain the entropy $S = k_B \ln \Gamma$ as a function of the magnetization $m = (N_\uparrow - N_\downarrow)/N$, where $N_\downarrow = N - N_\uparrow$. Combine this computation of the entropy with a mean field estimate of the average internal energy to obtain (3.16).

3.2 *Ising antiferromagnet:* We consider the Ising antiferromagnet on a square lattice. The Hamiltonian is

$$H_I = J \sum_{\langle ij \rangle} \sigma_i \sigma_j \tag{3.58}$$

where i, j extend over the sites of a square lattice, $\langle ij \rangle$ refers to nearest neighbors, and $\sigma_i = \pm 1$. Note that there is no minus sign in front of J. We take $J > 0$, so the ferromagnetic state, with all σ_i parallel, is the *highest* energy state. The ground states have the pattern of a chess board: $\sigma_i = 1$ on one sublattice (A) and $\sigma_i = -1$ on the other sublattice (B), and vice versa. Use mean field theory to describe the phase diagram of this model. Argue that the mean field Hamiltonian should have two fields, h_A and h_B, on the two sublattices, and correspondingly, two magnetizations m_A and m_B. Obtain equations for m_A and m_B and determine the value of T_c.

3.3 *XY model:*. We generalize the Ising model (with binary spin variables σ_i) to a model of *vector spins*, $\vec{S}_i$, of unit length ($\vec{S}_i^2 = 1$ at each i). This is a model of ferromagnetism in materials, *e.g.* iron, in which the electron spin is free to rotate in all directions (rather than being restricted to be parallel or anti-parallel to a given direction, as in the Ising model). For simplicity, let us assume that the spin is only free to rotate within the x-y plane, *i.e.* $\vec{S}_i = (\cos\theta_i, \sin\theta_i)$. So the degree of freedom is an angle $0 \leq \theta_i < 2\pi$ on each site i. The Hamiltonian of these XY spins is

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \tag{3.59}$$

and the partition function is

$$Z = \prod_i \int_0^{2\pi} d\theta_i e^{-\beta\mathcal{H}}. \tag{3.60}$$

Use the variational approach to obtain a mean field theory for the mean magnetization $\vec{m} = \langle \vec{S}_i \rangle$. at a temperature T. Use a trial Hamiltonian

$\mathcal{H}_0 = -\sum_i \vec{h} \cdot \vec{S}_i$, and bound the free energy by $F \leq \langle \mathcal{H} - \mathcal{H}_0 \rangle_0 + F_0$. Argue that because of spin rotational invariance (about the z axis) we can choose $\vec{h}$ and $\vec{m}$ to point along the x axis, and hence $\mathcal{H}_0 = -\sum_i h \cos\theta_i$. Also, in the variational approach we define

$$m \equiv \langle \cos\theta \rangle_0 = -\frac{1}{N}\frac{\partial F_0(h)}{\partial h}. \tag{3.61}$$

We use this equation for m to solve for h as a function of m, and so now we can consider F to be a function of the variational parameter m. Show that F equals

$$F = -\frac{NqJm^2}{2} + Nhm + F_0(h) \tag{3.62}$$

Use these equations to evaluate $\partial F/\partial m$ and so obtain the mean-field equation for m (note that you do not need an explicit form for F_0 to obtain this) and determine the critical temperature on a lattice with co-ordination number q.

3.4 *Critical exponents:* In mean field theory, the free energy density of the Ising model near its critical point can be obtained by minimizing the functional of the magnetization density m:

$$F = \frac{a}{2}m^2 + \frac{b}{4}m^4 - hm + F_0(T)$$

Here F represents the Helmholtz free energy density; $F_0(T)$ is a smooth background function of T, all of whose derivatives are finite and non-zero at $T = T_c$. Assume b is independent of T, while a is approximated by a linear T dependence $a = a_0(T - T_c)$.

(a) The critical exponent β is defined by the manner in which m vanishes at T_c at $h = 0$:

$$m \sim (T_c - T)^\beta$$

What is the mean field value of β ?

(b) The critical exponent δ is determined by the h dependence of m at $T = T_c$:

$$m \sim h^{1/\delta}$$

What is the mean field value of δ ?

(c) The critical exponents γ and γ' are defined by the manner in which the magnetic susceptibility $\chi = \partial m/\partial h|_{h\to 0}$ behaves above and below T_c:

$$\chi \sim A(T - T_c)^{-\gamma} \quad , \quad T > T_c$$
$$\chi \sim A'(T_c - T)^{-\gamma'} \quad , \quad T < T_c$$

Determine the values of γ, γ', and A/A'.

(d) Similarly, the behavior of the specific heat, $C_V = -T(\partial^2 F/\partial T^2)$ at $h = 0$ is specified by

$$C_V \sim B(T - T_c)^{-\alpha} \quad , \quad T > T_c$$
$$C_V \sim B'(T_c - T)^{-\alpha'} \quad , \quad T < T_c$$

Determine the values of α, α', and B and B'.

3.5 Establish (3.29) by changing variables of integration in (3.26) so that the matrix A is diagonal in the new basis. This will involve working with the eigenvectors and eigenvalues of A.

