Topology and Chern-Simons theories

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(Dated: April 8, 2018)

Abstract

We compute the ground-state degeneracy of theories with Chern-Simons terms on a torus. Then we consider Chern-Simons theories on space with a boundary, and describe the edge excitations. We also briefly describe quantum spin Hall insulators.

The previous lectures have described two gapped "topological" states: the $\mathbb{Z}_2$ spin liquid, and the fractional quantum Hall states. These states have quasiparticle excitations which are 'anyons' *i.e.* they pick up non-trivial phase factors upon encircling each other, even while they are separated by large distances. This long-distance physics of these phase was described by abelian Chern-Simons theory with imaginary time action

$$\mathcal{S}_{\rm CS} = \int d^3x \left[\frac{i}{4\pi} \epsilon_{\mu\nu\lambda} a_\mu^I \, K_{IJ} \, \partial_\nu a_\lambda^J \right], \tag{1}$$

where I, J are indices extending over N values $1 \ldots N$, and a_μ^I are N U(1) gauge fields. For the $\mathbb{Z}_2$ spin liquid, we had $N = 2$ and the symmetric K matrix

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \tag{2}$$

while the quantum Hall states had $N = 1$ and

$$K = m \tag{3}$$

with m an odd (even) integer for fermions (bosons). In addition to the structure described by $\mathcal{S}_{\rm CS}$, specification of a particular state of condensed matter requires the quasiparticle quantum numbers, and the transformations of the gauge fields and the quasiparticles under various global symmetries of the Hamiltonian. The latter information is part of the description of the 'symmetry enriched' topological (SET) phase, and we will defer discussion of SET issues to later in this chapter.

Almost all of the results of this chapter are contained in a remarkable paper by Witten [1]; unfortunately, this paper is written in a mathematical language that makes it difficult to extract the information relevant for condensed matter.

I. CHERN-SIMONS THEORY ON A TORUS

We now discuss the quantization of (1) on a spatial torus of size $L_x \times L_y$. One important property of (1) is that it is exactly invariant under the gauge transformations $a_\mu^I \to a_\mu^I - \partial_\mu \zeta^I$: there is no surface term upon integration by parts on a torus, and the variation in the action vanishes exactly.

For simplicity, we consider the case $N = 1$, with $K = m$; the methods below can be generalized to other values of N and K.

We work in the gauge $a_\tau = 0$. However, we cannot just set $a_\tau = 0$ in (1). We have to examine the equation of motion obtained by varying a_τ, which for the pure Chern-Simons theory is simply the zero flux condition

$$\epsilon_{ij} \partial_i a_j = 0. \tag{4}$$

But this does not imply the theory is pure gauge, and so trivial. We still have to consider fluxes around the cycles of the torus. So, up to a gauge transformation, we can choose the solutions of (4) as constants we parameterize as

$$a_x = \frac{\theta_x}{L_x} \quad , \quad a_y = \frac{\theta_y}{L_y} \tag{5}$$

in terms of new variables θ_x and θ_y. Now consider the influence of a'large' gauge transformations on (5), generated by

$$\zeta = \frac{2\pi \ell x}{L_x}, \tag{6}$$

where ℓ is an integer. Such a gauge transformation is permitted because $e^{i\zeta}$ is single-valued on the torus, and it is always $e^{i\zeta}$ that appears as a gauge transformation factor on any underlying particles. Under the action of (6) we have

$$\theta_x \to \theta_x - 2\pi\ell. \tag{7}$$

So only the value of θ_x modulo 2π can be treated as a gauge-invariant quantity, and θ_x is an 'angular' variable. A similar argument applies to θ_y. We therefore introduce the Wilson-loop operators

$$W_x \equiv e^{i\theta_x} \quad , \quad W_y \equiv e^{i\theta_y}. \tag{8}$$

These are the gauge-invariant observables which characterize Chern-Simons theory on a torus.

