Fractional quantum Hall states and chiral spin liquids

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Abstract

We describe quantum Hall states of bosons and fermions in a strong magnetic field using the field-theoretic methods of flux attachment and particle-vortex duality.

So far, we have met gapped states of bosons only at special densities: integer or half-integer filling fractions on lattices. We can imagine extending our previous analyses to obtain gapped states at other rational densities.

Now consider applying an electromagnetic field transverse to a layer of bosons in two spatial dimensions. If the bosons are in an insulating state, the magnetic flux will just penetrate the system, at the cost of a finite diamagnetic susceptibility. In contrast, in the superfluid state of bosons, the Meissner effect will attempt to expel the flux. Under suitable conditions, the magnetic flux can penetrate the superfluid, at the cost of creating an Abrikosov flux lattice: this is a lattice of vortices, with each vortex associated with the penetration of a single flux quantum. Such a flux lattice state can exist at a continuously varying boson density, and so this is a compressible and gapless state. The gapless excitations include 'spin-wave' fluctuations of the superfluid order, and phonon modes of the Abrikosov flux lattice. In the dual vortex language, the Abrikosov flux lattice is a Wigner crystal of vortices: we will note this realization in our analysis below.

However, there is a new class of gapped states possible at generic densities (relative to any possible lattice) in two spatial dimensions which allow full, uniform, penetration of the magnetic flux, and preserve translational symmetry. The existence of these states requires a different commensurability condition for the particle density: it has to be commensurate with the magnetic flux density, measured in units of the flux quantum. These are the quantum Hall states [1]. In an applied magnetic field B, the density of flux quanta is $B/(h/e)$. We use units with $\hbar = e = 1$, and hence the density of flux quanta is $B/(2\pi)$. It is conventional to measure the particle density, n, in terms of the filling fraction

$$\nu = \frac{2\pi n}{B}. \tag{1}$$

We will only consider quantum Hall states that appear when $\nu = 1/m$, where m is an integer. We will also discuss closely related states of quantum antiferromagnets called chiral spin liquids [2].

We will show that the quantum Hall states are described by a Chern-Simons topological field theory very similar to that obtained for $\mathbb{Z}_2$ spin liquids. The main difference is that time-reversal is no longer a symmetry of the Chern-Simons theory, and so a larger class of K-matrices is possible.

Our "derivation" of the quantum Hall states follows from a semi-microscopic model will require two formal techniques. One is the particle-to-vortex duality which we have already extensively employed for the $\mathbb{Z}_2$ spin liquid and other phases. The other is the method of "flux attachment" which we discuss in the first section below.

I. FLUX ATTACHMENT

We describe here a formally exact transformation that can be applied to the quantum theory of non-relativistic fermions or bosons.

In the Schrödinger formulation, a system of N particles is described by a wavefunction $\widetilde{\Psi}(\vec{x}_1, \vec{x}_2, \ldots \vec{x}_N)$, obeying an eigenvalue equation

$$H[\vec{\nabla}_i, \vec{x}_i]\, \widetilde{\Psi}(\vec{x}_i) = E\, \widetilde{\Psi}(\vec{x}_i), \tag{2}$$

where the Hamiltonian, H, depends upon the momenta $\vec{\nabla}_i$ and the positions $\vec{x}_i$. For bosons (fermions) the wavefunction $\widetilde{\Psi}$ is a totally symmetric (anti-symmetric) function of its arguments.

Now we perform a "singular gauge transformation" and introduce a new wavefunction via the unitary transformation

$$\Psi(\vec{x}_i) \equiv U^m\, \widetilde{\Psi}(\vec{x}_i) \tag{3}$$

where m is an integer, and U is a unitary transformation given by

$$U = \prod_{i<j} \frac{(z_i - z_j)}{|z_i - z_j|} \tag{4}$$

where $z_i \equiv x_i + iy_i$ are the complex co-ordinates of the particles. Note that for m even (odd) the wavefunction Ψ is a totally symmetric wavefunction for the case where the original Ψ was a wavefunction for bosons (fermions). We will always choose Ψ to be a bosonic wavefunction, and so m is correspondingly even or odd. The Ψ bosons are often referred to as 'composite bosons'.