3.6 Redo the computations in Section 3.3.1 for an N component field $y_{i\alpha}$.

3.7 *Landau theory for the XY model:* This is expressed in terms of a vector field $\vec{m}(r) = (m_1(r), m_2(r))$. In zero applied field, the Landau free energy has the form

$$\mathcal{F} = \int d^3r \left[\frac{K}{2} \sum_{i=x,y,z} \sum_{a=1,2} (\partial_i m_a)^2 + \frac{\alpha}{2} \left(\sum_{a=1,2} m_a^2 \right) + \frac{\beta}{4} \left(\sum_{a=1,2} m_a^2 \right)^2 \right] \tag{3.63}$$

and the critical point is at $\alpha = 0$ (Assume, $K, \beta > 0$). Determine the correlation functions $G_{11}(r) = \langle m_1(r) m_1(0) \rangle - \langle m_1(r) \rangle \langle m_1(0) \rangle$ and $G_{22}(r) = \langle m_2(r) m_2(0) \rangle - \langle m_2(r) \rangle \langle m_2(0) \rangle$ for both signs of α. You can compute these correlation functions by applying an external field $\vec{h}(r)$ to the system, under which

$$\mathcal{F} \to \mathcal{F} - \int d^3r \sum_{a=1,2} h_a(r) m_a(r), \tag{3.64}$$

and then computing the change in $\langle m_a \rangle$ due to the presence of the field. As for the Ising model, we have to linear order in $\vec{h}$

$$\langle m_a(r) \rangle|_{\vec{h}} = \langle m_a(r) \rangle|_{\vec{h}=0} + \frac{1}{k_B T} \int d^3r' G_{aa}(r - r') h_a(r') + \ldots \tag{3.65}$$

By writing $\vec{m}$ in the above form, you can read off the values of G_{aa}. Above the critical temperature ($\alpha > 0$), you should find $G_{11} = G_{22}$, which is a simple consequence of rotational invariance. Below the critical temperature, ($\alpha < 0$), choose the state with $\langle m_1(r) \rangle = \sqrt{|\alpha|/\beta}$ and

$\langle m_2(r) \rangle = 0$ in zero field. You should find $G_{11} \neq G_{22}$, and determine both functions in 3 dimensions.

4

The renormalization group

In Chapter 3 we developed the basic tools to describe the phase transition in the classical Ising model, and its cousins with N-component spins. We argued that the vicinity of the critical point at $K = K_c$ could be described by the classical field theory in (3.25). However, we observed that an expansion of the observable properties in powers of the quartic coupling u broke down near the critical point for dimensions $D < 4$. We will now show how the renormalization group circumvents this breakdown.

In its full generality, the renormalization group (RG) is a powerful tool with applications in many fields of physics, and covered at great lengths in other texts. Our treatment here will be relatively brief, and will be directed towards addressing the critical properties of the classical field theory in (3.25).

The key to the success of the RG is that it exposes a new symmetry of the critical point at $K = K_c$, which is not present in the underlying Hamiltonian. This symmetry can be understood to be a consequence of the divergence of the correlation length, ξ, at $K = K_c$, as indicated in (3.50). With the characteristic length scale equal to infinity, we may guess that structure of the correlations is the same at all length scales *i.e.* the physics is invariant under a *scaling transformation* under which the co-ordinates change as

$$x \to x' = x/b \tag{4.1}$$

where b is the rescaling factor. In other words, the basic structure of all correlations should be invariant under a transformation from the x to the x' co-ordinates. An example of this invariance appears in our result for the two-point correlation function $C(x) \sim x^{2-D}$ for $x \ll \xi$ in (3.49); this is only rescaled by an overall prefactor under the transformation (4.1). We should note here that the lattice statistical mechanics model does have a characteristic length scale, which is a the lattice spacing; the invariance under (4.1) holds only at lengths much larger than a, and only at the critical point.

We will see that the role of the scale invariance is similar to that of other, more familiar, symmetries. As an example, symmetry under rotations is due to an invariance of the Hamiltonian under a change in angular co-ordinates $\theta' = \theta + b$; (4.1) is an analogous co-ordinate transformation. Further, we know that rotational invariance classifies observables according to how they respond to the co-ordinate change: scalars, vectors, tensors, ..., which are labeled by different values of the angular momentum. We will similarly find that scale invariance labels observables by their *scaling dimension*.

4.1 Gaussian theory

It is useful to begin with a simplified model for which the scaling transformations can be exactly computed. This is the free field theory obtained from (3.25) at $u = 0$: let us write it down here explicitly for completeness:

$$\mathcal{Z} = \int \mathcal{D}\phi_\alpha(x) \exp\left(-\frac{1}{2}\int d^D x \left[(\nabla_x \phi_\alpha)^2 + r\phi_\alpha^2(x)\right]\right). \tag{4.2}$$

It is now easy to see that all correlations associated with this ensemble will maintain their form if we combine the rescaling transformation (4.1) with the definitions

$$\begin{aligned} \phi'_\alpha(x') &= b^{(D-2)/2}\phi_\alpha(x) \\ r' &= b^2 r \end{aligned} \tag{4.3}$$

The powers of b appearing in Eq. (4.3) are the scaling dimensions of the respective variables, which we denote as

$$\begin{aligned} \dim[\phi] &= (D-2)/2 \\ \dim[r] &= 2. \end{aligned} \tag{4.4}$$

Also, by definition, we always have from (4.1) that $\dim[x] = -1$.

These scaling transformations now place strong restrictions on the form of the correlation functions. Thus for the two-point correlation of ϕ_α, we have from Eq. (4.3) that

$$\left\langle \phi'_\alpha(x')\phi'_\beta(0)\right\rangle = b^{(D-2)} \left\langle \phi_\alpha(x)\phi_\beta(0)\right\rangle \tag{4.5}$$

However both correlators are evaluated in the same ensemble, and ϕ_α and ϕ'_α are merely dummy variables of integration which can be relabeled at will.

