Integer and fractional quantum Hall effects

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Abstract

We present an elementary discussion of the properties of the integer quantum Hall states of non-relativistic electrons in GaAs, and the electrons with the massless Dirac spectrum in graphene. Then we use the fermion-fermion duality to obtain the Dirac composite fermion theory of the Jain fractional quantum Hall states.

I. INTEGER QUANTUM HALL EFFECT

A. Non-relativistic particles

1. Landau levels

We consider the single particle Hamiltonian

$$H_0 = -\frac{1}{2M}(\vec{\nabla} - ie\vec{A})^2 \,. \tag{1}$$

We work in a gauge for the vector potential which preserves the translational symmetry along the x direction

$$\vec{A} = (-By, 0). \tag{2}$$

Then the eigenstates of H_0 are of the form

$$\psi_n(x,y) = \frac{1}{\sqrt{L_x}} e^{ikx} \phi_{n,k}(y) \tag{3}$$

where $n = 0, 1, 2, \ldots$ labels the energy eigenvalues, and $\phi_{n,k}(y)$ obeys

$$-\frac{1}{2M}\frac{d^2\phi_{n,k}}{dy^2} + \frac{1}{2M}(k + eBy)^2\,\phi_{n,k}(y) = E_n \phi_{n,k}(y), \tag{4}$$

where the eigenvalue

$$E_n = (n + 1/2)\omega_c \quad , \quad n = 0, 1, 2, \ldots \quad , \quad \omega_c = eB/M \,, \tag{5}$$

where ω_c is the cyclotron frequency. These are the dispersionless (independent of k_x) Landau levels. We also introduce the dimensionless co-ordinate $\overline{y} = y/\ell$, where

$$\ell = \frac{1}{\sqrt{M\omega_c}} = \frac{1}{\sqrt{eB}} \,. \tag{6}$$

Then the eigenvalue equation becomes

$$-\frac{1}{2}\frac{d^2\phi_{n,k}}{d\,\overline{y}^2} + \frac{1}{2}(k\ell + \overline{y})^2\,\phi_{n,k}(\overline{y}) = (n + 1/2)\,\phi_{n,k}(\overline{y}) \,. \tag{7}$$

The eigenfunctions are the harmonic oscillator eigenstates

$$\phi_{n,k}(y) = \frac{\pi^{-1/4}}{\sqrt{2^n n!}} H_n(y + k\ell) \exp\left(-\frac{(y + k\ell)^2}{2}\right) , \tag{8}$$

where $H_n(y)$ are the Hermite polynomials.

In a sample of size $L_x \times L_y$, k is quantized in integer multiples of $2\pi/L_x$. And the shift in the harmonic oscillator eigenstates implies that the range of allowed values of k is $-L_y/(2\ell^2)$ to $L_y/(2\ell^2)$. So the degeneracy of each Landau level is

$$\frac{L_x}{2\pi}\frac{L_y}{\ell^2} = \frac{AB}{h/e} \tag{9}$$

where A is the area, and h/e is the flux quantum *i.e.* the degeneracy is the number of flux quanta in the sample.

For the computation of the conductivity, we need the matrix elements:

$$\begin{aligned} \int dy\, \phi_{n,k}(y)\, \partial_y \phi_{m,k}(y) &= \sqrt{\frac{m}{2}}\, (\delta_{m,n+1} - \delta_{m,n-1}) \\ \int dy\, \phi_{n,k}(y)(y + k\ell)\phi_{m,k}(y) &= \sqrt{\frac{m}{2}}\, (\delta_{m,n+1} + \delta_{m,n-1}) \end{aligned} \tag{10}$$

2. Hall conductivity

We compute the conductivity σ_{xy} by the Kubo formula.

We introduce an operator $c_{n,k}$ which annihilates the electron in the Landau level state $\psi_{n,k}(x,y)$. Then the electron field operator is

$$\Psi(x,y) = \int \frac{dk}{2\pi} \sum_n \psi_{n,k}(x,y) c_{n,k} \tag{11}$$

and the electron current operator is

$$\vec{J}(x,y) = \frac{1}{2Mi}\left[\Psi^\dagger\, (\vec{\nabla} - i\vec{A})\Psi - (\vec{\nabla} + i\vec{A})\Psi^\dagger\, \Psi\right] \tag{12}$$