Note that the unitary transformation in (3) produces a single-valued wavefunction for all integer m. However, U is not well defined when two particles are at the same spatial point. So we will always consider 'hard-core' particles *i.e.* we will assume that Ψ and $\widetilde{\Psi}$ vanish whenever any pair of particle positions come within some small cutoff distance of each other.

We would now like to write down the Schrödinger equation that is obeyed by Ψ, given (2). to obtain this, we need the useful identity

$$\begin{aligned} U^{-1}\, \vec{\nabla}_i\, U &= -i \sum_{j \neq i} \frac{\hat{z} \times (\vec{x}_i - \vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^2} \\ &\equiv -i\, \vec{b}_i. \end{aligned} \tag{5}$$

It is easy to check that the vector $\vec{b}_i$ obeys

$$\begin{aligned} \vec{\nabla}_i \cdot \vec{b}_i &= 0 \\ \vec{\nabla}_i \times \vec{b}_i &= 2\pi \sum_{j \neq i} \delta^2(\vec{x}_i - \vec{x}_j). \end{aligned} \tag{6}$$

Using this unitary transformation, we can deduce the Schrödinger equation obeyed by $\widetilde{\Psi}$:

$$H[\vec{\nabla}_i - im\, \vec{b}_i, \vec{x}_i]\, \Psi(\vec{x}_i) = E\Psi\,(\vec{x}_i), \tag{7}$$

So in the wavefunction Ψ, each particle i sees a flux tube strength $2\pi m$ attached to every other particle.

We can write the Schrödinger equation (7) in a path integral with the imaginary time Lagrangian

$$\mathcal{L}_\Psi = \Psi^\dagger(x,\tau)\left(\frac{\partial}{\partial\tau} - \frac{(\vec{\nabla} - im\,\vec{b}(\vec{x}) - i\vec{A}(\vec{x}))^2}{2M} - \mu\right)\Psi(x,\tau) + \frac{u}{2}\left|\Psi(x,\tau)\right|^4, \tag{8}$$

where we have introduced simple Hamiltonian with quadratic dispersion with mass M, contact repulsion u, chemical potential, μ, and an external applied magnetic field $\vec{A}$. The 'internal' field $\vec{b}$ now obeys the constraint operator equations

$$\vec{\nabla}\times\vec{b} = 2\pi\,\Psi^\dagger\Psi \tag{9}$$

$$\vec{\nabla}.\vec{b} = 0 \tag{10}$$

at all points in spacetime. Th constraint in (9) can be implemented by introducing a conjugate Lagrange multiplier, mb_τ in the Lagrangian so that (8) becomes

$$\begin{aligned}\mathcal{L}_\Psi = \Psi^\dagger(x,\tau)&\left(\frac{\partial}{\partial\tau} - imb_\tau - \frac{(\vec{\nabla} - im\,\vec{b}(\vec{x}) - i\vec{A}(\vec{x}))^2}{2M} - \mu\right)\Psi(x,\tau) + \frac{u}{2}\left|\Psi(x,\tau)\right|^4 \\ &+ \frac{im}{2\pi}b_\tau\left(\vec{\nabla}\times\vec{b}\right).\end{aligned} \tag{11}$$

The integral over b_τ directly reproduces the constraint in (9). Now we notice that all terms in the first line of (11) are invariant under spacetime-dependent gauge transformations in which $b_\mu \equiv (b_\tau, \vec{b})$ transforms as a U(1) gauge field. The last term in (11) is not invariant under such gauge transformations, but we will now show that it does become invariant after we impose the gauge condition in (10). In particular, we note that this last term is almost a Chern-Simons term because

$$\epsilon_{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda = 2b_\tau\left(\vec{\nabla}\times\vec{b}\right) + (b_y\partial_\tau b_x - b_x\partial_\tau b_y) + \text{total derivatives}. \tag{12}$$