Therefore the correlators must have the same functional dependence on the spatial co-ordinates and the couplings constants, and so

$$C(x/b;b^2r) = b^{D-2}C(x;r) \tag{4.6}$$

where we have indicated the dependence of the correlator on the coupling r, which was previously left implicit. This is the payoff equation from the RG transformation, and places a non-trivial constraint on the form of the correlations; it can be checked that the results (3.47) and (3.48) do obey (4.6).

Although it is strictly not necessary here, it will be useful to build the rescaling transformation by the factor b via a series of infinitesimal transformations. This is analogous to the use of angular momentum to generate infinitesimal rotation in quantum mechanics. Here, we set $b = 1 + d\ell$, where $d\ell \ll 1$, and build up to a finite rescaling $b = e^{\ell}$ by a repeated action of infinitesimal rescalings. These rescalings are most conveniently represented in terms of differential equations representing the RG flow of the coupling constants. In the present case, we only have the coupling r, and by (4.3) or (4.4), its flow is represented by

$$\frac{dr}{d\ell} = 2r. \tag{4.7}$$

The constraint on the correlator in (4.6) can be written as

$$C(e^{-\ell}x, r(\ell)) = e^{(D-2)\ell}C(x,r). \tag{4.8}$$

The result of the RG flow (4.7) is very simple. For an initial value $r > 0$, we have $r \to \infty$ as ℓ increases: this represents the physics of the paramagnetic phase. Similarly for $r < 0$, we have $r \to -\infty$ as ℓ increases, representing the ferromagnetic phase. In between these two divergent flows, we have the *fixed point* $r = r^* = 0$, where r is ℓ-independent. This fixed point represents the phase transition between the two phases. Note that at a RG fixed point, we can set $r = r^*$ on both sides of a rescaling equation like (4.8), and so then the correlations are invariant under a rescaling of co-ordinates alone. Here, we have $C(e^{-\ell}x) = e^{(D-2)\ell}C(x)$, whose solution is

$$C(x) \sim x^{2-D} \quad ; \quad r = r^*, \tag{4.9}$$

which agrees with (3.49).

The above connection between RG fixed points and scale-invariance at phase transitions is very general, and we will meet numerous other examples. The 'homogeneity' relationship (4.8) for the correlation function is also in a form that applies many more complex situations.

Our main result so far has been the identification of the 'Gaussian fixed point', $r^* = 0$, of the RG transformation. Let us now look at the stability of this fixed point to other perturbations to the action. The simplest, and most important, is the quartic coupling u already contained in (3.25). Applying the transformation (4.3) we now see that u transforms as

$$u' = b^{4-D}u. \tag{4.10}$$

Equivalently, this can be written as the RG flow equation

$$\frac{du}{d\ell} = (4 - D)u \tag{4.11}$$

Thus in the more complete space of the two couplings r and u, the Gaussian fixed point is identified by $r^* = 0$ and $u^* = 0$.

We also notice a crucial dichotomy in the flow of u away from this Gaussian fixed point. For $D > 4$, the flow of u is back towards $u^* = 0$ as ℓ increases. In RG parlance, the Gaussian fixed point is *stable* towards u perturbations for $D > 4$. This result is entirely equivalent to our more pedestrian observation in (3.56), where we found that perturbation theory in u did not change the leading critical singularity for $D > 4$. Conversely, for $D < 4$, the Gaussian fixed point is *unstable* to u perturbations. This means that we have to flow away from $u^* = 0$. As we will see below, the flow is towards a new fixed point with $u^* \neq 0$, known as the Wilson-Fisher fixed point. It is the Wilson-Fisher fixed point which will cure the problem with perturbation theory in $D < 4$. identified in (3.57).

Before embarking on our search for the stable fixed point for $D < 4$, let us also consider other possible perturbations to the Gaussian fixed point. A simple example is the six-order coupling v defined by

$$\mathcal{S}_\phi \to \mathcal{S}_\phi + v \int d^D x \, \left(\phi_\alpha^2(x)\right)^3 \tag{4.12}$$

Application of (4.3) easily yields

$$\frac{dv}{d\ell} = (6 - 2D)v, \tag{4.13}$$

and so $v^* = 0$ is stable for $D > 3$. In alternative common terminology, v is 'irrelevant' for $D > 3$. A similar analysis can be applied to all possible local couplings which are invariant under $\mathrm{O}(N)$ symmetry, and it is found that they are all irrelevant for $D \geq 3$. This was the underlying reason for our focus on the field theory in (3.25), with only a single quartic non-linearity. The coupling v is 'marginal' at the Gaussian fixed point in $D = 3$, but we will see that it does not play an important role at the Wilson-Fisher fixed point.

4.2 Momentum shell RG

Our RG analysis so far has not been much more than glorified dimensional analysis. Nevertheless, the simple setting of the Gaussian field theory has been a useful place to establish notation and introduce key concepts. We are now ready to face the full problem of classical field theory (3.25) in $D \leq 4$ in the presence of the quartic non-linearity.

The structure of the perturbation expansion in Section 3.3.2 shows that well-defined expressions for the fluctuation corrections are obtained only in the presence of a wavevector cutoff Λ. So in our scaling transformations we will also have to keep track of the length scale Λ^{-1}. True scale invariance at the critical point appears only at length scales much larger than Λ^{-1}, and we need a more general RG procedure which allows such an asymptotic scale invariance to develop.

The needed new idea is that of decimation of the degrees of freedom. The first step in the RG will be a partial integration (or 'decimation') of some of the short-distance degrees of freedom. This results in a new problem with a smaller number of degrees of freedom. In our formulation with a momentum cutoff Λ, the new problem has a smaller cutoff $\widetilde{\Lambda} < \Lambda$. We will choose $\widetilde{\Lambda} = \Lambda/b$, and integrate degrees of freedom in the shell in momentum space between these momenta.

