AdS/CFT correspondence

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Abstract

This review of the AdS/CFT correspondence is based upon a forthcoming article by S. Hartnoll, A. Lucas, and S. Sachdev.
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I. THE 'T HOOFT MATRIX LARGE $N$ LIMIT

The essential fact about useful matrix large $N$ limits of quantum fields theories is that there exists a set of operators $\{\mathcal{O}_i\}$ with no fluctuations to leading order at large $N$ \cite{1, 2}:

$$\langle \mathcal{O}_{i_1} \mathcal{O}_{i_2} \cdots \mathcal{O}_{i_n} \rangle = \langle \mathcal{O}_{i_1} \rangle \langle \mathcal{O}_{i_2} \rangle \cdots \langle \mathcal{O}_{i_n} \rangle.$$ \hfill (1)

Operators that obey this large $N$ factorization are classical in the large $N$ limit. One might generally hope that, for a theory with many degrees of freedom 'per site', each configuration in the path integral has a large action, and so the overall partition function is well approximated by a saddle point computation. However, the many local fields appearing in the path integral will each be fluctuating wildly. To exhibit the classical nature of the large $N$ limit one must therefore identify a set of 'collective' operators $\{\mathcal{O}_i\}$ that do not fluctuate, but instead behave classically according to (1). Even if one can identify the classical collective operators, it will typically be challenging to find the effective action for these operators whose saddle point determines their expectation values. The complexity of different large $N$ limits depends on the complexity of this effective action for the collective operators.

A simple set of examples are given by vector large $N$ limits. One has, for instance, a bosonic field $\vec{\Phi}$ with $N$ components and, crucially, interactions are restricted to be $O(N)$ symmetric. This means the interaction terms must be functions of

$$\sigma \equiv \vec{\Phi} \cdot \vec{\Phi}.$$ \hfill (2)

(Slightly more generally, they could also be functions of $\vec{\Phi} \cdot \partial_{\mu_1} \cdots \partial_{\mu_2} \vec{\Phi}$.) While the individual vector components of $\vec{\Phi}$ fluctuate about zero, $\sigma$ is a sum of the square of $N$ such fluctuating fields. We thus expect that $\sigma$ behaves classically up to fluctuations which are subleading in $N$, analogous to the central limit theorem. Indeed, by means of a Hubbard-Stratonovich transformation that introduces $\sigma$ as an auxiliary field, the action can be represented as a quadratic function of $\vec{\Phi}$ coupled to $\sigma$. Performing the path integral over $\vec{\Phi}$ exactly, one then obtains an effective action for $\sigma$ alone with an order $N$ overall prefactor (from the $N$ functional determinants arising upon integrating out $\vec{\Phi}$). This is the desired effective action for the collective field $\sigma$ that can now be treated in a saddle point approximation in the large $N$ limit, see e.g. \cite{3}.

For our purposes the vector large $N$ limits are too simple. The fact that one can obtain the effective action for $\sigma$ from a few functional determinants translates into the fact that the theory is essentially a weakly interacting theory in the large $N$ limit. As we will emphasize repeatedly throughout this review, weak interactions means long lived quasiparticles, which in turn mean essentially conventional phases of matter and conventional Boltzmann descriptions of transport.
The point of holographic condensed matter is precisely to realize inherently strongly interacting phases of matter and transport that are not built around quasiparticles. Thus, while vector large $N$ limits do admit interesting holographic duals \cite{4}, they will not be further discussed here. As we will explain below, strongly interacting large $N$ theories are necessary to connect directly with the classical dissipative dynamics of event horizons discussed in the previous section.

The ’t Hooft matrix large $N$ limit \cite{5}, in contrast to the vector large $N$ limit, has the virtue of admitting a strongly interacting saddle point description. The fields in the theory now transform in the adjoint rather than the vector representation of some large group such as $U(N)$. Thus the fields are large $N \times N$ matrices $\Phi_I$ and interactions are functions of traces of these fields

$$O_i = \text{tr}(\Phi_{I_1} \Phi_{I_2} \cdots \Phi_{I_m}) .$$

These ‘single trace’ operators are straightforwardly seen to factorize and hence become classical in the large $N$ limit \cite{1, 2}. There are vastly more classical operators of the form (3) than there were in the vector large $N$ limit case (2). Furthermore there is no obvious prescription for obtaining the effective action whose classical equations of motion will determine the values of these operators. For certain special theories in the ’t Hooft limit, this is precisely what the holographic correspondence achieves. The fact that was unanticipated in the 70s is that the effective classical description involves fields propagating on a higher dimensional curved spacetime.

In the simplest cases, the strength of interactions in the matrix large $N$ limit is controlled by the ’t Hooft coupling $\lambda$ \cite{5}. When the Lagrangian is itself a single trace operator then this coupling appears in the overall normalization as, schematically,

$$\mathcal{L} = \frac{N}{\lambda} \text{tr}\left( \partial^\mu \Phi \partial_\mu \Phi + \cdots \right) .$$

When $\lambda$ is small, the large $N$ limit remains weakly coupled and can be treated perturbatively. In this limit the theory is similar to the vector large $N$ limit. When $\lambda$ is large, however, solving the theory would require summing a very large class of so-called planar diagrams. Through the holographic examples to be considered below, we will see that the large $\lambda$ limit is indeed strongly coupled: there are no quasiparticles and the single trace operators, while classical, acquire significant anomalous dimensions. This review is a study of the phenomenology of these strongly interacting large $N$ theories.

II. THE ESSENTIAL DICTIONARY

A. The GKPW formula

While string theory is useful in furnishing explicit examples of holographic duality, much of the machinery of the duality is quite general and can be described using only concepts from quantum field theory (QFT) and gravity.
The basic observables that characterize the QFT are the multi-point functions of operators in the QFT. In particular, at large $N$, the basic observables are multi-point functions of the single trace operators $O_i$, described above in (3). As examples of such operators, we can keep in mind charge densities $j'$ and current densities $\bar{j}$, associated with symmetries of the theory. The multi-point functions can be obtained if we know the generating functional

$$Z_{\text{QFT}}[\{h_i(x)\}] \equiv \left\langle e^{i \sum_i \int dx h_i(x) O_i(x)} \right\rangle_{\text{QFT}}.$$ (5)

Observables in gravitating systems can be difficult to characterize, because the spacetime itself is dynamical. In the case where the spacetime has a boundary, however, observables can be defined on the boundary of the spacetime. We can, for instance, consider a Dirichlet problem in which the values of all the ‘bulk’ fields (i.e. the dynamical fields in the theory of gravity) are fixed on the boundary. The boundary itself is not dynamical, giving the observer a ‘place to stand’. We can then construct the partition function of the theory as a function of the boundary values $\{h_i(x)\}$ of all the bulk fields $\{\phi_i\}$

$$Z_{\text{Grav.}}[\{h_i(x)\}] \equiv \int_{\phi_i \rightarrow h_i} \left( \prod_i \mathcal{D}\phi_i \right) e^{iS[\phi_i]}.$$ (6)

It is a nontrivial mathematical problem to show that the Dirichlet problem for gravity is in fact well-posed, even in the classical limit (see e.g. [6] for a recent discussion). However, we will see below how in practice the boundary data determines the bulk spacetime.

Suppose that a given QFT and theory of gravity are holographically dual. The essential fact relating observables in the two dual descriptions (of the same theory) is that there must be a one-to-one correspondence between single trace operators $O_i$ in the QFT, and dynamical fields $\phi_i$ of the bulk theory. For example, a given scalar operator such as $\text{tr}(\Phi^2)$ in the QFT will be dual to a particular scalar field $\phi$ in the bulk theory. The scalar $\phi$ will have its own bulk dynamics given by the action $S$ in (6). We will be more explicit about the bulk action shortly. Having matched up bulk operators and boundary fields in this way, we can write the essential entry in the holographic dictionary, as first formulated by Gubser-Klebanov-Polyakov and Witten (GKPW) [7, 8]:

$$Z_{\text{QFT}}[\{h_i(x)\}] = Z_{\text{Grav.}}[\{h_i(x)\}].$$ (7)

That is, the generating functional of the QFT with source $h_i$ for the single trace operator $O_i$ is equal to the bulk partition function with the bulk field $\phi_i$ corresponding to $O_i$ taking boundary value $h_i$. This relationship is illustrated in figure 1 below.

The reader may have many immediate questions: How do we know which bulk field corresponds to which operator? There are a very large number of single trace operators, and so won’t the bulk description involve a very large number of fields and hence be unwieldy? Before addressing these questions, we will illustrate the dictionary (7) at work.
FIG. 1. Essential dictionary: The boundary value $h$ of a bulk field $\phi$ is a source for an operator $\mathcal{O}$ in the dual QFT.

