Field theory of $\mathbb{Z}_2$ topological order in XY models in $D = 3$

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Abstract

We use the field theory of boson-vortex duality to obtain a continuum descriptions of the XY models in Lec9.

I. VORTEX FIELD THEORY OF XY MODEL

We present a field theoretic approach to the $D = 3$ XY models of Lec9. The same field theories are also expected to apply to suitable quantum models of bosons at integer filling.

Near its phase transition, we saw in previous sections that the ordinary $D = 3$ XY model is described by the Wilson-Fisher field theory of a complex 'boson' field $\psi \sim e^{i\theta}$. We also developed a dual field theory of a field ϕ representing a 2π vortex in ψ.

In Lec9, we showed that the XY model can develop a phase with topological order by condensing doubled 4π vortices. In the field theory, this doubled vortex is $\sim \phi^2$. It is convenient to introduce another quantum field Φ which creates a double vortex, and extend our theory of vortices to include both ϕ and Φ:

$$\mathcal{S}_{\Phi\phi} = \int d^3x \Bigg[|(\partial_\mu - i a_\mu)\phi|^2 + s|\phi|^2 + \frac{v}{2}|\phi|^4 + |(\partial_\mu - 2ia_\mu)\Phi|^2 + S|\Phi|^2 + \frac{V}{2}|\Phi|^4 \\ - \lambda(\Phi^* \phi^2 + \Phi\, \phi^{*2}) + \frac{1}{2K}\left(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda\right)^2 + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \Bigg]. \quad (1)$$

We have written down a general theory consistent with the symmetries and gauge invariance. As in Lec2, a_μ is an emergent gauge field whose flux measures the boson number current, and A_μ is the fixed external electromagnetic gauge field. When both s and S are large and positive, then both ϕ and Φ are gapped, and this theory continues to describe a conventional superfluid. The superfluid is however slightly unusual in that two parameters, s and S, can be tuned to separately determine the energies of single and double vortices.

Now we transition to the insulator by reducing S so that it is below a critical value $S < S_c$ where Φ condenses. In doing so, we are imagining that s remains large and positive so that a single vortex, ϕ, remain gapped. So the 'core' energy of a double vortex must become lower than that of a single vortex, an unusual situation which is not realized for the Hubbard model. But we will turn to boson models with longer range interactions shortly, in which this situation can be realized.

The phase with $S < S_c$ with

$$\langle \Phi \rangle = \Phi_0 \neq 0 \quad , \quad \langle \phi \rangle = 0, \quad (2)$$

is nevertheless an insulator, and not a superfluid. With Φ condensed, $\mathcal{S}_{\Phi\phi}$ shows that the gauge field a_μ acquires a gap by the Higgs mechanism. We choose a gauge in which Φ_0 is real and positive, and we can then write the low energy action for the gauge fields in the Φ-condensed phase as

$$\mathcal{S} = \int d^3x \left[4\Phi_0^2 \, a_\mu^2 + \frac{i}{2\pi} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \right]. \tag{3}$$

Integrating out a_μ we obtain

$$\mathcal{S} = \int d^3x \left[\frac{1}{64\pi^2 \Phi_0^2} \left(\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda \right)^2 \right]. \tag{4}$$

This is just renormalizes the Maxwell terms of electrodynamics, and so represents the dielectric and magnetostatic response of an insulator; in other words, there is no Meissner effect associated with the external field.

We now discuss the gapped excitations in the insulator associated with the vortex and boson fields in the following subsections.

A. Visons

The ϕ vortex remains a legitimate gapped excitation in the insulator described by (2). Extracting the terms from (1), we can write the quadratic action for the ϕ vortex as

$$\mathcal{S}_\phi = \int d^3x \Big[|(\partial_\mu - i a_\mu)\phi|^2 + s|\phi|^2 - \lambda \Phi_0 (\phi^2 + \phi^{*2}) \Big]. \tag{5}$$