Inserting (5) into (1), we find that the dynamics of $\theta_{x,y}$ is described by the simple action

$$\mathcal{S}_\theta = \frac{im}{2\pi} \int d\tau \, \theta_y \frac{d\theta_x}{d\tau} \tag{9}$$

This is a purely kinematical action, and it shows that $(m/(2\pi))\theta_y$ is the canonically conjugate momentum to θ_x. There is no Hamiltonian, and so the energy of all states is zero. Upon promoting $\theta_{x,y}$ to operators, this action implies the commutation relation

$$[\hat{\theta}_x, \hat{\theta}_y] = \frac{2\pi i}{m}. \tag{10}$$

In terms of the gauge-invariant Wilson loop operators, this commutation relation is equivalent to

$$\hat{W}_x \hat{W}_y = e^{-2\pi i/m} \, \hat{W}_y \hat{W}_x. \tag{11}$$

This is the fundamental operator relation which controls the quantum Chern-Simons theory on a torus.

For the simplest non-trivial case of $m = 2$, we see that $\hat{W}_x$ and $\hat{W}_y$ anti-commute. So they must act on a Hilbert space which is at-least 2-fold degenerate, because the smallest matrices which anti-commute are the Pauli matrices: we can choose $\hat{W}_x = \sigma^x$ and $\hat{W}_y = \sigma^z$. So the U(1) Chern-Simons theory on the torus at level $m = 2$ has a 2-dimensional Hilbert space at zero energy.

It is not difficult to generalize the above argument to general integer m. As $(\hat{W}_y)^m$ commutes with all other Wilson loop operators, we can demand that it equal the unit matrix. Then, the eigenvalues of $\hat{W}_y$ can only be $e^{2\pi i\ell/m}$ with $\ell = 0, 1, \dots m-1$. So we introduce the m states $|\ell\rangle$ obeying

$$\hat{W}_y \, |\ell\rangle = e^{2\pi i\ell/m} \, |\ell\rangle \,. \tag{12}$$

The relationship (11) can be satisfied by demanding that $\hat{W}_x$ is a cyclic 'raising' operator on these states

$$\hat{W}_x \, |\ell\rangle = |(\ell+1)(\mathrm{mod}\ m)\rangle \,. \tag{13}$$

So the U(1) Chern-Simons theory on the torus at level m has a m-fold ground state degeneracy.

A. Path integral quantization

It is instructive to also obtain the above results by regularizing the action (9) by adding higher derivative terms, so that the Hamiltonian does not vanish, and all states are not exactly at zero energy. By adding a bare Maxwell term to the Chern-Simons theory, we can extend (9) to

$$\mathcal{S}_\theta = \int d\tau \left[\frac{\mathcal{M}}{2} \left(\frac{d\theta_x}{d\tau} \right)^2 + \frac{\mathcal{M}}{2} \left(\frac{d\theta_y}{d\tau} \right)^2 + i\mathcal{A}_x \frac{d\theta_x}{d\tau} + i\mathcal{A}_y \frac{d\theta_y}{d\tau} \right], \tag{14}$$

with

$$(\mathcal{A}_x, \mathcal{A}_y) = (m\theta_y/(2\pi), 0). \tag{15}$$

But this is precisely the (imaginary time) Lagrangian of a fictitious particle with co-ordinates (θ_x, θ_y) and mass $\mathcal{M}$ moving in the presence of 'magnetic field' specified by a vector potential $(\mathcal{A}_x, \mathcal{A}_y)$. We are interested in the spectrum in the limit $\mathcal{M} \to 0$, when (14) reduces to (9). The strength of the magnetic field is $\mathcal{B} = \partial_{\theta_x}\mathcal{A}_y - \partial_{\theta_y}\mathcal{A}_x = -m/(2\pi)$. We can now introduce a wavefunction $\psi(\theta_x, \theta_y)$ obeying the Schrödinger equation

$$\mathcal{H}\, \psi(\theta_x, \theta_y) = \mathcal{E}\, \psi(\theta_x, \theta_y) \tag{16}$$

where the Hamiltonian is

$$\mathcal{H} = \frac{1}{2\mathcal{M}} \left(\frac{1}{i} \frac{\partial}{\partial\theta_x} - \mathcal{A}_x \right)^2 + \frac{1}{2\mathcal{M}} \left(\frac{1}{i} \frac{\partial}{\partial\theta_y} - \mathcal{A}_y \right)^2 \tag{17}$$