B. Fields in AdS spacetime

The large $N$ limit is supposed to be a classical limit, so let us evaluate the bulk partition function semiclassically

$$Z_{\text{Grav.}}[\{h_i(x)\}] = e^{iS[\{\phi^*_i \rightarrow h_i\}]}.$$  

Here $S[\{\phi^*_i \rightarrow h_i\}]$ is the bulk action evaluated on a saddle point, i.e. a solution to the bulk equations of motion, $\{\phi^*_i\}$, subject to the boundary condition that $\phi^*_i \rightarrow h_i$.

We need to specify the bulk action. The most important bulk field is the metric $g_{ab}$. The QFT operator corresponding to this bulk field will be the energy momentum tensor $T^{\mu\nu}$. This means that the boundary value of the bulk metric (more precisely, the induced metric on the boundary) is a source for the energy momentum tensor in the dual QFT, which seems natural. In writing down an action for the bulk metric, we will consider an expansion in derivatives, as one usually does in effective field theory. Later in this section we describe the circumstances in which this is a correct approach. The general bulk action for the metric involving terms that are at most quadratic in derivatives of the metric is

$$S[g] = \frac{1}{2\kappa^2} \int d^{d+2}x \sqrt{-g} \left( R + \frac{d(d+1)}{L^2} \right).$$  

The first term is the Einstein-Hilbert action, with $R$ the Ricci scalar and $\kappa^2 = 8\pi G_N \propto L_p^d$ defines the Planck length in the bulk $d + 2$ dimensions (i.e. the boundary QFT has $d$ spatial dimensions). The second term is a negative cosmological constant characterized by a lengthscale $L$. The equations of motion following from this action are

$$R_{ab} = -\frac{d+1}{L^2} g_{ab}.$$  

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The simplest solution to these equations is $AdS_{d+2}$:

$$ds^2 = L^2 \left( -dt^2 + \frac{d\vec{x}_d^2 + dr^2}{r^2} \right). \tag{11}$$

This spacetime has multiple symmetries. The easiest to see is the dilatation symmetry: $\{r, t, \vec{x}\} \to \lambda \{r, t, \vec{x}\}$. In fact there is an $SO(d+1, 2)$ isometry group [9]. This is precisely the conformal group in $d+1$ spacetime dimensions. The solution (11) describes the vacuum of the dual QFT. If we now perturb the solution by turning on sources $\{h_i\}$, these perturbations will need to transform under the $SO(d+1, 2)$ symmetry group of the background. This suggests that the $d+2$ dimensional metric (11) dually describes a ‘boundary’ conformal field theory (CFT) in $d+1$ spacetime dimensions. Let us see this explicitly.

Perturbing the metric itself is a little complicated, so consider instead an additional field $\phi$ in the bulk with an illustrative simple action

$$S[\phi] = -\int d^{d+2}x \sqrt{-g} \left( \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 \right). \tag{12}$$

As noted above, $\phi$ will correspond to some particular scalar operator in the dual quantum field theory. For concreteness we can have in the back of our mind an operator like $\tr (\Phi^2)$. To see the effects of adding a source $h$ for this operator, we must solve the classical bulk equations of motion

$$\nabla^2 \phi = m^2 \phi, \tag{13}$$

subject to the boundary condition $\phi \to h$, and then evaluate the action on the solution as specified in (8). We have already noted that the horizon of $AdS_{d+2}$ is the surface $r \to \infty$ in (11) – this is properly called the Poincaré horizon. This is the surface of infinite redshift where $g_{tt} \to 0$. The ‘conformal boundary’ at which we impose $\phi \to h$ is the opposite limit, $r \to 0$. In the metric (11) this corresponds to an asymptotic region where the volume of constant $r$ slices of the spacetime are becoming arbitrarily large. This should be thought of as imposing boundary conditions ‘at infinity’.

The equation of motion (13) is a wave equation in the $AdS_{d+2}$ background (11). We can solve this equation by decomposing the field into plane waves in the $t$ and $x$ directions

$$\phi = \phi(r)e^{-i\omega t + ik \cdot \vec{x}}. \tag{14}$$

The equation for the radial dependence then becomes

$$\phi'' - \frac{d}{r} \phi' + \left( \omega^2 - k^2 - \frac{(mL)^2}{r^2} \right) \phi = 0. \tag{15}$$

To understand the asymptotic boundary conditions, we should solve this equation in a series expansion as $r \to 0$. This is easily seen to take the form[10]

$$\phi = \frac{\phi^{(0)}}{L^{d/2}} r^{d+1-\Delta} + \ldots + \frac{\phi^{(1)}}{L^{d/2}} r^\Delta + \ldots \quad (\text{as} \quad r \to 0), \tag{16}$$
where
\[ \Delta(\Delta - d - 1) = (mL)^2. \] (17)

There are two constants of integration \( \phi_{(0)} \) and \( \phi_{(1)} \). The first of these is what we previously called the boundary value \( h \) of \( \phi \). We shall understand the meaning of the remaining constant \( \phi_{(1)} \) shortly. In (16) we see that in order to extract the boundary value \( h \) of the field, one must first strip away some powers of the radial direction \( r \). These powers of \( r \) will now be seen to have an immediate physical content. We noted below (11) that rescaling \( \{r, t, \vec{x}\} \rightarrow \lambda \{r, t, \vec{x}\} \) is a symmetry of the background \( AdS_{d+2} \) bulk spacetime. Like the metric, the scalar field \( \phi \) itself must remain invariant under this transformation (the symmetry is a property of a particular solution, not of the fields themselves). Therefore the source must scale as \( \phi_{(0)} = h \rightarrow \lambda^{\Delta - d - 1}h \) in order for (16) to be invariant. Recalling that the source \( h \) couples to the dual operator \( \mathcal{O} \) as in (5), we learn that the dual operator must rescale as
\[ \mathcal{O}(x) \rightarrow \lambda^{-\Delta} \mathcal{O}(\lambda x). \] (18)

Therefore \( \Delta \) is nothing other than the scaling dimension of the operator \( \mathcal{O} \). Equation (17) is a very important result that relates the mass of a bulk scalar field to the scaling dimension of the dual operator in a CFT. We can see that for relevant perturbations (\( \Delta < d + 1 \)), the \( \phi_{(0)} \) term in (16) goes to zero at the boundary \( r \rightarrow 0 \), while for irrelevant perturbations (\( \Delta > d + 1 \)), this term grows towards the boundary. This fits perfectly with the interpretation of the radial direction as capturing the renormalization group of the dual quantum field theory, with \( r \rightarrow 0 \) describing the UV.

For a given bulk mass, equation (17) has two solutions for \( \Delta \). For most masses we must take the larger solution \( \Delta_+ \) in order for \( \phi_{(0)} \) to be interpreted as the boundary value of the field in (16). Exceptions to this statement will be discussed later. Note that \( d + 1 - \Delta_\pm = \Delta_\mp \).

C. Simplification in the limit of strong QFT coupling

We can now address the concern above that a generic matrix large \( N \) quantum field theory has a very large number of single trace operators. For instance, with only two matrix valued fields one can construct operators of the schematic form \( \text{tr}(\Phi_1^{n_1} \Phi_2^{n_2} \Phi_1^{n_3} \Phi_2^{n_4} \cdots) \). These each correspond to a bulk field. The bulk action should therefore be extremely complicated. In theories with tractable gravity duals an unexpected simplification takes place: in the limit of large 't Hooft coupling, \( \lambda \rightarrow \infty \), all except a small number of operators acquire parametrically large anomalous dimensions. According to the relation (17) this implies that all but a small number of the bulk fields have very large masses. The heavy fields can be neglected for many purposes. This leads to a tractable bulk theory.
The lesson of the previous paragraph is that a holographic approach is useful for QFTs in which two simplifications occur:

Large $N$ limit

$\Leftrightarrow$ Classical bulk theory. \hfill (19)

Gap in single trace operator spectrum (e.g. large $\lambda$ limit)

$\Leftrightarrow$ Derivative expansion in bulk, small number of bulk fields. \hfill (20)

While the second condition may appear restrictive, string theory constructions show that these simplifications do indeed arise in concrete models. The most important point is that this is a way (a large $N$ limit with a gap in the operator spectrum) to simplify the description of QFTs without going to a weakly interacting quasiparticle regime. We will repeatedly see below how this fact allows holographic theories to capture generic behavior of, for instance, transport in strongly interacting systems that is not accessible otherwise. In contrast, weakly interacting large $N$ limits, such as the vector large $N$ limit, cannot have such a gap in the operator spectrum. This is because to leading order in large $N$ the dimensions of operators add, and in particular operators such as $\bar{\Phi} \cdot \partial_{\mu_1} \cdots \partial_{\mu_2} \Phi$ give an evenly spaced spectrum of conserved currents with no gap. Beyond the holographic approach discussed here, the development of purely quantum field theoretic methods exploiting the existence of an operator gap has recently been initiated [11–14].