Ignoring the gapped a_μ field, this quadratic form is diagonalized by the real and imaginary parts of ϕ. Writing

$$\phi = v + iw, \tag{6}$$

we have

$$\mathcal{S}_\phi = \int d^3x \Big[(\partial_\mu v)^2 + (s - 2\lambda\Phi_0) v^2 + (\partial_\mu w)^2 + (s + 2\lambda\Phi_0) w^2 \Big]. \tag{7}$$

We assume $\lambda > 0$, and then So we observe that the v particle has dispersion $\varepsilon_k = \sqrt{k^2 + s - 2\lambda\Phi_0}$, while the w particle has a larger mass gap with $\varepsilon_k = \sqrt{k^2 + s + 2\lambda\Phi_0}$. We ignore the heavier particle, and focus on the real particle v. This a vortex excitation with an Ising-like character, because it is its own anti-particle, and has been named the "vison" [1]. A vortex can turn into an anti-vortex by absorbing a double vortex from the Φ condensate, and so the distinction between vortices and anti-vortices has now disappeared: there is only a single low-energy vortex-like excitation: the vison.

It is remarkable that an insulator has a vortex-like vison excitation; normally vortices are found only in superfluids. Moreover the vison is a linear combination of a vortex and an anti-vortex, and so is even under time-reversal. Such time-reversal-even vortices of insulators were first introduced in Ref. 2.

B. Bosons with half-integer boson number

Recall that in Lecture 5, we obtained the gapped boson excitations of the Mott insulator by considering 'vortices-in-vortices' *i.e.* vortex configurations in the condensed ϕ. Here, it is the double vortex Φ that is condensed, and we have already shown above that the uncondensed ϕ field is now a vison excitation. By exactly the same argument leading to (41-44) in Lecture 5, we now have far from the core of the Φ vortex

$$\Phi(|x| \to \infty) = \Phi_0 e^{i\Theta} \tag{8}$$

and

$$\oint dx_i \partial_i \Theta = 2 \int d^2x \, \epsilon_{ij} \partial_i a_j = \pm 2\pi \tag{9}$$

So we have the remarkable result that the boson number of this excitation is $Q = \pm 1/2$: doubling the vortices, fractionalizes the bosons in the insulator.

We can also obtain this result in a more field-theoretic manner by applying the particle-to-vortex mapping of Lecture 5 to the action for the double vortex Φ in (1):

$$\mathcal{S}_\Phi = \int d^3x \left[|(\partial_\mu - 2ia_\mu)\Phi|^2 + S|\Phi|^2 + \frac{V}{2}|\Phi|^4 + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \right] \tag{10}$$

We now apply the mapping from (36) to (37) in Lecture 5 to $\mathcal{S}_\Phi$ above while viewing $2a_\mu$ here as the analog of an external gauge field. (In such a mapping we assume that Φ in the terms with co-efficient λ can be fully absorbed by the replacement $\Phi \to \Phi_0$ in (5)—this is more fully justified in [3].) This mapping introduces a 'half-boson' field h, and another U(1) gauge field b_μ:

$$\mathcal{S}_h = \int d^3x \left[|(\partial_\mu - ib_\mu)h|^2 + s_1|h|^2 + \frac{i}{\pi}\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu b_\lambda + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \right], \tag{11}$$

where we have omitted higher order and Maxwell terms which will be unimportant in the state where h is gapped (s_1 is large and positive). This action shows that the h boson carries boson number $Q = \pm 1/2$: the equation of motion of b_μ shows that each h particle carries π flux of a_μ, and the term linear in A_μ shows that this flux has half charge. The combination $\mathcal{S}_\phi + \mathcal{S}_h$ in (5) and (11) now present a nearly complete description of the excitations of the insulator: it has gapped visons described by (5), and half-charged bosons described by (11). These matter fields are coupled with unit charge to the U(1) gauge fields a_μ and b_μ respectively, and there is a 'mutual Chern-Simons' term between the gauge fields (the second-to-last term in (11)).

C. Mutual Semions

The mutual Chern-Simons term in (11) endows the ϕ and h excitations with novel 'anyonic statistics' when they are adiabatically move around each other. Specifically, adiabatically moving a vison around a h particle accumulates a π Berry phase.