A subtle feature in the solution of this familiar Hamiltonian is the nature of the periodic boundary conditions on θ_x and θ_y. This fictitious particle moves on a torus of size $(2\pi)\times(2\pi)$, not to be confused by the torus of size $L_x \times L_y$ for the original Chern-Simons theory. The total 'magnetic' flux is therefore $4\pi^2\mathcal{B}$, and the total number of 'magnetic' flux quanta is $4\pi^2|\mathcal{B}|/(2\pi) = m$. So we expect that the eigenstates of $\mathcal{H}$ are m-fold degenerate, just as we concluded from the arguments above using the Wilson loop operators. In computing the eigenstates of $\mathcal{H}$, we run into the difficulty that the vector potential in (15) is not explicitly a periodic function of θ_y, and instead obeys

$$\begin{aligned}\mathcal{A}_x(\theta_x, \theta_y + 2\pi) &= \mathcal{A}_x(\theta_x, \theta_y) + m \\ \mathcal{A}_y(\theta_x, \theta_y + 2\pi) &= \mathcal{A}_y(\theta_x, \theta_y).\end{aligned} \tag{18}$$

But we can make the vector potential periodic by using the gauge transformation

$$\mathcal{A}_i \to \mathcal{A}_i - \partial_i \zeta \tag{19}$$

with $\zeta = m\theta_x$. So we need to solve (16) and (17) subject to the boundary conditions

$$\psi(\theta_x + 2\pi, \theta_y) = \psi(\theta_x, \theta_y) \tag{20}$$

$$\psi(\theta_x, \theta_y + 2\pi) = e^{im\theta_x}\psi(\theta_x, \theta_y). \tag{21}$$

The Landau level eigenstates of (17) in an infinite plane are, of course, very familiar. We focus only on the lowest Landau level states, as these are the only ones that will survive the $\mathcal{M} \to 0$ limit. Imposing only the boundary condition (20) we obtain the unnormalized eigenstates

$$\phi_\ell(\theta_x, \theta_y) = \exp\left(i\ell\theta_x - \frac{m}{4\pi}\left(\theta_y - \frac{2\pi\ell}{m}\right)^2\right), \tag{22}$$

where ℓ is any integer. Notice that these states obey

$$\phi_\ell(\theta_x, \theta_y + 2\pi) = e^{im\theta_x}\phi_{\ell-m}(\theta_x, \theta_y). \tag{23}$$

Now it is evident that we can also satisfy the second boundary condition (21) with m different orthogonal wavefunctions $\psi_\ell(\theta_x, \theta_y)$, with $\ell = 0, 1, \ldots m-1$, which are given by

$$\psi_\ell(\theta_x, \theta_y) = \sum_{p=-\infty}^{\infty} \phi_{\ell+mp}(\theta_x, \theta_y). \tag{24}$$

These are related to Jacobi Theta functions. We have again reached the conclusion that the U(1) Chern-Simons theory at level m has a m-fold degenerate ground state on the torus.

To conclude this section, we note the straightforward extension of this computation to the case of the $\mathbb{Z}_2$ spin liquid described by the K matrix in (2). This model factorizes into two copies of the problem solved above, both with $m = 2$. The first copy involves the Wilson loops of a_x^1 and a_y^2, while the second copy involves the Wilson loops of a_y^1 and a_x^2. Each factor yields a degeneracy of 2, for a total degeneracy of 4.

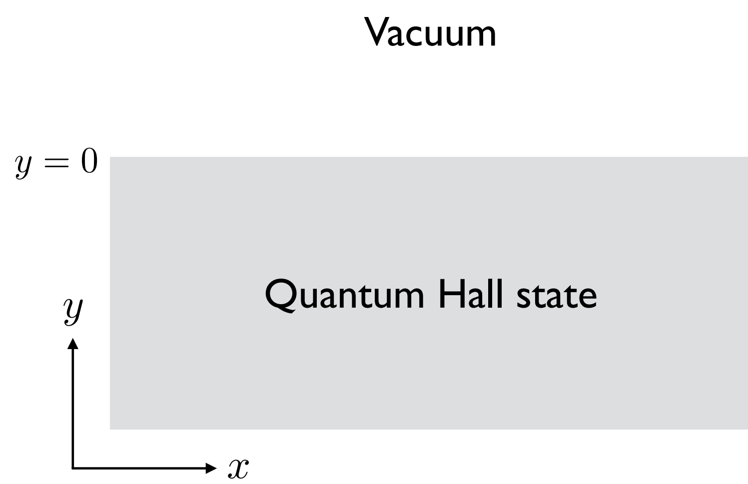

FIG. 1. Edge of a semi-infinite quantum Hall state, described by a Chern Simons theory, at $y = 0$.