Then applying the Kubo formula at a frequency ω_n, we obtain

$$\begin{aligned} \sigma_{xy}(\omega_n) = \frac{1}{\omega_n}\frac{1}{L_x L_y} \sum_k \sum_{n,m} &\left[\frac{f(E_n) - f(E_m)}{i\omega_n - E_n + E_m}\right] \\ &\times \left[\int dy\, \phi_{n,k}(y)\, \partial_y \phi_{m,k}(y)\right] \left[\int dy\, \phi_{n,k}(y)(y + k\ell)\phi_{m,k}(y)\right]. \end{aligned} \tag{13}$$

where $f(E)$ is the Fermi function. In the limit $T \to 0$, we find that only 2 levels contribute to the sum: the ones just above and below the Fermi level. Carefully evaluating this expression use the results above, we find

$$\sigma_{xy} = \frac{ne^2}{h} \tag{14}$$

where n is the number of filled Landau levels.

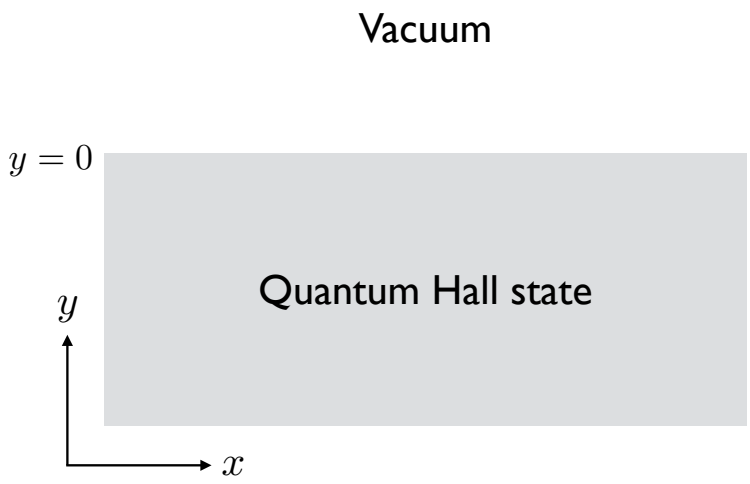

FIG. 1. Edge of a semi-infinite quantum Hall state at $y = 0$.

3. *Edge states*

Let us look at the situation in which the sample is only present for $y < 0$, and so has a edge at $y = 0$ (see Fig. 1). We consider the single particle Hamiltonian

$$H_0 = -\frac{1}{2M}(\vec{\nabla} - i\vec{A})^2 + V(y). \tag{15}$$

As translationally symmetry is preserved along the x direction, we can continue to work with the vector potential in Eq. (2). The eigenstates are as in Eq. (3)

$$\psi_n(x, y) = \frac{1}{\sqrt{L_x}} e^{ik_x x} \phi_{n,k}(y) \tag{16}$$

and $\phi_n(y)$ obeys

$$-\frac{1}{2M}\frac{d^2\phi_n}{dy^2} + \left[\frac{1}{2M}(k_x + By)^2 + V(y)\right]\phi_n(y) = E_n(k_x)\phi_n(y), \tag{17}$$

with $E_n(k_x)$ the energy eigenvalue which disperses as a function of k_x.

More generally, we can take $V(y) = 0$ in the bulk of the sample, far from the edge, without generality. Here, the eigenstates in (17) are harmonic oscillator states centered at $y = -k_x/B$. So we expect the eigenstates at large positive k_x to be within the sample, and insensitive to the edge. But near the edge of the sample at $y = 0$, we expect $V(y)$ to increase rapidly to confine the electrons within the sample. So as k_x decreases through 0, we expect the eigenstates to approach the edge of the sample, and for $E_n(k_x)$ to increase. We can estimate the change in E_n by perturbation theory

$$E_n(k_x) = (n + 1/2)\frac{B}{m} - \frac{k_x}{B}\int dy |\phi_n^0(y)|^2 \frac{dV}{dy} + \ldots, \tag{18}$$