To compute this, first examine the equation of motion of the b_μ gauge field $\mathcal{S}_h$. This yields a relationship analogous to (38) in Lecture 5, but with a factor of 2 difference

$$h^* \partial_\mu h - h \partial_\mu h^* = \frac{1}{\pi} \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda \tag{12}$$

So a stationary h particle carries a flux tube of a_μ gauge flux, with total flux π. As the ϕ particle has unit a_μ charge, moving the ϕ particle around the h particle yields a Aharonov-Bohm phase of π. Similarly, moving the h particle around a ϕ particle also yields a Aharonov-Bohm phase of π.

So we have the important conclusion that the vison and half-boson excitations are mutual semions in the insulator. This identifies the present insulator as a state with bulk 'topological order'. We will see later that the mutual Chern-Simons term has important consequences for the degeneracy of the ground state on manifolds of non-trivial topology, and hence the label 'topological order'. The Ising-nature of the vison is one of the reasons for labeling this order as $\mathbb{Z}_2$ topological order.

D. Fermion excitations

The insulator with $\mathbb{Z}_2$ topological order has one more excitation with a topological character *i.e.* an excitation that cannot be created by a local operator on its own, and can only appear in pairs (this is also true for the ϕ and h particles).

Consider a bound state, of a vison and a half-boson:

$$\chi \sim \phi \, h. \tag{13}$$

As local operators, ϕ and h are bosons, so as defined χ is also a bosonic operator *i.e.* χ operators at well-separated spacetime points commute with each other. However, the remarkable fact is that the particle created by the χ operator turns out to be a *fermion* [2]: this is a consequence of the long-range Aharonov-Bohm phase factors associated with the Chern-Simons term.

To see this, let us write down an effective theory for χ, based on symmetry and gauge invariance:

$$\mathcal{S}_\chi = \int d^3x \left[|(\partial_\mu - i a_\mu - i b_\mu)\chi|^2 + s_2 |\chi|^2 \right] \tag{14}$$

Now consider the adiabatic process where one particle, χ_1, is exchanged with another particle χ_2. We can perform this exchange by moving χ_2 half-way adiabatically around a stationary χ_1, and then translating. In such a process, the ϕ_2 particle performs a half circuit around h_1, and the h_2 particle performs a half-circuit around ϕ_1: both half-circuits yield a phase factor of $\pi/2$, for a total Aharonov-Bohm phase factor of π. Hence the χ particles are fermions.

E. Model Hamiltonians

Model boson Hamiltonians which realize the $\mathbb{Z}_2$ spin liquid were discussed in Refs. 4–7.

The boson models can also be connected to quantum spin systems via the Holstein-Primakoff tranformation. This transformation maps the $2S+1$ states of a quantum spin $\hat{S}_\alpha$ ($\alpha = x, y, z$) to those of a boson, b, satisfying $b^\dagger b \leq 2S$. The operator representation of the quantum spins is

$$\begin{aligned} \hat{S}_z &= S - b^\dagger b \\ \hat{S}_+ = \hat{S}_x + i\hat{S}_y &= \sqrt{2S}\left(1 - \frac{b^\dagger b}{2S}\right)^{1/2} b \end{aligned} \tag{15}$$

We can apply this transformation to a general quantum spin model of the type

$$H = \sum_{i<j} J_{ij} \left(\hat{S}_{ix}\hat{S}_{jx} + \hat{S}_{iy}\hat{S}_{jy} \right) + \sum_{i<j} J'_{ij} \hat{S}_{iz}\hat{S}_{jz}. \tag{16}$$

The important feature of H is that it commutes with the total spin $\sum_i \hat{S}_{iz}$, which is clearly related to the conservation of boson number in the representation (15). We are interested in antiferromagnets with a ground state with $\sum_i \hat{S}_{iz} = 0$, and this corresponds to the average boson number

$$\left\langle b^\dagger b \right\rangle = S \tag{17}$$

So the integer boson number case considered so far corresponds to integer values of S in quantum spin systems. Such integer spin models were considered in Refs. 8 and 9, and were shown to have $\mathbb{Z}_2$ spin liquid ground states by very different methods. This connection also makes it clear that we should also consider bosons at half-integer filling to describe spin liquids for the important case of $S = 1/2$ antiferromagnets, and we will do so in the following section.