II. EDGE EXCITATIONS OF CHIRAL CHERN-SIMONS THEORY

We return to the Chern-Simons theory in (1), and describe its quantization in the geometry of Fig. 1. For simplicity, we will just consider the U(1) theory with $N = 1$, and with $K = m$. For $m = 1$ we will find that we reproduce the results of Lec13, and obtain a chiral edge of free fermions. For the fractional quantum Hall states with $m > 1$, we obtain a different edge theory which cannot be expressed in terms of free fermions.

The first important property of $\mathcal{S}_{CS}$ is that it is not invariant under a gauge transformation $a_\mu \to a_\mu - \partial_\mu \zeta$ in the presence of an edge. Instead we obtain a surface term

$$\mathcal{S}_{\rm CS} \to \mathcal{S}_{\rm CS} - \frac{im}{4\pi} \int dx d\tau \, \zeta \left(\partial_\tau a_x - \partial_x a_\tau \right) \Big|_{y=0}. \tag{25}$$

We need additional degrees of freedom on the edge to cancel this surface term, and obtain a properly gauge invariant theory.

Let us try to deduce the needed degrees of freedom by working directly with the Chern-Simons action in the geometry of Fig. 1. The variation of the action is [2]

$$\delta \mathcal{S}_{\rm CS} = \frac{im}{2\pi} \int d^3x \Big[\delta a_\mu (\epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda) \Big] + \frac{im}{4\pi} \int dx d\tau \, \left(a_x \delta a_\tau - a_\tau \delta a_x \right) \Big|_{y=0}. \tag{26}$$

To make the variation vanish, we require the usual zero flux condition, $\epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda = 0$, in the bulk. But on the boundary, we must also impose a secondary condition to define the theory: a convenient choice is to set $a_\tau = 0$ (and hence also $\delta a_\tau = 0$) at $y = 0$. We will find that the fluctuations of the gauge field near the boundary are no longer pure gauge, in contrast to the situation in the bulk.

Let us quantize the system by choosing the gauge $a_\tau = 0$ in the bulk. Then (4) continues to hold for the spatial components of the gauge field, and so we can solve this constraint by the choice

$$a_i = \partial_i \varphi \tag{27}$$

in terms of a scalar field φ. As in (7), we can use large gauge transformations to argue that φ should be physically equivalent to $\varphi + 2\pi$, and so φ takes values on a unit circle. Inserting (27) into $\mathcal{S}_{\rm cs}$, and integrating over y, we obtain the edge action

$$\mathcal{S}_e = -\frac{im}{4\pi}\int dx d\tau\, \partial_\tau\varphi\partial_x\varphi, \tag{28}$$

where the fields are now implicitly evaluated at $y = 0$. Now we notice that at $m = 1$ this is precisely the kinematic term in the bosonic representation of a free chiral fermion. For general m, following the arguments in the chapter on Luttinger liquids, we can write (28) as a commutation relation

$$[\varphi(x_1), \varphi(x_2)] = -i\frac{\pi}{m}\mathrm{sgn}(x_1 - x_2). \tag{29}$$

In addition to the kinematic term in (28), non-zero energetic terms are also permitted at the boundary, provided they are consistent with the residual shift symmetry $\varphi \to \varphi$+constant. In an operator language, including the lowest order spatial gradient, we obtain the Hamiltonian

$$\mathcal{H}_\varphi = \frac{mv}{4\pi}\int dx\, (\partial_x\varphi)^2, \tag{30}$$

where v is a coupling constant with units of velocity. The interpretation of v becomes clearer in the action for the path integral, which is the final form of the edge theory [3]

$$\mathcal{S}_e = \frac{m}{4\pi}\int dx d\tau \Big[-i\partial_\tau\varphi\partial_x\varphi + v(\partial_x\varphi)^2\Big]. \tag{31}$$

This is a theory of left-moving chiral bosons at velocity v, and is also known as the U(1) Kac-Moody theory at level m. At $m = 1$, we can conclude from our previous analysis of Luttinger liquids that (31) is precisely the bosonized version of the chiral fermion theory in Lec13.