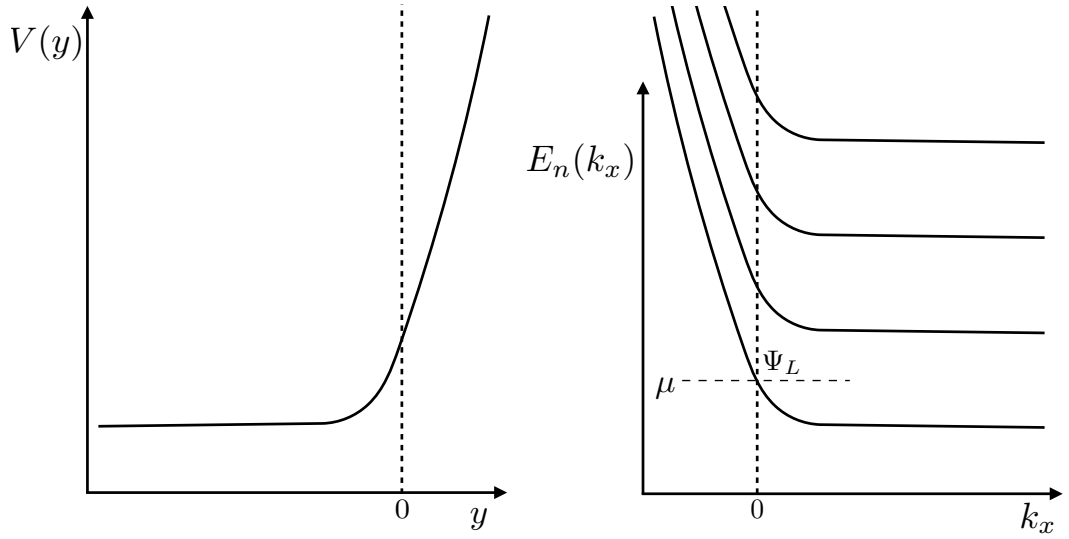

FIG. 2. Energy levels, $E_n(k_x)$ of electrons in a potential $V(y)$ and a magnetic field. The chemical potential, μ, is chosen so that only the $n = 0$ level is occupied. The left-moving chiral fermions, Ψ_R, describe the excitations on the $y = 0$ edge.

where $\phi_n^0(y)$ are the eigenstates for $V = 0$. See the sketch of energy levels in Fig. 2.

Now, we consider the situation for the $m = 1$ case, where only the lowest, $m = 0$, Landau level is fully occupied in the bulk. We see from Fig. 2, that in such a situation $E_0(k_x)$ will necessarily cross the chemical potential once as a function of decreasing k_x. This implies the existence of gapless, one-dimensional, fermionic excitations on the edge of the sample. The fermions all move with velocity $v = dE_0(k_x)/dk_x$, and so are left-moving chiral fermions, (18). Notice that there is no right-moving counterpart, at least on the edge near $y = 0$. If the sample had another edge far away at some $y < 0$, that edge would support a right-moving chiral fermion.

It is interesting to note that a single left-moving chiral fermion cannot appear by itself in any strictly one-dimensional system. In the presence of a lattice, the fermion dispersion, $E(k_x)$, of such a system must be a periodic function of k_x, and no periodic function can cross the Fermi level only once. But on the edge of a two-dimensional system, it is possible for $E(k_x)$ to cross the Fermi level just once, as we have shown above.

It is simple to write down a low energy effective theory for the left-moving chiral fermion at the edge of the simple. Using the notation and methods of the chapter on Luttinger liquids, we have the imaginary action

$$\mathcal{S}_L = \int dx d\tau \, \Psi_L^\dagger \left(\frac{\partial}{\partial \tau} + iv \frac{\partial}{\partial x} \right) \Psi_L. \tag{19}$$

This is the universal low-energy theory of the edge of a quantum Hall sample at $m = 1$. Note that, unlike non-chiral-Luttinger liquids, there are no marginal interaction corrections to the free theory. All such interaction corrections involve right-moving fermions too, which are absent in the

present system.

B. Relativistic particles (graphene)

Now we consider a Dirac fermion, with unit Fermi velocity, which describes the low spectrum of graphene with Hamiltonian

$$H = -i\vec{\sigma} \cdot (\vec{\nabla} - i\vec{A}) \tag{20}$$

The eigenvalue equation can be written as

$$\begin{pmatrix} 0 & -i(\partial_x - iA_x) - (\partial_y - iA_y) \\ -i(\partial_x - iA_x) + (\partial_y - iA_y) & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{21}$$

This translates into a Schroedinger-like equation for each component:

$$\left[-(\vec{\nabla} - i\vec{A})^2 + (\vec{\nabla} \times \vec{A}) \right] \psi_{1,2} = E^2 \, \psi_{1,2} \tag{22}$$

The eigenvalues and eigenfunctions are easily obtained from the solution in Section I A 1:

$$E_n = \mathrm{sgn}(n)\sqrt{2B|n|} \quad , \quad n = \ldots -2, -1, 0, 1, 2 \ldots \tag{23}$$

Note that the Landau levels now have both positive and negative energies, as does the Dirac dispersion in zero magnetic field. Also note the special Landau level at exactly zero energy. The degeneracy of each Landau level is still given by Eq. (9).