We can also map Kitaev's 'toric code' model [10] to bosons using (15): this yields a model in which the number of bosons is only conserved modulo 2. Its topological structure, as described by the Chern-Simons gauge theory in Section II, is identical.

F. Global phase diagram

It is useful to collect all the phases we have described so far for bosons at integer filling in a single phase diagram. The original action for the single and double vortices is in (1). We compute the mean field phase diagram as a function of s and S, while ignore the gauge field a_μ: this reduces to minimizing $\mathcal{S}_{\Phi\phi}$ with respect to Φ and ϕ, and leads to the phase diagram in Fig. 1. Note that there is a line of first-order phase transitions near the point where the 3 phases meet.

We can also describe the universality classes of the three lines of second order phase transitions:

- The transition from the superfluid to the trivial insulator is described by the condensation of ϕ. Here Φ can be neglected is $\mathcal{S}_{\Phi\phi}$, and the remaining action is that of a complex scalar, ϕ, coupled to a U(1) gauge field. By boson-vortex duality this theory is in the XY universality class.

- The transition from the superfluid to the $\mathbb{Z}_2$ spin liquid is similar to that to the trivial insulator, with a theory of the condensation of ϕ being replaced by the condensation of Φ. This is almost the same theory, but the doubling of the vortex leads to some subtle differences in the set of allowed operators and boundary conditions, and the resulting theory is called an XY* theory.

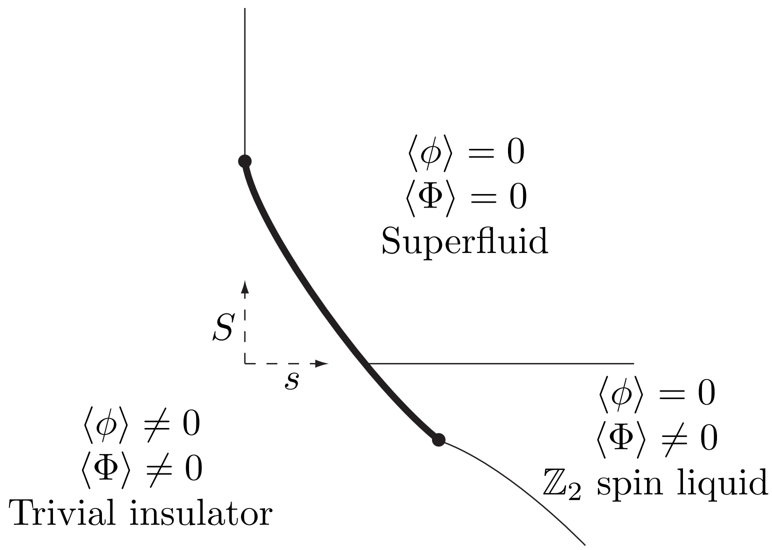

FIG. 1. Mean field phase diagram of the action of single (ϕ) and double (Φ) vortices in (1) for bosons at integer filling. Thin (thick) lines represent second (first) order phase transitions.

- Finally, we consider the transition between the two insulators. The gauge field is Higgsed in both phases, and so can be neglected at the critical point. The transition is then described by the condensation of the real vison field v in (7), and is in the Ising* universality class (all physical operators and boundary conditions must be expressed in terms of operators which contain even powers of v.)

II. CHERN-SIMONS THEORY

It is useful to collect all the ingredients of our complete theory of the excitations of an insulator with $\mathbb{Z}_2$ topological order, obtained for the case of bosons at integer filling, and zero applied magnetic field. We have three bosonic fields ϕ, h, and χ, two fluctuating U(1) gauge fields a_μ and b_μ, and the fixed external electromagnetic gauge field A_μ described by the combination of (5), (11), and (14)

$$\mathcal{S} = \int d^3x \Big[|(\partial_\mu - ia_\mu)\phi|^2 + s|\phi|^2 - \lambda\Phi_0(\phi^2 + \phi^{*2}) + |(\partial_\mu - ib_\mu)h|^2 + s_1|h|^2 + |(\partial_\mu - ia_\mu - ib_\mu)\chi|^2 + s_2|\chi|^2 + \frac{i}{\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu b_\lambda + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda \Big]. \quad (18)$$