The second step of the RG will be the rescaling transformation already discussed. Given the mapping of length scale $x' = x/b$, the mapping of the cutoff Λ is

$$\Lambda' = b\widetilde{\Lambda} = \Lambda \tag{4.14}$$

Note that after the complete RG transformation, the initial and final cutoffs are equal. Thus our RG will be defined at a *fixed* cutoff Λ. This is very useful, because we need no longer keep track of factors of cutoff in any of the scaling relations, and can directly compare theories simply by comparing the values of coupling constants like r, u, ...

Now we only need an implementation of the first decimation step. The second rescaling step will proceed just as in Section 4.1.

The key to the decimation procedure is the decomposition

$$\phi_\alpha(x) = \phi_\alpha^<(x) + \phi_\alpha^>(x) \tag{4.15}$$

where the two components lie in different regions of momentum space

$$\phi_\alpha^<(x) = \int_0^{\Lambda/b} \frac{d^Dk}{(2\pi)^D}\phi_\alpha^<(k)e^{ikx} \quad ; \quad \phi_\alpha^>(x) = \int_{\Lambda/b}^{\Lambda} \frac{d^Dk}{(2\pi)^D}\phi_\alpha^>(k)e^{ikx}. \tag{4.16}$$

Then we evaluate the partition function as usual, but integrate only over the fields at large momenta $\phi^>$:

$$\exp\left(-\mathcal{S}_{\phi^<}\right) = \int \mathcal{D}\phi^>_\alpha(x) \exp\left(-\mathcal{S}_\phi\right). \tag{4.17}$$

The resulting functional integral defines a functional $\mathcal{S}_{\phi^<}$ of the low momentum fields alone, which we will now compute to the needed order. An important property which facilitates the present analysis is that the $\phi^<$ and $\phi^>$ decouple at the Gaussian fixed point *i.e.* the Gaussian action is a sum of terms which involve only $\phi^<$ or $\phi^>$, but not both. This decoupling is a consequence of their disjoint support in the momentum space, and consequently we can write

$$\mathcal{S}_\phi = \int d^D x \left\{ \frac{1}{2} \left[(\nabla_x \phi^<_\alpha)^2 + r\phi^{<2}_\alpha(x) + (\nabla_x \phi^>_\alpha)^2 + r\phi^{>2}_\alpha(x) \right] \right.$$
$$\left. + \frac{u}{4!} \left(\left(\phi^<_\alpha(x) + \phi^>_\alpha(x) \right)^2 \right)^2 \right\}. \tag{4.18}$$

Inserting (4.18) into (4.17) we obtain

$$\mathcal{S}_{\phi^<} = \frac{1}{2} \int d^D x \left[(\nabla_x \phi^<_\alpha)^2 + r\phi^{<2}_\alpha(x) \right] - \ln \mathcal{Z}^>$$
$$- \ln \left\langle \exp\left(-\frac{u}{4!} \int d^D x \left(\left(\phi^<_\alpha(x) + \phi^>_\alpha(x) \right)^2 \right)^2 \right) \right\rangle_{\mathcal{Z}^>} \tag{4.19}$$

Here $\mathcal{Z}^>$ is the *free* Gaussian ensemble defined by

$$\mathcal{Z}^> = \int \mathcal{D}\phi^>_\alpha(x) \exp\left(-\frac{1}{2} \int d^D x \left[(\nabla_x \phi^>_\alpha)^2 + r\phi^{>2}_\alpha(x) \right] \right), \tag{4.20}$$

and the expectation value in the last term in (4.19) is taken under this Gaussian ensemble (as indicated by the subscript). The second term in (4.19) is an additive constant, and is important in computations of the free energy; we will not include it below because it plays no role in the renormalization of the coupling constants.

It now remains to evaluate the expectation value in the last term in (4.19). This is easily done in powers of u using the methods described in Section 3.3.1. Notice that $\phi^<$ appears as a source term, whose powers will multiply various correlation functions of the $\phi^>$; the latter can be evaluated by Wick's theorem, or equivalently by Feynman diagrams. Thus all internal lines in the Feynman diagram represent $\phi^>$, while external lines are $\phi^<$. We show the Feynman diagrams that will be important to us in Fig. 4.1 The spatial dependence of $\phi^<_\alpha(x)$ determines the momenta that are injected into the Feynman diagrams by the external vertices. We are ultimately interested in a spatial gradient expansion of the resulting action functional of

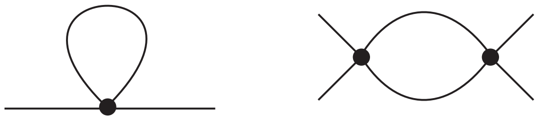

Figure 4.1 Feynman graphs important for the Wilson-Fisher RG equations. The external lines represent $\phi_<$, while the internal lines are $\phi^>$ propagators. The first graph renormalizes r, while the second renormalizes u.

$\phi^<$, and this means that we can expand the Feynman diagrams in powers of the external momenta.