Now, we can always solve the constraint (10) in two dimensions by the parameterization $\vec{b} = \hat{z}\times\vec{\nabla}\varrho$ for some scalar field ϱ; inserting this in (12), the second term on the right-hand-side is easily shown to vanish up to total derivatives. Focusing on just the last term in (11), this implies that for the path integral of interest to us we can write

$$\begin{aligned}&\int\mathcal{D}b_\mu\,\delta\left(\vec{\nabla}\cdot\vec{b}\right)\exp\left(-\frac{im}{2\pi}\int d^3x\,b_\tau(\vec{\nabla}\times\vec{b}) + \ldots\right) \\ &\qquad = \int\mathcal{D}b_\mu\,\delta\left(\vec{\nabla}\cdot\vec{b}\right)\exp\left(-\frac{im}{4\pi}\int d^3x\,\epsilon_{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda + \ldots\right),\end{aligned} \tag{13}$$

because the difference in the actions vanishes when $\vec{\nabla}\cdot\vec{b} = 0$. In the second form, all terms inside the exponential are invariant under gauge transformations. Now we make the change of variable in the functional integral $\vec{b}\to\vec{b} - \vec{\nabla}\zeta$. The gauge invariance of the action implies that it remains unchanged, while the constraint does require modification:

$$\int \mathcal{D}b_\mu \, \delta\left(\vec{\nabla}\cdot\vec{b}\right) \exp\left(-\frac{im}{4\pi}\int d^3x \, \epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda + \ldots\right)$$
$$= \int \mathcal{D}b_\mu \, \delta\left(\vec{\nabla}\cdot\vec{b} - \vec{\nabla}^2\zeta\right) \exp\left(-\frac{im}{4\pi}\int d^3x \, \epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda + \ldots\right). \quad (14)$$

This equivalence implies that the path integral is independent of the choice of ζ. Hence, we can just integrate over ζ (at the cost of an unimportant overall prefactor and Jacobian), and then the functional integral over b_μ does not have any delta-function constraint. We are left with a conventional path integral over a U(1) gauge field b_μ, and a boson Ψ carrying m units of U(1) charge, with the gauge-invariant action

$$\mathcal{S}_\Psi = \int d^3x \Bigg[\Psi^\dagger(x,\tau) \left(\frac{\partial}{\partial\tau} - im\, b_\tau - \frac{(\vec{\nabla} - im\,\vec{b}(\vec{x}) - i\vec{A}(\vec{x}))^2}{2M} - \mu \right) \Psi(x,\tau) + \frac{u}{2} |\Psi(x,\tau)|^4$$
$$+ \frac{im}{4\pi} \epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda \Bigg]. \quad (15)$$

This is the final form of the composite boson action of the quantum Hall states [3].

II. MEAN FIELD THEORY AND FLUCTUATIONS

We will now perform a simple mean field computation of the ground state (15), and show that it can yield the advertized translationally invariant ground state at generic particle number density, and with full flux penetration. We will find that such a solution exists only for particle number density commensurate with the flux quanta density. We will follow this up by an analysis of Gaussian fluctuations, and find that they indeed are fully gapped: this gap reassures us that the mean-field ansatz is a suitable starting point.

We look solutions of the saddle-point equations given by

$$\frac{\delta \mathcal{S}_\Psi}{\delta \Psi} = 0 \quad , \quad \frac{\delta \mathcal{S}_\Psi}{\delta b_\mu} = 0. \quad (16)$$

An ansatz with $b_\mu = (0, \vec{b}^0)$ and a spacetime independent $\Psi = \Psi_0$ is a solution of these equations provided

$$\Psi_0 = \sqrt{\frac{\mu}{u}}$$
$$\vec{\nabla}\times\vec{b}^0 = -\frac{1}{m}\vec{\nabla}\times\vec{A}$$
$$|\Psi_0|^2 = \frac{1}{2\pi}\vec{\nabla}\times\vec{b}^0. \quad (17)$$