At other values of m, $\mathcal{S}_e$ remains a Gaussian theory, and so it is possible to compute all correlators on the edge using the methods developed in the chapter on Luttinger liquid theory. In particular, a useful result that can be obtained by such methods is

$$\langle\varphi(x,\tau)\varphi(0,0)\rangle = -\frac{1}{m}\ln(x - iv\tau) + \ldots. \tag{32}$$

Quantum Hall systems also have a conserved U(1) charge in the bulk, and as we will discuss below, this is important for the stability of the chiral boson theory in (31) towards external perturbations on the edge. As we saw in the previous chapter, the external electromagnetic potential A_μ couples via the term

$$\mathcal{S}_{Aa} = \int d^3x \left[\frac{i}{2\pi}a_\mu\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda\right]. \tag{33}$$

Using a non-zero electrostatic potential A_τ which is independent of y, and integrating over y the term above reduces to $(i/(2\pi))\int dx d\tau A_\tau \partial_x \varphi$, and so we may identify the charge density as

$$\rho(x) = \frac{1}{2\pi}\partial_x \varphi, \tag{34}$$

which is precisely the relation obtained in the Luttinger liquid chapter.

We can also identify the fate of the quasiparticle operators on the boundary by considering the adiabatic transport of the quasiparticles via the bulk between two points, x_1 and x_2, on the boundary. Such a process would be accompanied by the Berry phase

$$\exp\left(i\int_{(x_1,0)}^{(x_2,0)} d\vec{x}\cdot\vec{a}\right) = e^{i(\varphi(x_2)-\varphi(x_1))} \tag{35}$$

where the integral on the left-hand-side is a along path in the bulk of the sample, and the right-hand-side follows from (27). So we can identify the operator $e^{i\varphi}$ as the quasiparticle operator on the edge. Using (29) and (34), we can verify the commutation relation

$$[\rho(x_1), e^{i\varphi(x_2)}] = \frac{1}{m}\delta(x_1 - x_2)e^{i\varphi(x_2)}, \tag{36}$$

which confirms that the quasiparticle carries charge $1/m$. Similarly, the operator of the underlying particles of the quantum Hall state (electrons (fermions) or bosons) becomes $e^{im\varphi}$ on the edge of the sample. The correlators of these operators can be easily computed by (32), along with the knowledge that φ is a harmonic free field.

At this point, it is important to note an important subtlety associated with the effective edge Hamiltonian in (30). In principle, we should be able to add any local operator which is consistent with the global symmetry. Among the particle creation operators considered above, any operator which creates a boson, and which also has trivial mutual statistics with all other particles is a legitimate local operator which can be added to $\mathcal{H}_\varphi$. One such operator is $e^{im\varphi}$ for m even, and $e^{2im\varphi}$ for m odd, and so we can consider extending $\mathcal{H}_\varphi \to \mathcal{H}_\varphi + \lambda\int dx \cos(m\varphi)$ for m even, and correspondingly for m odd. However, these operators carry non-zero U(1) charge, as follows from the extension of (36), and so charge conservation prohibits their appearance in $\mathcal{H}_\varphi$.

III. EDGE EXCITATIONS OF $\mathbb{Z}_2$ SPIN LIQUIDS

The Chern-Simons theory of $\mathbb{Z}_2$ spin liquids has $N = 2$ U(1) gauge fields and K matrix

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{37}$$

Proceeding just as in Section II, we now introduce 2 scalars θ and ϕ (both defined modulo 2π) so that

$$a_i^1 = \partial_i \theta \quad , \quad a_i^2 = \partial_i \phi \tag{38}$$

Then the boundary kinematic action is

$$\mathcal{S}_e = -\frac{i}{\pi} \int dx d\tau \, \partial_x \theta \partial_\tau \phi, \tag{39}$$

which implies the commutation relation

$$[\phi(x_1), \theta(x_2)] = i\frac{\pi}{2} \mathrm{sgn}(x_1 - x_2). \tag{40}$$

Remarkably, this is precisely the kinematics of the Luttinger liquid of spinless fermions we analyzed in Lecture 3! This theory in non-chiral, and has equal numbers of left- and right-moving excitations. The simplest terms in the Hamiltonian for these edge excitations are

$$\mathcal{H}_e = \int dx \left[\frac{K_1}{2} \left(\partial_x \phi\right)^2 + \frac{K_2}{2} \left(\partial_x \theta\right)^2 \right], \tag{41}$$

and this also coincides with the Hamiltonian for the Luttinger liquid in Lecture 3.