In the action functional for $\phi^<$ so obtained, we obtain terms which have the same form as those in the original $\mathcal{S}_\phi$ in (3.25). We also obtain a number of other terms which involve higher powers of $\phi^<$ or additional gradients: we have just argued at the end of Section 4.1 that these other terms are not important, and so we can safely drop them here. The first graph in Fig. 4.1 renormalizes terms that are quadratic in $\phi^<$, and this turns out to be independent of momentum: this is related to the k independence of the self energy $\Sigma(k)$ in (3.51). The second graph in Fig. 4.1 is quartic in $\phi^<$, and this can be evaluated at zero external momenta because we are not interested in gradients of the quartic term. Collecting terms in this manner, the final results of (4.19) can be written as

$$\mathcal{S}_{\phi^<} = \int d^D x \left\{ \frac{1}{2} \left[(\nabla_x \phi_\alpha^<)^2 + \widetilde{r} \phi_\alpha^{<2}(x) \right] + \frac{\widetilde{u}}{4!} \left(\phi_\alpha^{<2}(x) \right)^2 \right\} \tag{4.21}$$

Notice that the co-efficient of the gradient term does not renormalize: this is an artifact of the low order expansion, and will be repaired below. The other terms do renormalize, and have the modified values

$$\begin{aligned} \widetilde{r} &= r + u \frac{(N+2)}{6} \int_{\Lambda/b}^{\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 + r)} \\ \widetilde{u} &= u - u^2 \frac{(N+8)}{6} \int_{\Lambda/b}^{\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 + r)^2}. \end{aligned} \tag{4.22}$$

Finally, it should be reiterated that the theory $\mathcal{S}_{\phi^<}$ in (4.21) has an implicit momentum cutoff of $\widetilde{\Lambda} = \Lambda/b$.

We can now immediately implement the second rescaling step of the RG and obtain the new couplings

$$r' = b^2 \widetilde{r} \quad ; \quad u' = b^{4-D} \widetilde{u}. \tag{4.23}$$

Finally, we specialize to an infinitesimal rescaling $b = 1 + d\ell$, expand every-

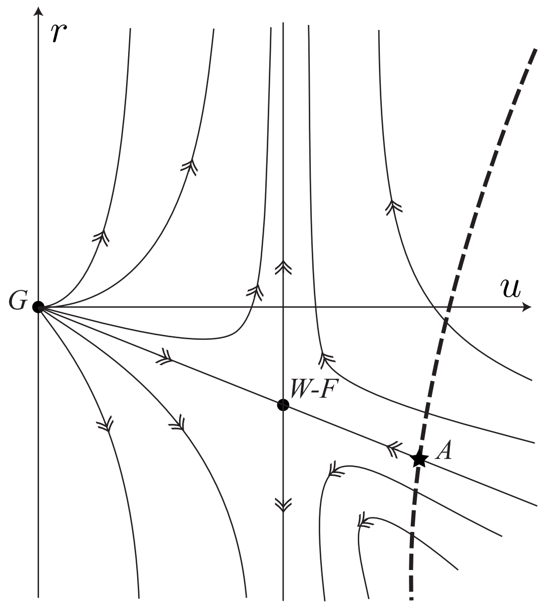

Figure 4.2 Plot of the flow equations in (4.24). There are 2 fixed points, the Gaussian (G) and Wilson-Fisher (W-F). The dashed line represents a possible line of initial values of r and u, as the coupling K is changed in the underlying lattice model. The critical point of such a lattice model is the point A, which flows into the Wilson-Fisher fixed point.

thing to first order in $d\ell$, and obtain the RG flow equations

$$\begin{aligned}\frac{dr}{d\ell} &= 2r + u\frac{(N+2)}{6}\frac{S_D}{(1+r)} \\ \frac{du}{d\ell} &= (4-D)u - u^2\frac{(N+8)}{6}\frac{S_D}{(1+r)^2}.\end{aligned} \tag{4.24}$$

Because the RG flow is computed at fixed Λ, we have conveniently set $\Lambda = 1$, and will henceforth measure all lengths in units of Λ^{-1}. The phase space factor $S_D = 2/(\Gamma(D/2)(4\pi)^{D/2})$ arises from the surface area of a sphere in D dimensions. The equations (4.24) update the equations (4.7) and (4.11) to next order in u, and are the celebrated Wilson-Fisher RG equations.

Let us momentarily ignore concerns about the range of validity of RG flow equations, and examine the consequences of integrating (4.24). The results are shown in Fig 4.2. In addition to $r^* = u^* = 0$ Gaussian fixed point already found, we observe there is second fixed point at a non-zero values r^* and u^*: this is the Wilson-Fisher fixed point, and it will be the focus of our attention.

However, before embarking upon a study of the Wilson-Fisher fixed point, we need a systematic method of assessing the reliability of our results. Wilson and Fisher pointed out that a very useful expansion is provided by

$$\epsilon \equiv 4 - D. \tag{4.25}$$

Even though the physical values of D are integer, all our expressions for the perturbative expansions, Feynman diagrams, and flow equations have been analytic functions of D. So we can consider an analytic continuation to the complex D plane, and hence to small ϵ. The expansion in powers of ϵ has since established itself as an invaluable tool in describing a variety of classical and quantum critical points.

We can now systematically determine the values of r^* and u^* by an ϵ expansion of (4.24). We find

$$u^* = \frac{6\epsilon}{(N+8)S_4} + \mathcal{O}(\epsilon^2) \;\; ; \;\; r^* = -\frac{\epsilon(N+2)}{2(N+8)} + \mathcal{O}(\epsilon^2) \tag{4.26}$$

All the omitted higher order terms in (4.24), and all other terms higher order in ϕ_α, or with additional gradients that could have been added to $\mathcal{S}_\phi$ will not modify the results in (4.26). This includes the effect of the coupling v in (4.12), which is actually irrelevant for small ϵ.