So we observe that such a solution exists only when the particle density constrained by

$$n = |\Psi_0|^2 = -\frac{\nabla \times \vec{A}}{2\pi m} \equiv -\frac{B}{2\pi m}, \tag{18}$$

where B is the strength of the magnetic flux penetrating the system. In other words, the density of flux quanta (in units of h/e) is an integer, m, times the particle density; this corresponds to the filling fraction, defined in (1), $\nu = 1/m$. Recall, we noted earlier that m is constrained to be even (odd) for bosonic (fermionic) particles in the underlying theory. This mean-field solution corresponds to the Laughlin quantum Hall states [1].

We are now ready to examine fluctuations about the solution in (17). We parameterize the fluctuations by

$$\begin{aligned}
\Psi(x,\tau) &= \sqrt{\frac{\mu}{u} + \varrho(x,\tau)}\, e^{i\theta(x,\tau)} \\
\vec{b}(x,\tau) &= \vec{b}^{\,0}(x) + \vec{b}^{\,1}(x,\tau) \\
b_\tau(x,\tau) &= b^1_\tau(x,\tau)
\end{aligned} \tag{19}$$

Substituting (19) in (15), and expanding to quadratic order, we obtain the action

$$\begin{aligned}
\mathcal{S}_{\theta,b} = \int d^3x \Bigg[& i\varrho\left(\frac{\partial\theta}{\partial\tau} - m b^1_\tau\right) + \frac{u}{2}\varrho^2 + \frac{u}{4M\mu}(\vec{\nabla}\varrho)^2 + \frac{\mu}{2Mu}\left(\vec{\nabla}\theta - m\vec{b}^{\,1}\right)^2 \\
& + \frac{im}{4\pi}\epsilon_{\mu\nu\lambda} b^1_\mu \partial_\nu b^1_\lambda \Bigg].
\end{aligned} \tag{20}$$

We can simplify our analysis of the normal modes by integrating over the quadratic density fluctuations ϱ, and then we obtain

$$\mathcal{S}_{\theta,b} = \int d^3x \left[\frac{1}{2u}\left(\frac{\partial\theta}{\partial\tau} - m b^1_\tau\right)^2 + \frac{\mu}{2Mu}\left(\vec{\nabla}\theta - m\vec{b}^{\,1}\right)^2 + \frac{im}{4\pi}\epsilon_{\mu\nu\lambda} b^1_\mu \partial_\nu b^1_\lambda\right]. \tag{21}$$

If we drop the b^1_μ gauge field, this is precisely the action we obtained in Lecture 2 for the fluctuations of the Bogoliubov theory about the BEC, and in that case we obtained a gapless phonon mode. In the present situation, it is simplest to work in the gauge $\theta = 0$, and then compute the poles of the quadratic form of b^1_μ. A simple computation of the determinant of the quadratic form in (20) shows that there is only one pole at the real excitation frequency

$$\omega = \frac{2m\pi\mu}{Mu}\sqrt{1 + \frac{Mu^2}{4m^2\pi^2\mu}\,\vec{k}^2}. \tag{22}$$

This is a collective density fluctuation and, unlike the Bogoliubov theory, it is gapped in the limit of zero momentum. The answer for the Bogoliubov theory is restored by taking the limit $m \to 0$, and then (22) becomes a linearly-dispersing gapless phonon. Also, notice that the energy gap, Δ, in (22) can be written as

$$\Delta = \frac{2\pi m\mu}{Mu} = \frac{B}{M}. \tag{23}$$

It is now evident that the gap Δ is just the cyclotron frequency. The appearance of the exact value of cyclotron frequency here is a consequence of the Galilean-invariant quadratic dispersion in (8) and Kohn's theorem [4].