However, unlike the case in Section II, the gapless non-chiral edge states described by $\mathcal{H}_e$ are generally not stable. For the $\mathbb{Z}_2$ spin liquid, both $e^{2i\phi}$ and $e^{2i\theta}$ are trivial bosonic excitations: this corresponds to the fact that in the bulk, two visons or two spinons can fuse into trivial excitations. Consequently, the general edge Hamiltonian is [4]

$$\mathcal{H}_e = \int dx \left[\frac{K_1}{2} \left(\partial_x \phi\right)^2 + \frac{K_2}{2} \left(\partial_x \theta\right)^2 - \lambda_1 \cos(2\phi) - \lambda_2 \cos(2\theta) \right]. \tag{42}$$

As we showed earlier in our study of Luttinger liquids, we have $(\dim[e^{2i\phi}])(\dim[e^{2i\theta}]) = 1$, and therefore at least one of the two cosine terms is always relevant. Consequently the edge spectrum is always gapped.

In the presence of additional global symmetries in the underlying lattice model, it is possible that the cosine terms conspire to leave at least one mode gapless. The symmetry constraints for the existence of $\mathbb{Z}_2$ spin liquids with gapless edge states have been explored in recent work [5–7]. None of the specific $\mathbb{Z}_2$ spin liquids we have covered in these lectures so far satisfy these constraints, and they all have gapped edges.

IV. EDGE EXCITATIONS OF QUANTUM SPIN HALL INSULATORS

A great deal of recent work has focused on a particular electron 'symmetry protected topological' (SPT) state: the quantum spin Hall insulator [8, 9]. This can also be described in the framework of Chern-Simons theory [10] with $N = 2$ U(1) gauge fields and K matrix

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{43}$$

Such a K matrix only allows fermionic excitations in the bulk, or their composites: therefore, the bulk is trivial. A direct extension of the analysis of Section II shows that the edge state of such a system has 2 copies of the integer quantum Hall edge, one left-moving and the other right-moving. So we can write the edge theory in terms of a left-moving fermion Ψ_L and a right-moving fermion Ψ_R

$$\mathcal{S}_e = \int dx d\tau \left[\Psi_L^\dagger \left(\frac{\partial}{\partial \tau} + iv \frac{\partial}{\partial x} \right) \Psi_L + \Psi_R^\dagger \left(\frac{\partial}{\partial \tau} - iv \frac{\partial}{\partial x} \right) \Psi_R \right]. \tag{44}$$

In the proposed experimental realization, such edge states can appear in the free-electron dispersion of two-dimensional lattices in the presence of spin-order coupling, with Ψ_L a spin up fermion (say), and Ψ_R a spin down fermion.

Now the key question is whether there are allowed edge operators which can gap out this pair of edge states. In the absence of any symmetries, we could imagine a back-scattering term like

$$\mathcal{H}_e' = \int dx \, \xi(x) \Psi_L^\dagger(x) \Psi_R(x) + \text{c.c.}, \tag{45}$$

where we have even allowed for the breaking of translational symmetry on the edge by a disordered coupling $\xi(x)$. Such a term, if present, would certainly gap out the edge states of (44). However, the key observation is that a term like (45) is forbidden in systems with time-reversal symmetry. Under time-reversal, the electron operator transforms as $c_\uparrow \rightarrow ic_\downarrow$, $c_\downarrow \rightarrow -ic_\uparrow$, and with the spin assignments noted above, (45) is time-reversal odd. A single-fermion backscattering terms is therefore forbidden, and the edge states remain gapless.

The situation is more complex when we allow for processes in which 2 fermions are backscattered [10, 11]. Such terms are allowed by time-reversal, and can be relevant for sufficiently strong interactions.

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