Let us now examine the nature of the flows near r^* and u^*, and their implication for the critical properties of $\mathcal{S}_\phi$, and the underlying O(N) spin models. The phases of the spin models are accessed by varying a single coupling K. The initial values of r and u depend upon the value of K, and so varying K will move us along a line in the r,u plane: this line of initial values is shown as the dashed line in Fig. 4.2. Now notice that the RG flows predict to very distinct final consequences as we vary K. On the right of the point A in Fig. 4.2, all points ultimately flow to $r \to \infty$; it is natural to identify all such points as residing in the high temperature paramagnetic phase. In contrast, points to the left of A flow to $r \to -\infty$, as is natural in the ferromagnetic phase. The only point on the line of initial values to avoid these fates is the point A itself, and it flows directly into the Wilson-Fisher fixed point. It is therefore natural to identify A as the point $K = K_c$. Thus we have established, that for essentially all realizations of the O(N) spin model, the physics of the critical point is described by the field theory of the Wilson-Fisher fixed point. This is a key step in the 'proof' of the hypothesis of universality: independent of the set of microscopic couplings in H, the critical point is described by the same universal field theory.

Let us examine the structure of the flows near the Wilson-Fisher fixed point. Defining, $r = r^* + \delta r$ and $u = u^* + \delta u/S_D$, (4.24) yields the linearized equations

$$\frac{d}{d\ell}\begin{pmatrix}\delta r\\ \delta u\end{pmatrix} = \begin{pmatrix} 2-\frac{(N+2)\epsilon}{(N+8)} & \frac{(N+2)}{6}+\frac{(N+2)^2\epsilon}{(N+8)} \\ 0 & -\epsilon \end{pmatrix}\begin{pmatrix}\delta r\\ \delta u\end{pmatrix}. \quad (4.27)$$

These equations are most easily integrated by diagonalizing the co-efficient matrix: this is done by defining two new eigen-couplings $w_{1,2}$ which are different linearly-independent combinations of δr and δu. The flow of the eigen-couplings is simply

$$\frac{dw_i}{d\ell} = \lambda_i w_i \quad , \quad i = 1,2 \quad (4.28)$$

with the eigenvalues

$$\lambda_1 = 2 - \frac{(N+2)\epsilon}{(N+8)} + \mathcal{O}(\epsilon^2) \quad , \quad \lambda_2 = -\epsilon + \mathcal{O}(\epsilon^2). \quad (4.29)$$

The flows in (4.29) are now finally as simple as those for the Gaussian theory in Section 4.1. Regardless of its initial value, the coupling w_2 is attracted to $w_2^* = 0$ and so can be regarded as irrelevant. Setting $w_2^* = 0$ puts us on the track to identifying the universal properties of the critical point. Notice however, that the eigenvalue λ_2 has a small magnitude, and so the flow towards the universal theory will be slow: we expect this flow of w_2 to provide the leading corrections to the universal critical behavior.

Our entire RG analysis has therefore been reduced to the flow of a *single* relevant coupling w_1, with the simple flow equation in Eq. (4.28). Just as was the case with the coupling r in Section 4.1, the critical point is at $w_1 = w_1^* = 0$, and so may identify this coupling

$$w_1 \sim K_c - K \quad (4.30)$$

as a measure of the deviation from criticality. The flow of w_1 when combined with the rescaling of the field ϕ_α leads to a number of interesting physical consequences which we will explore below.

4.3 Field renormalization

The entire analysis in Section 4.2 did not modify the original rescaling of the field ϕ_α defined in Eq. (4.3). This is actually an artificat of working to first order in ϵ. At higher orders in ϵ we do find a momentum in the self energy, and consequently a renormalization of the gradient term in $\mathcal{S}_\phi$, leading to change in the RG rescaling of ϕ_α. A complete treatment of these effects is far more efficiently carried out using a more formal field-theoretical

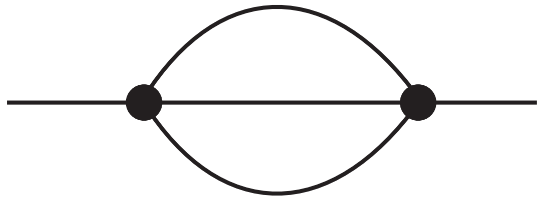

Figure 4.3 Graph contributing at order ϵ^2 to the anomalous dimension η.

RG which we will not describe now. Rather we will be satisfied with a shortcut which yields the leading non-vanishing renormalization after some reasonable assumptions.

Anticipating the field scale renormalization, we define the RG rescaling of ϕ_α by

$$\phi'_\alpha(x') = b^{(D-2+\eta)/2}\phi_\alpha(x) \tag{4.31}$$

where η is known as the anomalous dimension of the field ϕ_α. This is of course equivalent to

$$\dim[\phi_\alpha] = (D-2+\eta)/2. \tag{4.32}$$

Assuming (4.32), the generalization of the relation (4.8) for the correlator $C(x)$ at the critical point $K = K_c$ or $w_1 = 0$ is

$$C(e^{-\ell}x) = e^{(D-2+\eta)\ell}C(x). \tag{4.33}$$

This implies $C(x) \sim x^{-(D-2+\eta)}$. Taking the Fourier transform, we have for the susceptibility

$$\chi(k) \sim \frac{1}{k^{2-\eta}}. \tag{4.34}$$

Let us see how (4.34) could emerge from an analysis of the Wilson-Fisher fixed point. We set $r = r^*$ and $u = u^*$, and compute $\chi(k)$ using (3.45) and a perturbative expansion for the self energy $\Sigma(k)$. Because we are at the critical point, $\chi^{-1}(0)$ should vanish, and hence we should have $r - \Sigma(0) = 0$. Thus $\chi^{-1}(k) = k^2 - \Sigma(k) + \Sigma(0)$. At first order in u, $\Sigma(k)$ is k-independent, and so we have no correction to the free field behavior. However, at second order in u, we do obtain a momentum dependent Σ given by the Feynman diagram in Fig. 4.3 (which corresponds to the last term in (3.38)), which yields

$$\Sigma(k) = \ldots + u^2\frac{(N+2)}{18}\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\frac{1}{(p^2+r)(q^2+r)((p+q+k)^2+r)} \tag{4.35}$$