In the Bogoliubov theory, we interpreted the gapless phonon mode as the elementary quasiparticle excitation of the superfluid. In the present quantum Hall case, the elementary quasiparticle excitations turn out to be vortex-like, as we will show in the following section; the density mode in (22) is better interpreted as a collective composite of quasiparticle excitations.

A. Quasiparticles

We identify the quasiparticles by examining vortex-like solutions of (16). The procedure closely parallels our earlier analysis for bosons in zero magnetic field. We look for a time-independent solution of the form

$$\Psi(x) = f(|\vec{x}|)e^{i\theta(x)} \tag{24}$$

where $f(|x| \to \infty) = \Psi_0 = \sqrt{\mu/u}$. Then the usual argument for the finiteness of the energy of such a vortex solution applied to (15) yields

$$\begin{aligned} \oint d\vec{x}\cdot\vec{\nabla}\theta &= m\oint d\vec{x}\cdot\vec{b} + \oint d\vec{x}\cdot\vec{A} \\ &= m\oint d\vec{x}\cdot\vec{b}^1. \end{aligned} \tag{25}$$

So an elementary vortex, in which θ winds by $\pm 2\pi$, the total flux is $\int d^2x(\vec{\nabla}\times\vec{b}^1) = \pm 2\pi/m$. From (9), we conclude that the vortex-like excitations are *fractionally* charged, with total boson number $\pm 1/m$ [5]. These are the Laughlin quasiparticles of the quantum Hall states.

III. PARTICLE-VORTEX DUALITY

We can now work out the effective action controlling the dynamics of the quasiparticles by following a route familiar from our analysis of $\mathbb{Z}_2$ spin liquids in Lecture 6: we will simply apply the particle-to-vortex mapping to the theory of the boson Ψ in (15). In doing so, it is convenient to work with a form of (15) obtained by applying a chemical potential to the relativistic theory of bosons at integer filling in a lattice. The chemical potential breaks the relativistic invariance, and the low energy theory has the same form as (15) as the density deviates from integer filling. The main utility of embedding (15) in a relativistic theory is that it makes application of the particle-to-vortex duality immediate. While previously we were considering a gas of bosons above an empty vacuum, in the relativistic theory we describe a gas of bosons above a Mott insulator with an integer number of bosons per site; we expect that the universal low energy properties of both theories should be the same.

So we consider the theory of a boson ψ (the "relativistic" parent of Ψ) in the presence of an electromagnetic potential

$$A_\mu = (-i\mu, \vec{A}) + \widetilde{A}_\mu. \tag{26}$$

Both μ and $B = \vec{\nabla} \times \vec{A}$ can be large, and we will see the reappearance of the commensurability condition between the B and the density below. Here $\widetilde{A}^1_\mu$ represents any additional small perturbations of the quantum Hall system used to perturbatively examine its response functions. We obtain from (15) the parent relativistic theory

$$\mathcal{S}_\psi = \int d^3x \left[|(\partial_\mu - iA_\mu - imb_\mu)\psi|^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4 + \frac{im}{4\pi}\epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda \right]. \tag{27}$$

(In the low energy limit, (27) reverts to (15) after dropping the second-order temporal derivative, rescaling ψ to Ψ, and shifting the zero of the chemical potential from the "Mott-lobe-tip" at integer filling to the particle vacuum.) Now we can directly apply the particle-to-vortex of Lecture 5 to (27), while treating the combination $A_\mu + mb_\mu$ as an external field. In this manner, we obtain a theory for a complex vortex field, ϕ, and a new gauge field

$$\begin{aligned}\mathcal{S}_\phi = \int d^3x \Big[& |(\partial_\mu - ia_\mu)\phi|^2 + s|\phi|^2 + \frac{v}{2}|\phi|^4 + \frac{1}{2K}\left(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda\right)^2 \\ & + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda + \frac{im}{2\pi}\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu b_\lambda + \frac{im}{4\pi}\epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda \Big]\end{aligned} \tag{28}$$