D. Expectation values and Green’s functions

A second result that can be obtained from the asymptotic expansion (16) is a formula for the expectation value of an operator. From (5), (7) and (8) above

$$\langle O_i(x) \rangle = \frac{1}{Z_{\text{QFT}}} \frac{\delta Z_{\text{QFT}}}{\delta h_i(x)} = i \left. \frac{\delta}{\delta h_i(x)} S \{ \phi^i \rightarrow h_i \} \right|.$$ \hfill (21)

The scalar action (12) becomes a boundary term when evaluated on a solution. Because the volume of the boundary is diverging as $r \to 0$, we take a cutoff boundary at $r = \epsilon$. Then

$$\langle O(x) \rangle = -i \frac{\delta}{2 \delta \phi(0)} \int_{r=\epsilon} d^{d+1}x \sqrt{-\gamma} \phi n^a \nabla_a \phi$$

$$= \frac{i}{2 \delta \phi(0)} \int_{r=\epsilon} d^{d+1}x \left( \frac{L}{\epsilon} \right)^d \left( (d + 1 - \Delta) \frac{2d+1-2\Delta}{L^d} \phi^2(0) + (d+1) \frac{\phi(0) \phi(1)}{L^d} + \cdots \right).$$ \hfill (22)

In the first line $\gamma$ is the induced metric on the boundary (i.e. put $r = \epsilon$ in (11)) and $n$ is an outward pointing unit normal (i.e. $n^r = -\epsilon/L$). In the second line we have used the boundary expansion (16). Upon taking $\epsilon \to 0$, the first term diverges, the second is finite, and the remaining terms all go to zero. The divergent term is an uninteresting contact term, to be dealt with more carefully.
in the following section. We ignore it here. In evaluating the second term, note that because the bulk field $\phi$ satisfies a linear equation of motion, then $\delta \phi(1)/\delta \phi(0) = \phi(1)/\phi(0)$. Thus we obtain

$$\langle \mathcal{O}(x) \rangle \propto \phi(1).$$  \hfill (24)

We have not given the constant of proportionality here, as obtaining the correct answer requires a more careful computation, which will appear in the following section. The important point is that we have discovered the following basic relations:

Field theory source $h$

$$\Rightarrow$$ Leading behavior $\phi(0)$ of bulk field. \hfill (25)

Field theory expectation value $\langle \mathcal{O} \rangle$

$$\Rightarrow$$ Subleading behavior $\phi(1)$ of bulk field. \hfill (26)

This connection will turn out to be completely general. For instance, the source could be the chemical potential and the expectation value the charge density. Or, the source could be an electric field and the expectation value could be the electric current.

For a given set of sources $\{h_i\}$ at $r \to 0$, regularity of the fields in the interior of the spacetime (e.g. at the horizon $r \to \infty$) will fix the bulk solution completely. In particular, the subleading behavior near the boundary will be fixed. Therefore, by solving the bulk equations of motion subject to specified sources and regularity in the interior, we will be able to solve for the expectation values $\langle \mathcal{O}_i \rangle$. We can illustrate this with the simple case of the scalar field. The radial equation of motion (15) is solved by Bessel functions. The boundary condition at the horizon is that the modes of the bulk wave equation must be ‘infalling’, that is, energy flux must fall into rather than come out of the horizon. We will discuss infalling boundary conditions in detail below. For now, we quote the fact that the boundary conditions at the horizon and near the asymptotic boundary pick out the solution (let us emphasize that $t$ here is real time, we will also be discussing imaginary time later)

$$\phi \propto h \, r^{(d+1)/2} K_{\Delta - \frac{d+1}{2}} \left( r \sqrt{k^2 - \omega^2} \right) e^{-i\omega t + ik \cdot x}.$$  \hfill (27)

The overall normalization is easily computed but unnecessary and not illuminating. Here $K$ is a modified Bessel function and $\Delta$ is the larger of the two solutions to (17). By expanding this solution near the boundary $r \to 0$, we can extract the expectation value (24). Taking the ratio of the expectation value by the source gives the retarded Green’s function of the boundary theory

$$G^R_{\mathcal{O}\mathcal{O}}(\omega, k) = \frac{\langle \mathcal{O} \rangle}{\hat{h}} \propto \frac{\hat{\phi}(1)}{\hat{\phi}(0)} \propto (k^2 - \omega^2)^{\Delta - (d+1)/2}.$$  \hfill (28)

This is precisely the retarded Green’s function of an operator $\mathcal{O}$ with dimension $\Delta$ in a CFT. This confirms our expectation that perturbations about the $AdS_{d+2}$ background (11) will dually describe the excitations of a CFT.
E. Bulk gauge symmetries are global symmetries of the dual QFT

The computation of the Green’s function (28) illustrates how knowing the bulk action and the bulk background allows us to obtain correlators of operators in the dual strongly interacting theory. For this to be useful, we would like to know which bulk fields correspond to which QFT operators. This can be complicated, even when the dual pair of theories are known explicitly. Symmetry is an important guide. In particular, gauge symmetries in the bulk are described by bulk gauge fields. These include Maxwell fields $A_a$, the metric $g_{ab}$ and also nonabelian gauge bosons. The theory must be invariant under gauge transformations of these fields, including ‘large’ gauge transformations, where the generator of the transformation remains constant on the asymptotic boundary. We can illustrate this point easily with a bulk gauge field $A_a$, which transforms to $A_a \rightarrow A_a + \nabla_a \chi$ for a scalar function $\chi$. If $\chi$ is nonzero on the boundary, then the boundary coupling in the action transforms to

$$
\int d^{d+1}x \sqrt{-\gamma} (A_\mu + \nabla_\mu \chi) J^\mu = \int d^{d+1}x \sqrt{-\gamma} (A_\mu J^\mu - \chi \nabla_\mu J^\mu),
$$

where we integrated by parts in the last step. Invariance under the bulk gauge transformation requires that the current is conserved (in all correlation functions). Therefore

$$
\text{Bulk gauge field (e.g. } A_a, g_{ab}) \quad \Leftrightarrow \quad \text{Conserved current of global symmetry in QFT (e.g. } J^\mu, T^{\mu\nu}).
$$

Similarly, fields that are charged under a bulk gauge field will be dual to operators in the QFT that carry the corresponding global charge. Thus quantities such as electric charge and spin must directly match up in the two descriptions.

Matching up operators beyond their symmetries is often not possible in practice, and indeed is not really the right question to ask. The bulk is a self-contained description of the strongly coupled theory. The spectrum and interactions of bulk fields define the correlators and all other properties of a set of dual operators. Reference to a weakly interacting QFT Lagrangian description is not necessary and potentially misleading. Nonetheless, it can be comforting to have in the back of our minds some familiar operators. Thus a low mass scalar field $\phi$ in the bulk might be dual to QFT operators such as $\text{tr} F^2$, $\text{tr} \Phi^2$ or $\text{tr} \bar{\Psi} \Psi$. A low mass fermion $\psi$ in the bulk will be dual to an operator such as $\text{tr} \Phi \Psi$.

F. Bulk volume divergences and boundary counterterms

We have already encountered a divergence when we tried to evaluate the on-shell action in (23). This divergence is due to the infinite volume of the bulk spacetime, that is integrated over in evaluating the action. In (23) we regulated the divergence by cutting off the spacetime close
to the boundary at $r = \epsilon$. We will see momentarily that this cutoff appears in the dual field theory as a short distance regulator of UV divergences. This connection between infinitely large scales in the bulk and infinitesimally short scales in the field theory is sometimes called the UV/IR correspondence \[15\]. Taking our cue from perturbative renormalization in field theory, we can regulate the bulk action by adding a ‘counterterm’ boundary action. For the case of the scalar, let

$$S \rightarrow S + S_{\text{ct.}} = S + \frac{a}{2L} \int_{r=\epsilon} \, d^{d+1}x \sqrt{-\gamma} \phi^2.$$  

(31)

Choosing $a = \Delta - d - 1$, the counterterm precisely cancels the divergent term in (23). Because it is a boundary term defined in terms of purely intrinsic boundary data, it neither changes the bulk equations of motion nor the boundary conditions of the bulk field. The counterterm action furthermore contributes to the finite term in the expectation value of the dual operator: We can now write (24) more precisely as

$$\langle \mathcal{O}(x) \rangle = (2\Delta - d - 1) \phi^{(1)}.$$  

(32)

The counterterm boundary action is a crucial part of the theory if we wish the dual field theory to be well-defined in the ‘continuum’ limit $\epsilon \rightarrow 0$. A bulk action that is finite as $\epsilon \rightarrow 0$ allows us to think of the dual field theory as being defined by starting from a UV fixed point. This is the first of several instances we will encounter in holography where boundary terms in the action must be considered in more detail than one might be accustomed to.