This theory implies that the self-statistics of the ϕ and h particles is bosonic, the self-statistics of χ is fermionic, and the mutual statistics of any particle with the other two is semionic. This turns out to be the simplest theory with anyonic excitations and bulk topological order which preserves time-reversal symmetry: such a theory was first proposed (in a very different language) in Refs. 8 and 11 as a 'spin liquid' ground state of quantum antiferromagnets: hence the name '$\mathbb{Z}_2$ spin liquid'.

With an eye to subsequent discussions, where we will find other topological states, it is useful to write the Chern-Simons terms in (18) in the general form

$$\mathcal{S}_{\rm CS} = \int d^3x \left[\frac{i}{4\pi} \epsilon_{\mu\nu\lambda} a_\mu^I \, K_{IJ} \, \partial_\nu a_\lambda^J + \frac{i}{2\pi} t_I A_\mu \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda^I \right] \tag{19}$$

where I, J are indices extending over N values $1 \ldots N$, and a_μ^I are N U(1) gauge fields. This turns out to be a general description of the degenerate ground states of states with abelian topological order on manifolds with non-trivial topology. Here we have obtained (19) by integrating out the gapped matter degrees of freedom in (18). Such an insulator, realizing a $\mathbb{Z}_2$ spin liquid, has $N = 2$, $a_\mu^1 \equiv a_\mu$, $a_\mu^2 \equiv b_\mu$, and the 'K-matrix' given by

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{20}$$

The vector t_I determines the electromagnetic charges of the quasiparticles, and for the $\mathbb{Z}_2$ spin liquid

$$t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{21}$$

The n'th quasiparticle excitation of the theory is labeled by a ℓ-vector $\ell_I^{(n)}$ of integers representing the charges under a^I gauge fields. For the present theory, we have the vectors

$$\ell^\phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \ell^h = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad \ell^\chi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{22}$$

By repeating the Aharonov-Bohm flux-tube arguments presented above for this general case, it is not difficult to show that the self-statistics for the particle with a given ℓ-vector is specified by the statistical angle

$$\theta_\ell = \pi \, \ell^T \, K^{-1} \, \ell, \tag{23}$$

where $\theta = 0 \bmod (2\pi)$ corresponds to bosons, and $\theta = \pi \bmod (2\pi)$ corresponds to fermions. Similarly, the mutual statistics between distinct particles ℓ and ℓ' is given by the statistical angle

$$\theta_{\ell,\ell'} = 2\pi\, \ell^T K^{-1} \ell', \tag{24}$$

where $\theta = \pi \bmod (2\pi)$ corresponds to mutual semions. Finally, the electromagnetic charge of the quasiparticle ℓ is

$$Q = \ell^T K^{-1} t. \tag{25}$$

It is simple to verify these expressions for the case of the $\mathbb{Z}_2$ spin liquid, where Q becomes the boson number.

Finally, we note that it is possible to "integrate out" the a^I_μ from (19), and obtain an effective action for A_μ (*e.g.* by adding Maxwell terms for a^I_μ to regularize the integral)

$$\mathcal{S}_A = (t^T K^{-1} t) \int d^3x \left[\frac{i}{4\pi} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \right]. \tag{26}$$

The value of $(t^T K^{-1} t)/(2\pi)$ is the Hall conductance, in units of $e^2/\hbar$, where e is the charge of the underlying boson. In the $\mathbb{Z}_2$ spin liquid, the Hall conductance vanishes (as it must, in a theory with time-reversal symmetry), but we will later meet models in which this is not the case.

Our discussion above of Chern-Simons theory will generalize to a large class of topological phases of matter. Although, strictly speaking, the theory in (19) is not purely topological, as it has a global U(1) symmetry, associated with the conservation of the number of bosons in the underlying model, and we have coupled an external gauge field A_μ to the conserved U(1) current.

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