As in Lecture 5, the flux of the a_μ field is the particle number current, J^ψ_μ of the boson ψ:

$$J^\psi_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda. \tag{29}$$

Finally, we can integrate out the b_μ gauge field (after introducing a suitable Maxwell term to regulate high energies, as in Lecture 5) to obtain

$$\begin{aligned}\mathcal{S}_\phi = \int d^3x \Big[& |(\partial_\mu - ia_\mu)\phi|^2 + s|\phi|^2 + \frac{v}{2}|\phi|^4 + \frac{1}{2K}\left(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda\right)^2 \\ & + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda + \frac{im}{4\pi}\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \Big].\end{aligned} \tag{30}$$

This is our effective theory of the gapped quantum Hall states in nearly its final form [6]. The last two terms describe the topological field theory in terms of an emergent U(1) gauge field a_μ with a Chern-Simons term and the external electromagnetic field A_μ. The remaining terms describe gapped quasiparticles created by the complex field ϕ.

An important property of (30) is the equation of motion obtained by varying $\mathcal{S}_\phi$ with respect to a_μ; this yields (we are ignoring the Maxwell term in a_μ here, as it does modify the long distance considerations)

$$J^\phi_\mu = \frac{m}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda + \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda, \tag{31}$$

where J^ϕ_μ is the current of the vortex excitations. We are interested here in a gapped state in which there is no density, J^ϕ_τ, of the vortex-like ϕ excitations: a non-zero value of J^ϕ_τ would correspond to a Wigner crystal with modulations in the density of the underlying particles, and that is not the class of translationally invariant state we are describing. So with $J^\phi_\tau = 0$, (29) and (31) imply that for the ground state we should choose $a_\mu = a^0_\mu$ so that the particle number density

$$n \equiv \left\langle J^\psi_\tau \right\rangle = \frac{1}{2\pi}(\vec{\nabla} \times \vec{a}^{\,0}) = -\frac{1}{2\pi m}(\vec{\nabla} \times \vec{A}). \tag{32}$$

This is precisely the commensurability condition in (1) and (18) between the particle density and the magnetic flux.

We can now expand about the saddle point in (32) by defining

$$a_\mu = a^0_\mu + \widetilde{a}_\mu. \tag{33}$$

Then (30) becomes

$$\mathcal{S}_\phi = \int d^3x \left[|(\partial_\mu - i a^0_\mu + \widetilde{a}_\mu)\phi|^2 + s|\phi|^2 + \frac{v}{2}|\phi|^4 + \frac{1}{2K}\left(\epsilon_{\mu\nu\lambda}\partial_\nu \widetilde{a}_\lambda\right)^2 \right. \\ \left. + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}\widetilde{a}_\mu \partial_\nu \widetilde{A}_\lambda + \frac{im}{4\pi}\epsilon_{\mu\nu\lambda}\widetilde{a}_\mu\partial_\nu\widetilde{a}_\lambda \right]. \tag{34}$$

This is the final form of the theory. Note that the external electromagnetic field term does not contain the background magnetic field, only the additional perturbation $\widetilde{A}_\mu$ needed to determine response functions. However, note that the ϕ excitations move in a background flux specified by a^0_μ. This accounts for the Berry phase accumulated by these excitations from the underlying particles in the ground state. This flux is analogous to that described in Lecture 7 in the motion of single vison states.

The theory in (34) is of the canonical form for topological phases introduced earlier in our considerations in Lecture 6 of $\mathbb{Z}_2$ spin liquids. It corresponds to a 1×1 K matrix with $K = m$, and a unit charge vector $t = 1$. From the application of the results obtained there, we conclude that the quasiparticles have charge $\pm 1/m$ (as obtained earlier above), obey fractional self-statistics with angle $\theta = \pi/m$ (so they are anyons), and the Hall conductivity of the ground state is $1/(2\pi m)$ (in units of $e^2/\hbar$).

- For $m \neq 1$, there are non-trivial excitations in the bulk, and these are fractional quantum Hall states.
- For $m = 1$, the elementary bulk excitations are fermionic: then this is the integer quantum Hall state.