The counterterm action (31) is a local functional of boundary data. It is a nontrivial fact that all divergences that arise in evaluating on shell actions in holography can be regularized with counterterms that are local functionals of the boundary data. The terms that arise precisely mimic the structure of UV divergences in quantum field theory in one lower dimension, even though the bulk divergences are entirely classical. This identification of the counterterm action is called holographic renormalization. Entry points into the vast literature include \[16, 17\]. An important conceptual fact is that in order for the bulk volume divergences to be local in boundary data, the growth of the volume towards the boundary must be sufficiently fast. Spacetimes that asymptote to AdS spacetime are the best understood case where such UV locality holds.

III. ENTANGLEMENT ENTROPY

Shortly after the discovery of the holographic correspondence, it was realized that the bulk can be thought of as being made up of order $N^2$ QFT degrees of freedom per AdS radius \[15\]. The recent discovery \[18–20\] and proof \[21\] of the Ryu-Takayangi formula for entanglement entropy in theories with holographic duals suggests that a more refined understanding of how the bulk reorganizes the QFT degrees of freedom is within reach. In the simplest case, in which the bulk is
described by classical Einstein gravity coupled to matter, the Ryu-Takayanagi formula states that the entanglement entropy of a region $A$ in the QFT is given by

$$S_E = \frac{A_\Gamma}{4G_N},$$

(33)

where $G_N$ is the bulk Newton’s constant and $A_\Gamma$ is the area of a minimal surface (i.e. a soap bubble) $\Gamma$ in the bulk that ends on the boundary of the region $A$. The region $A$ itself is at the boundary of the bulk spacetime. We illustrate this formula in the following figure 2. The formula (33) generalizes the Bekenstein-Hawking entropy formula for black holes. This last statement is especially clear for black holes describing thermal equilibrium states, which correspond to entangling the system with a thermofield double [22].

Let us now illustrate the Ryu-Takayanagi formula by computing the entanglement entropy of a spherical region of radius $R$ in a CFT, following [19]. By spherical symmetry, we know that the minimal surface will be of the form $r(\rho)$, where $\rho$ is the radial coordinate on the boundary (so that $d\vec{x}_d^2 = d\rho^2 + \rho^2 d\Omega_{d-1}^2$). The induced metric on this surface is given by putting $r = r(\rho)$ and $t = 0$ in the $AdS_{d+2}$ spacetime (11) to obtain:

$$ds^2 = \frac{L^2d\rho^2}{r^2} \left[ 1 + \left( \frac{dr}{d\rho} \right)^2 \right] + \frac{L^2\rho^2}{r^2} d\Omega_{d-1}^2. \quad (34)$$

From the determinant of this induced metric, the area of the surface is given by:

$$A_\Gamma = L^d \Omega_{d-1} \int \frac{d\rho}{\rho} \rho^{d-1} \sqrt{1 + \left( \frac{dr}{d\rho} \right)^2}. \quad (35)$$

$\Omega_{d-1}$ denotes the volume of the unit sphere $S^{d-1}$. It is simple to check that

$$r(\rho) = \sqrt{R^2 - \rho^2}, \quad (36)$$
solves the Euler-Lagrange equations of motion found by minimizing the area (35). This is therefore the Ryu-Takayanagi surface. Note that \( dr/d\rho = -\rho/r \).

The entanglement entropy (33) is given by evaluating (35) on the solution (36). For dimensions \( d > 1 \),

\[
A_G = L^d \Omega_{d-1} \int_0^R d\rho \frac{R^{d-1}}{(R^2 - \rho^2)^{(d+1)/2}}. \tag{37}
\]

This integral is dominated near \( r = 0 \). Switching to the variable \( y = R - \rho \), we obtain that

\[
A_G \approx L^d \Omega_{d-1} \int_{\delta}^R dy \frac{y^d}{(2Ry)^{(d+1)/2}} \sim L^d \left( \frac{R}{\delta} \right)^{(d-1)/2}, \tag{38}
\]

where \( \delta \) is a cutoff on the bulk radial dimension near the boundary that, as in the previous Wilsonian section, we will want to interpret as a short distance cutoff in the dual field theory. This result becomes more transparent in terms of the equivalent cutoff on the bulk coordinate \( r = \epsilon \); using (36) one has that \( \epsilon = \sqrt{2R\delta} \). Hence, the entanglement entropy is

\[
S_E \sim \frac{L^d}{G_N} \left( \frac{R}{\epsilon} \right)^{d-1} + \text{subleading}. \tag{39}
\]

The first term is proportional to the area of the boundary sphere \( R \), and hence (39) is called the area law of entanglement [23]. It can be intuitively understood as follows – entanglement in the vacuum is due to excitations which reside partially inside the sphere, and partially outside. In a local QFT, we expect that the number of these degrees of freedom is proportional to the area of the sphere in units of the short distance cutoff, \( (R/\epsilon)^{d-1} \). This is true regardless of whether or not the state is gapped. Equation (39) therefore gives another perspective on the identification of a near-boundary cutoff on the bulk geometry with a short-distance cutoff in the dual QFT. In the holographic result (39), the proportionality coefficient \( L^d/G_N \sim (L/L_p)^d \gg 1 \). This is consistent with the general expectation that there must be a large number of degrees of freedom “per lattice site” in microscopic QFTs with classical gravitational duals. While the coefficient of the area law is sensitive to the UV cutoff \( \epsilon \) (since we have to count the number of degrees of freedom residing near the surface), some subleading coefficients are universal [19].

In \( d = 1 \), the story is a bit different. Now,

\[
A_G = 2L \int_0^R d\rho \frac{R}{R^2 - \rho^2} \approx 2L \int_{\delta}^R d\rho \frac{R}{2\rho} = L \log \frac{R}{\delta} + \cdots = 2L \log \frac{R}{\epsilon} + \cdots. \tag{40}
\]

The prefactor of 2 is related to the fact that the semicircular minimal surface has two sides. Using the result (beyond the scope of this review) that the central charge \( c \) of the dual CFT2, which counts the effective degrees of freedom, is given by [24]

\[
c = \frac{3L}{2G_N}, \tag{41}
\]
we recover the universal CFT2 result [25]

\[ S = \frac{c}{3} \log \frac{R}{\epsilon} + \cdots. \]  

(42)

Note that the area law, which would have predicted \( S = \text{constant} \), has logarithmic violations in a CFT2. We will see an intuitive argument for this shortly, as shown in Figure 3.

A. Analogy with tensor networks

It was pointed out in [26] that the Ryu-Takayanagi formula resembles the computation of the entanglement entropy in quantum states described by tensor networks. We will now describe this connection, following the exposition of [27]. While, at the time of writing, these ideas remain to be fleshed out in any technical detail, they offer a useful way to think about the emergent spatial dimension in holography. For a succinct introduction to the importance of entanglement in real space renormalization, see [28].

Consider a system with degrees of freedom living on lattice sites labelled by \( i \), taking possible values \( s_i \). For instance each \( s_i \) could be the value of a spin. Tensor network states (see e.g. [29]) are certain wavefunctions \( \psi(\{s_i\}) \). The simplest example of a tensor network state is a Matrix Product State (MPS) for a one dimensional system with \( L \) sites. For every value that the degrees of freedom \( s_i \) can take, construct a \( D \times D \) matrix \( T_{s_i} \). Here \( D \) is called the bond dimension. The physical wavefunction is now given by

\[ \psi(s_1, \ldots, s_L) = \text{tr}(T_{s_1}T_{s_2} \cdots T_{s_L}). \]  

(43)

For the case that each \( s_i \) describes a spin half, these Matrix Product States parametrize a \( 2LD^2 \) dimensional subspace of the full \( 2^L \) dimensional Hilbert space. The importance of these states is that they capture the entanglement structure of the ground state of gapped systems, as we now recall.