The Hall conductivity computation can be carried out in a manner similar to Section I A 2. We find now

$$\sigma_{xy} = (n + 1/2)\frac{e^2}{h} \tag{24}$$

when the chemical potential is just above the Landau level with energy E_n. Although this has a half-integer conductance, the additional valley degeneracy of the Dirac fermions always yields integer values (except on the surface of a topological insulator). Note that just above and below the zero energy Landau level, the Hall conductivity $= \pm e^2/(2h)$. This is similar to the Hall conductivity of a massless Dirac fermion with time-reversal symmetry broken by a mass term $m\overline{\psi}\psi$.

II. FRACTIONAL QUANTUM HALL EFFECT: DIRAC COMPOSITE FERMIONS

The fractional quantum Hall effect occurs when Landau levels are fractionally filled. The most famous and simplest example is the Laughlin state at $\nu = 1/3$, for which Laughlin proposed a trial wavefunction with all the correct properties. The analysis based on the trial wavefunction is reviewed in many books, and we will not go into it here.

The physics of the fractional quantum Hall states is believed to be fully captured by projecting the Hamiltonian down to an effective Hamiltonian that only acts on states within any given Landau level. We shall follow a field theoretic approach here, and we will choose a convenient choice for the physics outside the Landau level. It turns out that the best starting point [1, 2] is to consider graphene at its charge neutrality point in the presence of a magnetic field. Because of the particle-hole symmetry of graphene, the $n = 0$ Landau level in graphene is exactly half-filled, and we are at $\nu = 1/2$. By particle-hole symmetry, this state has $\sigma_{xy} = 0$. If we are considering a non-relativistic fermion, as in GaAs, then quantum Hall states would have their conductivity shifted by $e^2/(2h)$, *i.e.* the Hall conductivity of the half-filled Landau level would be $e^2/(2h)$.

So starting from graphene at its charge-neutrality point, we consider the theory the

$$\mathcal{L}_{\text{graphene}} = \overline{\psi}\gamma_\mu \left(\partial_\mu - iA_\mu\right)\psi + \dots \tag{25}$$

where ψ creates an electron in the form a two-component Dirac fermion. The spatial components of A_μ, represent the applied magnetic field with $\vec{\nabla} \times \vec{A} = B$. The time component, iA_τ, is the applied chemical potential which allows us to consider electron densities away from the charge neutrality point of graphene. If we were considering the non-relativistic quantum Hall states, as in GaAs, then we would shift the Hall conductivity (and the density) by adding a Chern-Simons term in the external gauge field

$$\mathcal{L}_{\text{GaAs}} = \overline{\psi}\gamma_\mu \left(\partial_\mu - iA_\mu\right)\psi + \frac{i}{8\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \dots \tag{26}$$

The remainder of the discussion here will be carried out using $\mathcal{L}_{\text{GaAs}}$, as that is the case considered in most of the literature. It is easy to translate back to graphene, by appropriate shifts in the Hall conductivity and the density.

To describe the $\nu = 1/2$ state and its vicinity, we perform the fermion-fermion duality on $\mathcal{L}_{\text{GaAs}}$, to obtain the theory of *Dirac composite fermions*, ψ_c:

$$\mathcal{L}_c = \overline{\psi}_c\gamma_\mu \left(\partial_\mu - ia_\mu\right)\psi_c + \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda + \frac{i}{8\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \dots . \tag{27}$$

As has been our convention, the field a_μ is a dynamical U(1) gauge field that has to be integrated over, while A_μ is the external, "background", electromagnetic gauge field.

We now introduce some basic notation, and obtain important relations by comparing saddle point equations of Eq. (26) and (27). Let ρ be the density of electrons. Then filling factor of the Landau level ν is (in units with $\hbar = e = 1$)

$$\nu = \frac{2\pi\rho}{B} \tag{28}$$

Taking the derivative of $\mathcal{L}_{\text{GaAs}}$ with respect to A_τ, we have

$$\rho = -\left\langle\overline{\psi}\gamma_\tau\psi\right\rangle + \frac{B}{4\pi} \tag{29}$$

So at $\nu = 1/2$, the density of Dirac electrons in graphene, $-\left\langle \overline{\psi}\gamma_\tau\psi \right\rangle$, vanishes. Let us also represent the average internal gauge field on the composite fermions by $b = \vec{\nabla}\times\vec{a}$. Now we take the derivative of $\mathcal{L}_c$ with respect to A_τ. This yields

$$\rho = \frac{b}{4\pi} + \frac{B}{4\pi}\,, \tag{30}$$

which we can also write as

$$b = B(2\nu - 1)\,. \tag{31}$$

So at $\nu = 1/2$, the average magnetic field on the Dirac composite fermions, b, vanishes. This is the primary advantage of the composite fermion formulation, we have mapped a problem at high magnetic field to one at vanishing magnetic field.