To leading non-vanishing order in ϵ, we can set $u = u^*$, $r = 0$, and $D = 4$ above. The resulting two-loop integral requires some technical results from the mathematical theory of Feynman graphs to evaluate; the reader can find a concise and useful discussion, with a valuable table of integrals, in the book by Ramond [398]. Evaluating the integral in this manner we find

$$\Sigma(k) = \ldots - (u^* S_4)^2 \frac{(N+2)}{72} k^2 \ln \frac{\Lambda}{k}. \tag{4.36}$$

Inserting this into $\chi^{-1}(k) = k^2 - \Sigma(k) + \Sigma(0)$, and assuming the logarithm is the first term in a series which exponentiates, we obtain the form in (4.34). Using the value of u^* in (4.26), we obtain our needed result for the anomalous dimension of ϕ_α:

$$\eta = \frac{(N+2)}{2(N+8)^2}\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{4.37}$$

4.4 Correlation functions

The results of Sections 4.2 and 4.3 will now be collected and applied to determine the form of correlations near the critical point.

Given the form of the field scaling relation (4.32), and the flow of the relevant coupling w_1 in (4.28), the most complete form of the homogeneity relation (4.8) for the two-point ϕ_α correlator is

$$C(e^{-\ell}x; w_1 e^{\lambda_1 \ell}) = e^{(D-2+\eta)\ell} C(x; w_1). \tag{4.38}$$

Recall that we have already set the leading irrelevant coupling $w_2 = w_2^* = 0$, and so we are dealing exclusively with the 'universal theory'. The relation (4.38) holds for any ℓ, and so let us evaluate it at $\ell = \ell^*$ where $w_1 e^{\lambda_1 \ell^*} = \pm 1$; the choice in sign will depend upon the sign of the initial value of w_1, *i.e.* whether we are above or below the critical point. Then it takes the form

$$C(x; w_1) = \xi^{-(D-2+\eta)} F_\pm(x/\xi) \tag{4.39}$$

where we have set $\xi = e^{\ell^*}$ and $F_\pm$ are as yet undetermined functions, known as the scaling functions; the subscript indicates the distinct forms of the scaling function on the two sides of the critical point. The structure of (4.39) is very suggestive: it indicates that as we change the value of the coupling w_1, the x dependence of the correlations changes at the characteristic scale ξ. It is therefore natural to identify ξ with the correlation length, the analog of the quantity that appeared in (3.47) in the Gaussian theory; indeed, it is easy to see that (3.47) is of the form (4.39), with an explicit result for the scaling function F_+, and an exponent $\eta = 0$ as expected at the Gaussian fixed point. We will mainly be interested in (4.39) at the Wilson-Fisher fixed point, where η is given by (4.37), and we can make an analogous ϵ expansion for the scaling functions $F_{\pm}$.

The above results also specify the divergence of ξ at the critical point. Using the expressions just before and after (4.39), we have

$$\xi \sim |w_1|^{-1/\lambda_1} \sim |K_c - K|^{-\nu}. \tag{4.40}$$

We meet again the correlation exponent ν, whose value is given quite generally by

$$\nu = \frac{1}{\lambda_1}. \tag{4.41}$$

In other words, the exponent ν is the inverse of the relevant RG eigenvalue. Implicit in our discussion, is the assumption that there is only *one* such relevant eigenvalue, and that *all* other perturbations are irrelevant at the Wilson-Fisher fixed point. The reader will recognize that we have in fact established this strong and powerful result to leading order in ϵ. There is overwhelming numerical evidence that such a result is also true at the physically important value of $\epsilon = 1$, and this is behind the success of the ϵ expansion.

We will also have occasion to meet RG fixed points with either zero or more than one relevant perturbation. Those with zero relevant perturbations describe *critical phases* rather than critical points; for all couplings not too far from the fixed point, the flow is towards the fixed point, and hence the long distance correlations have characteristics independent of all microscopic parameters. The extent of such critical phases is determined by the domain of attraction of the RG fixed point. RG fixed points with more than one relevant perturbations describe 'multicritical points'. Reaching such multicritical points requires that we tune more than one linearly independent coupling in the underlying Hamiltonian: the number of tuning parameters is equal to the number of relevant perturbations.

Finally, let us show that (4.39) also allows us to determine all other critical exponents we have defined so far. Integrating (4.39) over all x, we obtain using (3.42) the result for the wavevector dependent susceptibility

$$\chi(k) = \xi^{2-\eta} \widetilde{F}_{\pm}(k\xi) \tag{4.42}$$

where the scaling functions $\widetilde{F}_{\pm}$ are Fourier transforms of $F_{\pm}$. This result clearly generalizes (4.34) away from the critical point. As long as we are not at the critical point, we expect the susceptibility to be non-singular as $k \to 0$, and so the $\widetilde{F}_{\pm}$ will approach non-zero constants in the limit of zero argument. Consequently, the uniform static susceptibility behaves as

$$\chi \sim \xi^{2-\eta} \tag{4.43}$$

or the critical exponent γ in (3.46) is given by the exact 'scaling relation'