To every site \( i \) in the lattice, associate two auxiliary vector spaces of dimension \( D \). Let these have basis vectors \( |n\rangle_{i1} \) and \( |n\rangle_{i2} \) respectively, with \( n = 1, \ldots, D \). Now form a state, in this auxiliary space, made of maximally entangled pairs between neighboring sites

\[ |\Psi\rangle = \prod_{i=1}^L \sum_{n=1}^D |n\rangle_{i2}|n\rangle_{(i+1)1}. \]  

(44)

For periodic boundary conditions we can set \( |n\rangle_{(L+1)1} = |n\rangle_{11} \). It is clear that if we trace over some number of adjacent sites, the entanglement entropy of the reduced density matrix will be \( 2 \log D \), corresponding to maximally entangled pairs at each end of the interval we have traced over. To obtain a state in the physical Hilbert space, we now need to project at each site. That
is, let $P_i : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}^2$ be a projection, then the physical state

$$|\psi\rangle = \prod_{i=1}^{L} P_i \Psi.$$  \hspace{1cm} (45)

If we write the projection operators as $P_i = \sum_{m,n,s} T_{s_i}^{mn} |s_i\rangle \langle i | m \langle n |$, then we recover precisely the MPS state (43). The advantage of this perspective is that we now see that if we trace out a number of adjacent sites in this physical state $\psi$, the entanglement entropy of the reduced density matrix is bounded above by $S_E \leq 2 \log D$. This follows from the fact that, focusing on a single cut between two sites for simplicity, the state (44) in the auxiliary space can be written $|\Psi\rangle = \sum_{p=1}^{D} |p\rangle_L |p\rangle_R$ for some set of states $|p\rangle_L$ and $|p\rangle_R$ to the left and right of the cut, respectively. The projectors act independently on the left and right states, and necessarily map each $D$ dimensional subspace onto a vector space of dimension $D$ or lower. Schmidt decomposing the projected state, we have $|\psi\rangle = \sum_{p=1}^{D} c_p |\tilde{p}\rangle_L |\tilde{p}\rangle_R$, for coefficients $c_p$ and states $|\tilde{p}\rangle_L/R$ in the physical Hilbert space on either side of the cut. Hence the entanglement entropy is bounded above by $\log D$. In gapped systems, the entanglement entropy of the ground state grows with the correlation length. At a fixed correlation length, the entanglement structure of the state can therefore be well approximated by a MPS state with some fixed bond dimension $D$ (if the inequality $S_E \leq 2 \log D$ is approximately saturated).

For gapless systems such as CFTs, a more complicated structure of tensor contractions is necessary to capture the large amount of entanglement. A structure that realizes a discrete subgroup of the conformal group is the so-called MERA network, illustrated in figure 3 below. Viewing the network as obtained by projecting maximally entangled pairs, the entanglement bound discussed above for MPS states generalizes to the statement that the entanglement entropy of a sequence of adjacent sites is bounded above $\ell \log D$, where $\ell$ is the number of links in the network that must be cut to separate the entangled sites from the rest of the network. The strongest bound is found from the minimal number of such links that can be cut. This effectively defines a `minimal surface’ within the network. As with the minimal surface appearing in the Ryu-Takayanagi formula, the entanglement entropy is proportional (if the bound is saturated) to the length of the surface. Such a surface in a MERA network is illustrated in figure 3.

The similarity between the tensor network computation of the entanglement entropy and the Ryu-Takayanagi formula suggests that the extended tensor network needed to capture the entanglement of gapless states can be thought of as an emergent geometry, analogous to that arising in holography [26]. From this perspective, the emergent radial direction is a consequence of the large amount of entanglement in the QFT ground state, and the tensor network describing the highly entangled state is the skeleton of the bulk geometry. Further discussion may be found in [26, 30, 31]. A key challenge for future work is to show how the large $N$ limit can flesh out this skeleton to provide a local geometry at scales below the AdS radius.
FIG. 3. MERA network and a network geodesic that cuts a minimal number of links allowing it to enclose a region of the physical lattice at the top of the network. Each line between two points can be thought of as a maximally entangled pair in the auxiliary Hilbert space, while the points themselves correspond to projections that glue the pairs together. At the top of the network, the projectors map the auxiliary Hilbert space into the physical Hilbert space. The entanglement entropy of the physical region is bounded by the number of links cut by the geodesic: $S_E \leq \ell \log D$.

IV. CONDENSED MATTER SYSTEMS AT INTEGER DENSITY

The simplest examples of systems without quasiparticle excitations are realized in quantum matter at special densities which are commensurate with an underlying lattice \cite{32}. These are usually modeled by quantum field theories which are particle-hole symmetric, and have a vanishing value of a conserved U(1) charge linked to the particle number of the lattice model — hence ‘zero’ density. Even in the lattice systems, it is conventional to measure the particle density in terms of deviations from such a commensurate density.

The simplest example of a system modeled by a zero density quantum field theory is graphene. This QFT is furthermore relativistic at low energies.\cite{33} Graphene consists of a honeycomb lattice of carbon atoms, and the important electronic excitations all reside on a single $\pi$-orbital on each atom. ‘Zero’ density corresponds here to a density of one electron per atom. A simple computation of the band structure of free electrons in such a configuration yields an electronic dispersion identical to massless 2-component Dirac fermions, $\Psi$, which have a ‘flavor’ index extending over 4 values. The Dirac ‘spin’ index is actually a sublattice index, while the flavor indices correspond to the physical $S = 1/2$ spin and a ‘valley’ index corresponding to 2 Dirac cones at different points in the Brillouin zone. The density of electrons can be varied by applying a chemical potential, $\mu$, by external gates, so that the dispersion is $\epsilon(k) = v_F|k| - \mu$, where $v_F$ is the Fermi velocity \cite{34}.
FIG. 4. Ground states of bosons on a square lattice with tunneling amplitude between the sites $w$, and on-site repulsive interactions $U$. Bosons are condensed in the zero momentum state in the superfluid, and so there are large number fluctuations in a typical component of the wavefunction shown above. The Mott insulator is dominated by a configuration with exactly one particle on each site.

Electronic states with $\epsilon(k) < 0$ are occupied at $T = 0$, and zero density corresponds to $\mu = 0$.

At first glance, it would appear that graphene is not a good candidate for holographic study. The interactions between the electrons are instantaneous Coulomb interactions, because the velocity of light $c \gg v_F$. These interactions are then seen to be formally irrelevant at low energies under the renormalization group, so that the electronic quasiparticles are well-defined. However, as we will briefly note later, the interactions become weak only logarithmically slowly [35, 36], and there is a regime with $T \gtrsim \mu$ where graphene can be modeled as a dissipative quantum plasma. Thus, insight from holographic models can prove quite useful. Experimental evidence for this quantum plasma has emerged recently [37], as we will discuss further in this review. More general electronic models with short-range interactions on the honeycomb lattice provide examples of quantum critical points without quasiparticle excitations even at $T = 0$.

For a more clear-cut realization of quantum matter without quasiparticles we turn to a system of ultracold atoms in optical lattices. Consider a collection of bosonic atoms (such as $^{87}\text{Rb}$) placed in a periodic potential, with a density such that there is precisely one atom for each minimum of the periodic potential. The dynamics of the atoms can be described by the boson Hubbard model: this is a lattice model describing the interplay of boson tunneling between the minima of the potential (with amplitude $w$), and the repulsion between two bosons on the same site (of energy $U$). For small $U/w$, we can treat the bosons as nearly free, and so at low temperatures $T$ they condense into a superfluid with macroscopic occupation of the zero momentum single-particle state, as shown in Fig. 4. In the opposite limit of large $U/w$, the repulsion between the bosons localizes them into a Mott insulator: this is a state adiabatically connected to the product state with one boson in each potential minimum. We are interested here in the quantum phase transition
between these states which occurs at a critical value of $U/w$. Remarkably, it turns out that the low energy physics in the vicinity of this critical point is described by a quantum field theory with an emergent Lorentz invariant structure, but with a velocity of 'light', $c$, given by the velocity of second sound in the superfluid [32, 38]. In the terms of the long-wavelength boson annihilation operator $\psi$, the Euclidean time ($\tau$) action for the field theory is

$$
S = \int d^2x \, d\tau \left[ \nabla_x \psi |^2 + c^2 |\partial_\tau \psi |^2 + s |\psi|^2 + u |\psi|^4 \right].
$$

Here $s$ is the coupling which tunes the system across the quantum phase transition at some $s = s_c$. For $s > s_c$, the field theory has a mass gap and no symmetry is broken: this corresponds to the Mott insulator. The gapped particle and anti-particle states associated with the field operator $\psi$ correspond to the ‘particle’ and ‘hole’ excitations of the Mott insulator. These can be used as a starting point for a quasiparticle theory of the dynamics of the Mott insulator via the Boltzmann equation. The other phase with $s < s_c$ corresponds to the superfluid: here the global U(1) symmetry of $S$ is broken, and the massless Goldstone modes correspond to the second sound excitations. Again, these Goldstone excitations are well-defined quasiparticles, and a quasiclassical theory of the superfluid phase is possible (and is discussed in classic condensed matter texts).