We also need some relations for the density of the Dirac composite fermions. Let us denote the density of the Dirac composite fermions by

$$\rho_c = -\left\langle \overline{\psi}_c\gamma_\tau\psi_c \right\rangle\,. \tag{32}$$

Then taking the derivative of $\mathcal{L}_c$ with respect to a_τ we obtain

$$\rho_c = -\frac{B}{4\pi} \tag{33}$$

So the density of Dirac composite fermions is determined by the applied magnetic field.

In general, the theory of composite fermions therefore has a non-zero density, ρ_c, and an applied average field, b. So, if anything, it is more complicated than the original problem of electrons, as we also have to consider a fluctuating gauge field a_μ. However, the composite fermion theory simplifies under two conditions: (i) if the average field b vanishes, or (ii) if exactly an integer number of its Landau levels are filled, so that there is an energy gap to composite fermion excitations.

Let us consider the case when the n'th Landau level of Dirac fermions is fully filled. Recall that the allowed values of n are $\ldots, -2, -1, 0, 1, 2, \ldots$, and that the corresponding Hall conductivity of the Dirac fermions is $(n + 1/2)e^2/h$. In this situation, the relationship between ρ_c and b is

$$\rho_c = \left(n + \frac{1}{2}\right)\frac{b}{2\pi} \tag{34}$$

From Eqs. (31), (33), and (34), we obtain the Jain filling fractions

$$\boxed{\nu = \frac{n}{2n+1}}\,. \tag{35}$$

The simplest cases of the fractional quantum Hall states are $n = 1$, $\nu = 1/3$ and $n = -2$, $\nu = 2/3$. Notice that for the Dirac composite fermions, the cases $n = 1$ and $n = -2$ are particle-hole symmetric: the chemical potential is between the adjacent Landau levels with $|n| = 1$ and $|n| = 2$, and the composite fermion Hall conductivity is $\pm(3/2)e^2/h$. This particle-hole symmetry is a crucial feature of the Dirac composite fermion theory.

This theory can also describes integer quantum Hall states at the special values $n = 0, -1$ corresponding $\nu = 0, 1$: in these cases the zero'th Dirac Landau level is either fully filled or empty, and these cases are also particle-hole symmetric.

A. Chern Simons theory

Let us assume that we are in one of the gapped Jain states give by Eq. (35). We would like to derive an effective theory of the gapped quasiparticle excitations of this state. To obtain this, we simply integrate out the filled Landau levels from the field theory in Eq. (27). The filled Landau levels will contribute a Chern-Simons term for the internal gauge field a_μ with a coefficient determined by the Hall conductivity of the filled Landau levels. In this manner we obtain the effective theory of the Jain states

$$\mathcal{L}_J = \mathcal{L}_{\rm qp}[a_\mu] + \frac{i(n+1/2)}{4\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda + \frac{i}{8\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \dots\,, \tag{36}$$

where $\mathcal{L}_{\rm qp}[a_\mu]$ is the Lagrangian for the gapped quasiparticles, expressed as fermions, minimally coupled with unit charge to the internal gauge field a_μ. Eq. (36) is of the generic form of a Chern-Simons topological quantum field theory that we discussed in Lec10. From that analysis we see that the Chern-Simons term endows the quasiparticles with fractional statistics. Also, ignoring the gapped quasiparticles, we can integrate out the internal gauge field a_μ, and obtain an effective Lagrangian for the external gauge field A_μ:

$$\mathcal{L}_A = \frac{i}{4\pi}\left(\frac{1}{2} - \frac{1}{4(n+1/2)}\right)\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \dots\,. \tag{37}$$

From Eq. (35), we find that the Hall conductivity is simply

$$\sigma_{xy} = \frac{\nu e^2}{h}\,. \tag{38}$$

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[1] D. T. Son, "Is the Composite Fermion a Dirac Particle?" Phys. Rev. X **5**, 031027 (2015), arXiv:1502.03446 [cond-mat.mes-hall].

[2] D. T. Son, "The Dirac composite fermion of the fractional quantum Hall effect," PTEP **2016**, 12C103 (2016), arXiv:1608.05111 [cond-mat.mes-hall].