$$\gamma = (2-\eta)\nu. \tag{4.44}$$

It remains to determine the critical exponent β for the ferromagnetic moment $\langle \phi_\alpha \rangle = N_0 \hat{e}_\alpha$, where $\hat{e}_\alpha$ is an arbitrary unit N-component vector. As discussed in (3.24), this moment vanishes as

$$N_0 \sim (K - K_c)^\beta \tag{4.45}$$

as we approach the critical point from low temperatures. Rather than deducing directly from (4.39), let us reapply the RG transformation from scratch. Application of the fundamental scaling relation (4.31) tells us that

$$N_0(w_1 e^{\lambda_1 \ell}) = e^{(D-2+\eta)\ell/2} N_0(w_1), \tag{4.46}$$

where again we neglect the influence of irrelevant couplings like w_2. Evaluating (4.46) at $\ell = \ell^*$ as before, we obtain (4.45) with the scaling relation

$$\beta = (D-2+\eta)\nu/2. \tag{4.47}$$

More generally, we can regard (4.47) as the most important application of a scaling relation determining the singular contribution to the average value of any observable $\mathcal{O}$

$$[\langle \mathcal{O} \rangle]_{\text{sing}} \sim \xi^{-\dim[\mathcal{O}]}. \tag{4.48}$$

An important case of the above result is the scaling of the free energy density $\widetilde{\mathcal{F}} = -(1/V)\ln \mathcal{Z}$, where V is the D-dimensional volume of the system. The partition function $\mathcal{Z}$ is a RG invariant, and so the scaling dimension of $\ln \mathcal{Z}$ must be zero. Consequently

$$\dim[\widetilde{\mathcal{F}}] = D, \tag{4.49}$$

and the singular part of the free energy density scales as ξ^{-D}. This result is often stated in terms of the exponent of the specific heat (see exercise 3.4) $-\partial^2 \widetilde{\mathcal{F}}/\partial r^2 \sim |K - K_c|^{-\alpha}$ for which we have the 'hyperscaling relation'

$$\alpha = 2 - D\nu. \tag{4.50}$$

Exercises

4.1 Consider the RG flow of the sixth-order coupling v in (4.12). Compute the one loop correction to the RG flow in (4.13) by determining the coefficient of the term of order uv on the right hand side. Hence show that Wilson-Fisher fixed point has $v^* = 0$, and the fixed-point eigenvalue of the six-order operator is

$$\begin{aligned} \lambda_v &= 6 - 2D - \frac{(n+14)}{2} S_4 u^* + \mathcal{O}(\epsilon^2) \\ &= -2 - \frac{(n+26)}{(n+8)}\epsilon + \mathcal{O}(\epsilon^2) \end{aligned}$$

4.2 This exercise is adapted from Ref. [388]. We consider the consequences of anisotropy in the O(N) symmetry of the Wilson-Fisher fixed point. In some applications to classical ferromagnets, spin-orbit interactions may introduce a weak anisotropy in which the $r\phi_\alpha^2$ term is replaced by

$$r_s \sum_{\alpha<N} \phi_\alpha^2 + r_N \phi_N^2, \tag{4.51}$$

while the quartic term is replaced by

$$\frac{u_1}{24} \sum_{\alpha,\beta<N} \phi_\alpha^2 \phi_\beta^2 + \frac{u_2}{12} \sum_{\alpha<N} \phi_\alpha^2 \phi_N^2 + \frac{u_3}{24} \phi_N^4. \tag{4.52}$$

Clearly, the original problem with full O(N) symmetry is the case $r_s = r_n$ and $u_1 = u_2 = u_3$. The model with $r_s = \infty$, $u_1 = u_2 = 0$ is the field theory of the Ising model, while the model with O($N-1$) symmetry is $r_n = \infty$, $u_2 = u_3 = 0$.

(a) Show that the one-loop RG flow equations for this model are:

$$\begin{aligned} \frac{dr_s}{d\ell} &= 2r_s + \frac{(N+1)}{6(1+r_s)} S_D u_1 + \frac{1}{6(1+r_n)} S_D u_2 \\ \frac{dr_n}{d\ell} &= 2r_n + \frac{(N-1)}{6(1+r_s)} S_D u_2 + \frac{1}{2(1+r_n)} S_D u_3 \\ \frac{du_1}{d\ell} &= \epsilon u_1 - \frac{(n+7)}{6(1+r_s)^2} S_D u_1^2 - \frac{1}{6(1+r_n)^2} S_D u_2^2 \\ \frac{du_2}{d\ell} &= \epsilon u_2 - \frac{2}{3(1+r_s)(1+r_n)} S_D u_2^2 - \frac{(N+1)}{6(1+r_s)^2} S_D u_1 u_2 - \frac{1}{2(1+r_n)^2} S_D u_2 u_3 \\ \frac{du_3}{d\ell} &= \epsilon u_3 - \frac{3}{2(1+r_n)^2} S_D u_3^2 - \frac{(N-1)}{6(1+r_s)^2} S_D u_2^2, \end{aligned} \tag{4.53}$$

(b) Show that these equations reduce to the expected equations in the limits corresponding to the models with O(N), Ising, and O($N-1$) symmetry just noted.

(c) Consider the fixed point of the flow equations with O(N) symmetry: $r_s = r_n = r^*$, and $u_1 = u_2 = u_3 = u^*$. Show that, to leading order in ϵ, and for $n \leq 4$, this fixed point has *two* relevant eigenvalues $2 - (n+2)\epsilon/(n+8)$ and $2 - 2\epsilon/(n+8)$.

(d) Assume the experimental conditions are such that the u couplings are close to the O(N) fixed point. Describe, qualitatively, the behavior of the susceptibility for $T > T_c$ for the two cases $r_s > r_n$ and $r_n > r_s$.