Our primary interest here is in the special critical point $s = s_c$, which provides the first example of a many body quantum state without quasiparticle excitations. And as we shall discuss momentarily, while this is only an isolated point at $T = 0$, the influence of the non-quasiparticle description widens to a finite range of couplings at non-zero temperatures. Renormalization group analyses show that the $s = s_c$ critical point is described by a fixed point where $u \rightarrow u^*$, known as the Wilson-Fisher fixed point [32, 39]. The value of $u^*$ is small in a vector large $M$ expansion in which $\psi$ is generalized to an $M$-component vector, or for small $\epsilon$ in a field theory generalized to $d = 3 - \epsilon$ spatial dimensions. Given only the assumption of scale invariance at such a fixed point, the two-point $\psi$ Green’s function obeys

$$
G^R_{\psi\psi}(\omega, k) = k^{-2+\eta} F \left( \frac{\omega}{k^2} \right)
$$

at the quantum critical point. Here $\eta$ is defined to be the anomalous dimension of the field $\psi$ which has scaling dimension

$$
[\psi] = (d + z - 2 + \eta)/2
$$

in $d$ spatial dimensions. The exponent $z$ determines the relative scalings of time and space, and is known as the dynamic critical exponent. The function $F$ is a scaling function which depends upon the particular critical point under study. In the present situation, we know from the structure of the underlying field theory in Eq. (46) that $G^R_{\psi\psi}$ has to be Lorentz invariant ($\psi$ is a Lorentz scalar), and so we must have $z = 1$ and a function $F$ consistent with

$$
G^R_{\psi\psi}(\omega, k) \sim \frac{1}{(c^2k^2 - \omega^2)^{(2-\eta)/2}}.
$$

(49)
Indeed, the Wilson-Fisher fixed point is not only Lorentz and scale invariant, it is also conformally invariant and realizes a conformal field theory (CFT), a property we will exploit below.

A crucial feature of $G^R$ in Eq. (49) is that for $\eta \neq 0$ its imaginary part does not have any poles in the $\omega$ complex plane, only branch cuts at $\omega = \pm ck$. This is an indication of the absence of quasiparticles at the quantum critical point. However, it is not a proof of absence, because there could be quasiparticles which have zero overlap with the state created by the $\psi$ operator, and which appear only in suitably defined observables. For the case of the Wilson-Fisher fixed point, no such observable has ever been found, and it seems highly unlikely that such an ‘integrable’ structure is present in this strongly-coupled theory. Certainly, the absence of quasiparticles can be established at all orders in the $\epsilon$ or vector $1/M$ expansions. In the remaining discussion we will assume that no quasiparticle excitations are present, and develop a framework to describe the physical properties of such systems. We also note in passing that CFTs in 1+1 spacetime dimensions (i.e. CFT2s) are examples of theories in which there are in fact quasiparticles present, but whose presence is not evident in the correlators of most field variables [40]. This is a consequence of integrability properties in CFT2s which do not generalize to higher dimensions.

We can also use a scaling framework to address the ground state properties away from the quantum critical point. In the field theory $S$ we tune away from quantum critical point by varying the value of $s$ away from $s_c$. This makes $s - s_c$ a relevant perturbation at the fixed point, and its scaling dimension is denoted as

$$[s - s_c] = 1/\nu.$$

In other words, the influence of the deviation away from the criticality becomes manifest at distances larger than a correlation length, $\xi$, which diverges as

$$\xi \sim |s - s_c|^{-\nu}.$$

We can also express these scaling results in terms of the dimension of the operator conjugate to $s - s_c$ in the action

$$[|\psi|^2] = d + z - \frac{1}{\nu}.$$

For $s > s_c$, the influence of the relevant perturbation to the quantum critical point is particularly simple: the theory acquires a mass gap $\sim c/\xi$ and so

$$G^R_{\psi\psi}(\omega, k) \sim \frac{1}{(c^2(k^2 + \xi^{-2}) - \omega^2)}.$$

at frequencies just above the mass gap. Note now that a pole has emerged in $G^R$, confirming that quasiparticles are present in the $s > s_c$ Mott insulator.

We now also mention the influence of a non-zero temperature, $T$, on the quantum critical point, and we will have much more to say about this later. In the imaginary time path integral for the field theory, $T$ appears only in boundary conditions for the temporal direction, $\tau$: bosonic (fermionic)
FIG. 5. Phase diagram of $S$ as a function of $s$ and $T$. The quantum critical region is bounded by crossovers at $T \sim |s - s_c|^{2\nu}$ indicated by the dashed lines. Conventional quasiparticle or classical-wave dynamics applies in the non-quantum-critical regimes (colored blue) including a Kosterlitz-Thouless phase transition above which the superfluid density is zero. One of our aims in this review is to develop a theory of the non-quasiparticle dynamics within the quantum critical region.

fields are periodic (anti-periodic) with period $\hbar/(k_B T)$. At the fixed point, this immediately implies that $T$ should scale as a frequency. So we can generalize the scaling form (47) to include a finite $\xi$ and a non-zero $T$ by

$$G_{\psi\psi}^R(\omega, k) = \frac{1}{T^{(2-\eta)/z}} F \left( \frac{k}{T^{1/\nu}}, \frac{\xi^{-1}}{T^{1/\nu}}, \frac{\hbar \omega}{k_B T} \right). \tag{54}$$

We have included fundamental constants in the last argument of the scaling function $F$ because we expect the dependence on $\hbar \omega/(k_B T)$ to be fully determined by the fixed-point theory, with no arbitrary scale factors. Note also that for $T > \xi^{-z} \sim |g - g_c|^{2\nu}$, we can safely set the second argument of $F$ to zero. So in this “quantum critical” regime, the finite temperature dynamics is described by the non-quasiparticle dynamics of the fixed point CFT3: see Fig. 5. In contrast, for $T < \xi^{-z} \sim |g - g_c|^{2\nu}$ we have the traditional quasiparticle dynamics associated with excitations similar to those described by (53). Our discussion in the next few sections will focus on the quantum-critical region of Fig. 5.
V. SCALE INVARIANT GEOMETRIES

We have already seen in §II above that the simplest solution to the most minimal bulk theory (9) is the $AdS_{d+2}$ spacetime

$$ds^2 = L^2 \left( -\frac{dt^2 + d\vec{x}_a^2}{r^2} + dr^2 \right).$$

(55)

We noted that this geometry has the $SO(d + 1, 2)$ symmetry of a $d + 1$ (spacetime) dimensional CFT. We verified that a scalar field $\phi$ in this spacetime was dual to an operator $O$ with a certain scaling dimension $d$ determined by the mass of the scalar field via (17). We furthermore verified that the retarded Green’s function (28) of $O$, as computed holographically, was indeed that of an operator of dimension $\Delta$ in a CFT. More generally, the dynamics of perturbations about the bulk background (55) gives a holographic description of zero temperature correlators in a dual CFT.

A. Dynamical critical exponent $z > 1$

A CFT has dynamic critical exponent $z = 1$. This section explains the holographic description of general quantum critical systems with $z$ not necessarily equal to 1. The first requirement is a background that geometrically realizes the more general scaling symmetry: $\{t, \vec{x}\} \rightarrow \{\lambda^z t, \lambda \vec{x}\}$. The geometry

$$ds^2 = L^2 \left( -\frac{dt^2 + d\vec{x}_a^2 + dr^2}{r^2} \right),$$

(56)

does the trick [41]. These are called ‘Lifshitz’ geometries because the case $z = 2$ is dual to a quantum critical theory with the same symmetries as the multicritical Lifshitz theory. These backgrounds do not have additional continuous symmetries beyond the scaling symmetry, spacetime translations and spatial rotations. They are invariant under time reversal ($t \rightarrow -t$). We will shortly see that correlation functions computed from these backgrounds indeed have the expected scaling form (47). Firstly, though, we describe how the spacetime (56) can arise in a gravitational theory.

All Lifshitz metrics have constant curvature invariants. It can be proven that the only effects of quantum corrections in the bulk is to renormalize the values of $z$ and $L$ [42]. In this sense the scaling symmetry is robust beyond the large $N$ limit. However, for $z \neq 1$ these metrics suffer from so-called pp singularities at the ‘horizon’ as $r \rightarrow \infty$. Infalling observers experience infinite tidal forces. See [43] for more discussion. Such null singularities are likely to be acceptable within a string theory framework, see e.g. [44, 45]. So far, no pathologies associated with these singularities has arisen in classical gravity computations of correlators and thermodynamics.