IV. CHIRAL SPIN LIQUIDS

As we discussed in Lecture 6, quantum antiferromagnets can also be interpreted as boson systems via the Holstein-Primakoff transformation. This mapping also applies to the problem of bosons in a magnetic field considered here: the magnetic field maps onto mutli-spin interaction terms which break time-reversal symmetry. More interestingly, it was proposed by Laughlin and Kalmeyer [2] that an effective magnetic field could also arise by spontaneous breaking of time-reversal symmetry, even in models with only simple exchange interactions.

We consider the case of a $S = 1/2$ antiferromagnet on the triangular lattice with nearest neighbor exchange (although in this specific model, we know from numerical studies that a chiral spin liquid is not the ground state). The Hamiltonian in the spin and boson formulation is

$$\begin{aligned} H &= J \sum_{\langle ij \rangle} \left(\hat{S}_{ix}\hat{S}_{jx} + \hat{S}_{iy}\hat{S}_{jy} \right) + J' \sum_{\langle ij \rangle} \hat{S}_{iz}\hat{S}_{jz} \\ &= J \sum_{\langle ij \rangle} \left(b_i^\dagger b_j + b_j^\dagger b_i \right) + J' \sum_{\langle ij \rangle} \left(\frac{1}{2} - b_i^\dagger b_i \right) \left(\frac{1}{2} - b_j^\dagger b_j \right), \end{aligned} \tag{35}$$

where the b_i are hard-core bosons. We are interested in states with spin zero, and so the boson density is $\langle b^\dagger b \rangle = 1/2$. Now notice that the boson hopping term in (35) has a positive sign, and so the bosons will not prefer to sit at zero momentum. On the square lattice, this problem can be circumvented by changing the sign of the boson operator on one sublattice. But no such transformation is possible on the triangular lattice. Instead, it was proposed [2] that we should view the bosons as hopping with amplitude $-J$ between nearest-neighbor sites, along with an additional π-flux applied to each triangle. As the triangular lattice has 2 triangles for each site, there is a net 2π flux per site. And as the boson density per site is $1/2$, we are reduced to a model of bosons moving on the triangular lattice in the presence of 4π flux density per boson. Hence the bosons could form a quantum Hall state with $m = 2$. This is the chiral spin liquid state of Ref. 2. There is recent numerical evidence [8] for time-reversal symmetry breaking leading to a chiral spin liquid in certain $S = 1/2$ spin models on the kagome lattice.

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[1] R. B. Laughlin, "Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations," Phys. Rev. Lett. **50**, 1395 (1983).

[2] V. Kalmeyer and R. B. Laughlin, "Equivalence of the resonating-valence-bond and fractional quantum Hall states," Phys. Rev. Lett. **59**, 2095 (1987).

[3] S. C. Zhang, T. H. Hansson, and S. Kivelson, "Effective-Field-Theory Model for the Fractional Quantum Hall Effect," Phys. Rev. Lett. **62**, 82 (1989).

[4] W. Kohn, "Cyclotron Resonance and de Haas-van Alphen Oscillations of an Interacting Electron Gas," Phys. Rev. **123**, 1242 (1961).

[5] S. M. Girvin, "Collective Excitations," in *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, 1986) pp. 353–380.

[6] X.-G. Wen, "Topological orders and edge excitations in fractional quantum Hall states," Advances in Physics **44**, 405 (1995), cond-mat/9506066.

[7] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, "Symmetry protected topological orders and the group cohomology of their symmetry group," Phys. Rev. B **87**, 155114 (2013), arXiv:1106.4772 [cond-mat.str-el].

[8] Y.-C. He, D. N. Sheng, and Y. Chen, "Chiral Spin Liquid in a Frustrated Anisotropic Kagome Heisenberg Model," Phys. Rev. Lett. **112**, 137202 (2014), arXiv:1312.3461 [cond-mat.str-el].