Unlike Anti-de Sitter spacetime, the Lifshitz geometries are not solutions to pure gravity. Therefore, a slightly more complicated bulk theory is needed to describe them. Two simple theories that have Lifshitz geometries (56) as solutions are Einstein-Maxwell-dilaton theory [46, 47] and
Einstein-Proca theory [46]. The Einstein-Proca theory couples gravity to a massive vector field:

$$S[g, A] = \frac{1}{2\kappa^2} \int d^{d+2}x \sqrt{-g} \left( R + \Lambda - \frac{1}{4} F^2 - \frac{m^2_A}{2} A^2 \right),$$

with

$$\Lambda = \frac{(d+1)^2 + (d+1)(z-2) + (z-1)^2}{L^2}, \quad m^2_A = \frac{z d}{L^2}. \quad (58)$$

It is easily verified that the equations of motion following from this action have (56) as a solution, together with the massive vector

$$A = \sqrt{\frac{2(z-1)}{z}} \frac{L}{r^z} dt. \quad (59)$$

This is only a real solution for $z \geq 1$. We will discuss physically allowed values of $z$ later. The Einstein-Maxwell-dilaton theory

$$S[g, A, \phi] = \frac{1}{2\kappa^2} \int d^{d+2}x \sqrt{-g} \left( R + \Lambda - \frac{1}{4} e^{2\alpha\phi} F^2 - \frac{1}{2} (\nabla \phi)^2 \right),$$

with

$$\Lambda = \frac{(z + d)(z + d - 1)}{L^2}, \quad \alpha = \sqrt{\frac{d}{2(z-1)}}, \quad (61)$$

has the Lifshitz metric (56) as a solution, together with the dilaton and Maxwell fields

$$A = \sqrt{\frac{2(z-1)}{z + d}} \frac{L}{r^{z+d}} dt, \quad e^{2\alpha\phi} = r^{2d}. \quad (62)$$

Again, these solutions make sense for $z > 1$. In these solutions the scalar field $\phi$ is not scale invariant. We will see below that these kinds of running scalars can lead to interesting anomalous scaling dimensions. Because the Einstein-Maxwell-dilaton theory involves a nonzero conserved flux for a bulk gauge field (dual to a charge density in the QFT), it is most naturally interpreted as a finite density solution. There is a large literature constructing Lifshitz spacetimes from consistent truncations of string theory, starting with [48, 49]. In our later discussion of compressible matter, we will describe some further ways to obtain the scaling geometry (56).

Repeating the analysis we did for the CFT case in §II, we can consider the two point function of a scalar operator in these theories. As before, consider a scalar field $\phi$ (not the dilaton above) that satisfies the Klein-Gordon equation (13), but now in the Lifshitz geometry (56). Once again writing $\phi = \phi(r)e^{-i\omega t + ik \cdot x}$, the radially dependent part must now satisfy

$$\phi'' - \frac{z + d - 1}{r} \phi' + \left( \omega^2 r^{2z-2} - k^2 - \frac{(mL)^2}{r^2} \right) \phi = 0. \quad (63)$$

This equation cannot be solved explicitly in general. Expanding into the UV region, as $r \to 0$, we find that the asymptotic behavior of (16) generalizes to

$$\phi = \phi_0 \left( \frac{r}{L} \right)^{D_{ext} - \Delta} + \cdots + \phi_1 \left( \frac{r}{L} \right)^{\Delta} + \cdots \quad (as \quad r \to 0) \quad (64)$$
We introduced the effective number of spacetime dimensions

\[ D_{\text{eff.}} = z + d, \]

because these theories can be thought of as having \( z \) rather than 1 time dimensions, and the (momentum) scaling dimension \( \Delta \) now satisfies

\[ \Delta(\Delta - D_{\text{eff.}}) = (mL)^2. \]

As in our discussion of AdS in §II, \( \phi(0) \) is dual to a source for the dual operator \( \mathcal{O} \), whereas \( \phi(1) \) is the expectation value. By rescaling the radial coordinate of the equation (63), setting \( \rho = r\omega^{1/z} \), and then using the asymptotic form (64), we can conclude that the retarded Green’s function has the scaling form required for an operator of dimension \( \Delta \) in a scale invariant theory with dynamical critical exponent \( z \) [46]:

\[ G^{R}_{\mathcal{O}\mathcal{O}}(\omega, k) = \frac{2\Delta - D_{\text{eff.}}}{L} \frac{\phi(1)}{\phi(0)} = \omega^{(2\Delta - D_{\text{eff.}})/z} F \left( \frac{\omega}{k^z} \right). \]

The scaling function \( F(x) \) appearing in (67) will depend on the theory, it is not constrained by symmetry when \( z \neq 1 \). Remarkably, a certain asymptotic property of this function has been shown to hold both in strongly coupled holographic models and also in general weakly interacting theories. Specifically, consider the spectral weight (the imaginary part of the retarded Green’s function) at low energies (\( \omega \to 0 \)), but with the momentum \( k \) held fixed. We expect this quantity to be vanishingly small, because the scaling of on-shell states as \( \omega \sim k^z \) implies that there are no low energy excitations with finite \( k \) [50, 51]. The precise holographic result is obtained from a straightforward WKB solution to the differential equation (63), in which \( k^z/\omega \) is the large WKB parameter. For the interested reader, we will go through some explicit WKB derivations of holographic Green’s functions (very similar to the computations that pertain here) in later parts of the review. The essential point is that if the wave equation (63) is written in the form of a Schrödinger equation, then the imaginary part of the dual QFT Green’s function is given by the probability for the field to tunnel from the boundary of the spacetime through to the horizon. Here we can simply quote the result [52]

\[ \text{Im} \ G^{R}_{\mathcal{O}\mathcal{O}}(\omega, k) \sim e^{-A(k^z/\omega)^{\frac{1}{z-1}}}, \quad (\omega \to 0, \ k \text{ fixed}). \]

The positive coefficient \( A \) is a certain ratio of gamma functions. Precisely the same relation (68) has been shown to hold in weakly interacting theories [52]. At weak coupling the constant \( A \) goes like the logarithm of the coupling. The weak coupling results follows from considering the spectrum of on-shell excitations that are accessible at some given order in perturbation theory. Taking the holographic and weak coupling results together suggests that the limiting behavior (68) may be a robust feature of the momentum dependence of general quantum critical response functions, beyond the existence of quasiparticles.
We should note also that when $z = 2$, the full Green’s function can be found explicitly in terms of gamma functions [41, 46], which is typically a sign of a hidden $SL(2, \mathbb{R})$ symmetry.

B. Hyperscaling violation

A more general class of scaling metrics than (56) take the form

$$ds^2 = L^2 \left( \frac{r}{R} \right)^{2g/d} \left( -\frac{dt^2}{r^{2z}} + \frac{d\vec{r}^2}{r^2} + d\rho^2 + dr^2 + \frac{d\phi^2}{r_r^2} \right).$$

(69)

Here $L$ is determined by the bulk theory while $R$ is a constant of integration in the solution. On the face of it, these metrics do not appear to be scale invariant. Under the rescaling $\{t, \vec{x}, r\} \rightarrow \{\lambda^z t, \lambda \vec{x}, \lambda r\}$, the metric transforms as $ds^2 \rightarrow \lambda^{2g/d} ds^2$. However, this covariant transformation of the metric amounts to the fact that the energy density operator (as well as other operators) in the dual field theory has acquired an anomalous dimension. Recall from §II that the bulk metric is dual to the energy-momentum tensor in the QFT. From the essential holographic dictionary (7), the boundary value $\delta \gamma_{\mu\nu}$ of a bulk metric perturbation sources the dual energy density $\varepsilon$ via the field theory coupling

$$\frac{1}{R^\theta} \int d^{d+1}x \left( \delta \gamma \right)_t^t \varepsilon .$$

(70)

The factors of $R$ appear because the boundary metric is obtained, analogously to (16), from (69) by stripping off the powers of $r$ and $L$ (but not $R$, which is a property of the solution) from each component of the induced boundary metric. As in our discussion around (18) above, the scaling action $\{t, \vec{x}, r\} \rightarrow \{\lambda^z t, \lambda \vec{x}, \lambda r\}$ must be combined with a scaling of parameters in the solution that leave the field (the metric in this case) invariant. Therefore, under this scaling we have $d^{d+1}x \rightarrow \lambda^{z+d} d^{d+1}x$, $\left( \delta \gamma \right)_t^t$ is invariant (because one index is up and the other down), $\varepsilon(x) \rightarrow \lambda^{-\Delta_\varepsilon} \varepsilon(\lambda x)$ and $R \rightarrow \lambda R$. Scale invariance of the coupling (70) then requires

$$\Delta_\varepsilon = z + d - \theta = \Delta_{\varepsilon,0} - \theta ,$$

(71)

where $\Delta_{\varepsilon,0}$ is the canonical dimension of the energy density operator. The existence of such an anomalous dimension is known as hyperscaling violation, and $\theta$ is called the hyperscaling violation exponent [53]. We will see later that this anomalous dimension indeed controls thermodynamic quantities and correlation functions of the energy-momentum tensor in the expected way. It is important to note, however, that because of the dimensionful parameter $R$ in the background, typically not all correlators of all operators will be scale covariant. An example is given by scalar fields with a large mass in the bulk [54]. In particular, when $z = 1$, theories with $\theta \neq 0$ are not CFTs. Hyperscaling violation arises when a UV scale does not decouple from certain IR quantities.

Hyperscaling violating metrics (69) will be important below in the context of compressible phases of matter. However, they can also describe zero density fixed points and therefore fit into
this section of our review. For instance, hyperscaling violating metrics of the form (69) with \( z = 1 \) arise as solutions to the Einstein-scalar action [55, 56]:

\[
S[g, \phi] = \frac{1}{2k^2} \int d^{d+2}x \sqrt{-g} \left( R + V_0 e^{\beta \phi} - \frac{1}{2} (\nabla \phi)^2 \right),
\]

(72)

with \( V_0 = \frac{(d+1-\theta)(d-\theta)}{L^2}, \quad \beta^2 = \frac{2\theta}{d(\theta - d)}, \)

(73)

where on the solution

\[
e^{\beta \phi} = r^{-2\theta/d}.
\]

(74)

Thus we find Lorentzian hyperscaling-violating theories. One sees in the above that for \( \beta \) to be real one must have \( \theta < 0 \) or \( \theta > d \). In fact, this is a special case of a more general requirement that we now discuss.

A spacetime satisfies the null energy condition if \( G_{ab}n^a n^b \geq 0 \) for all null vectors \( n \). Here \( G_{ab} \) is the Einstein tensor. The null energy condition is a constraint on matter fields sourcing the spacetime (because Einstein’s equations are \( G_{ab} = 8\pi G_N T_{ab} \)). In a holographic context, the null energy condition is especially natural because it seems to be necessary for holographic renormalization to make sense [57]. In particular, it ensures that the spacetime shrinks sufficiently rapidly as one moves into the bulk, cf. [58]. For the hyperscaling violating spacetimes (69), the null energy condition requires the following two inequalities to hold

\[
(d - \theta)[d(z - 1) - \theta] \geq 0, \quad (z - 1)(d + z - \theta) \geq 0.
\]

(75)

(76)

With \( \theta = 0 \), these inequalities require that \( z \geq 1 \), consistent with our observations above. While perhaps plausible, this condition on the dynamic critical exponent remains to be understood from a field theory perspective.

Unlike the Lifshitz geometries, hyperscaling-violating spacetimes do not have constant curvature invariants, and these invariants generically (with one potentially interesting exception [59, 60]) diverge at either large or small \( r \). These are more severe singularities than the pp-singularities of the Lifshitz geometries, and likely indicate the hyperscaling-violating spacetime only describes an intermediate energy range of the field theory, with new IR physics needed to resolve the singularity. For instance, microscopic examples of theories with zero density hyperscaling-violating intermediate regimes are non-conformal D-brane theories, such as super Yang-Mills theory in 2+1 dimensions [54].

VI. NONZERO TEMPERATURE

The most universal way to introduce a single scale into the scale invariant theories discussed above is to heat the system up. The thermal partition function is given by the path integral of the
theory in Euclidean time with a periodic imaginary time direction, $\tau \sim \tau + 1/T$:

$$Z_{\text{QFT}}(T) = \int_{S^1 \times \mathbb{R}^d} d\Phi e^{-I_E[\Phi]}.$$  

(77)

Here $I_E[\Phi]$ is the Euclidean action of the quantum field theory.

The essential holographic dictionary (7) identifies the space in which in the field theory lives as the asymptotic conformal boundary of the bulk spacetime. In particular, once we have placed the theory in a nontrivial background geometry ($S^1 \times \mathbb{R}^d$ in this case), then the bulk geometry must asymptote to this form. In order to compute the thermal partition function (77) holographically, in the semiclassical limit (8), we must therefore find a solution to the Euclidean bulk equations of motion with these boundary conditions.

A. Thermodynamics

The required solutions are Euclidean black hole geometries. Allowing for general dynamic critical exponent $z$ and hyperscaling violation, the solutions can be written in the form

$$ds^2 = L^2 \left( \frac{r}{R} \right)^{2\theta/d} \left( \frac{f(r)dr^2}{r^{2z}} + \frac{dr^2}{f(r)r^2} + \frac{d\vec{x}^2}{r^2} \right).$$  

(78)

The new aspect of the above metric relative to the scaling geometry (69) is the presence of the function $f(r)$. There is some choice in how the metric is written, because we are free to perform a coordinate transformation: $r \rightarrow \rho(r)$. The above form is convenient, however, because often the function $f(r)$ is simple in this case. We will see some examples of $f(r)$ shortly, but will first note some general features.

Asymptotically, as $r \rightarrow 0$, the metric must approach the scaling form (69), and therefore $f(0) = 1$. The near-boundary metric corresponds to the high energy UV physics of the dual field theory and, therefore, this boundary condition on $f(r)$ is the familiar statement that turning on a finite temperature does not affect the short distance, high energy properties of the theory. The temperature will, however, have a strong effect on the low energy IR dynamics. In fact, we expect the temperature to act as an IR cutoff on the theory, with long wavelength processes being Debye screened. It should not come as a surprise, therefore, that in the interior of the spacetime we find that at a certain radius $r_+$ we have $f(r_+) = 0$. At this radius the thermal circle $\tau$ shrinks to zero size and the spacetime ends. This is the Euclidean version of the black hole event horizon. The radial coordinate extends only over the range $0 \leq r \leq r_+$.

The radius $r_+$ is related to the temperature by imposing that the spacetime be regular at the Euclidean horizon. If we expand the metric near the horizon we find

$$ds^2 = A_+ \left( \frac{|f'(r_+)|(r_+ - r)dr^2}{r_+^{2z}} + \frac{dr^2}{|f'(r_+)|(r_+ - r)r_+^2} + \frac{d\vec{x}^2}{r_+^2} \right) + \cdots.$$  

(79)
Performing the change of coordinates

\[ r = r_+ - \frac{r_+^2 |f'(r_+)|}{4 A_+} \rho^2, \quad \tau = \frac{2 r_+^{z-1}}{|f'(r_+)|} \theta, \quad (80) \]

the near-horizon geometry becomes

\[ ds^2 = \rho^2 d\theta^2 + d\rho^2 + \frac{d\rho^2}{r_+^2} + \cdots \quad (81) \]

We immediately recognize this metric as flat space in cylindrical polar coordinates. In order to avoid a conical singularity at \( \rho = 0 \), we must have the identification \( \theta \sim \theta + 2\pi \). But \( \theta \) is defined in terms of \( \tau \) in (80), and \( \tau \) already has the periodicity \( \tau \sim \tau + 1/T \). Therefore

\[ T = \frac{|f'(r_+)|}{4\pi r_+^{z-1}}. \quad (82) \]

The (topological) plane parametrized by the \( \tau \) and \( r \) coordinates is called the cigar geometry and is illustrated in figure 6.

![Cigar geometry](image)

**FIG. 6.** **Cigar geometry.** The \( r \) coordinate runs from 0 at the boundary to \( r_+ \) at the horizon, where the Euclidean time circle shrinks to zero.

The Einstein-scalar theories with action (72) are an example of cases in which the function \( f(r) \) can be found analytically. The equations of motion following from this action are found to have a black hole solution given by the metric (78), with \( z = 1 \), with the scalar remaining unchanged from the zero temperature solution (74) and

\[ f(r) = 1 - \left( \frac{r}{r_+} \right)^{d+1-\theta}. \quad (83) \]

It follows from (82) that for these black holes

\[ T = \frac{d + 1 - \theta}{4\pi r_+}. \quad (84) \]
In particular, when $\theta = 0$ this gives the temperature of the (planar) AdS-Schwarzschild black hole in pure Einstein gravity. In other cases, such as the Einstein-Proca theory considered in the previous section, it will not be possible to find a closed form expression for $f(r)$. In these cases, the Einstein equations will give ODEs for $f(r)$, which will typically be coupled to other unknown functions appearing in the solution. These ODEs must then be solved numerically subject to the boundary conditions at $r = 0$ and $r = r_+$ that we gave in the previous paragraph. These numerics are straightforward and are commonly performed either with shooting or with spectral methods. In performing the numerics, because the constant $r_+$ is the only scale, it can be scaled out of the equations by writing $r = r_+ \hat{r}$. This scaling symmetry together with the formula (82) for the temperature implies that in generality

$$T \sim r_+^{-z}. \quad (85)$$

With the temperature at hand, we can now compute the entropy as a function of temperature. It is a general result in semiclassical gravity that the entropy is given by the area of the horizon divided by $4G_N [61, 62]$. In our conventions, Newton’s constant $G_N = \kappa^2 / (8\pi)$. Therefore, for the black hole spacetime (78) we have the entropy density

$$s = \frac{S}{V_d} = \frac{2\pi L_d^d \Theta^{-d}}{\kappa^2 R^\theta} \sim \frac{L_d^d T^{(d-\theta)/z}}{R^\theta}. \quad (86)$$

For the last relation we used (85) and also re-expressed $\kappa$ in terms of the $d + 2$ dimensional Planck length $\kappa^2 \sim L_p^d$. For a given theory with an explicit relation between temperature and horizon radius, such as (84), we easily get the exact numerical prefactor of the entropy density. Two observations should be made about the result (86) for the entropy. Firstly, the temperature scaling $s \sim T^{(d-\theta)/z}$ is exactly what is expected for a scale-invariant theory with exponents $z$ and $\theta$. Secondly, in the semiclassical limit, the radius of curvature $L$ of the spacetime is much larger than the Planck length $L_p$. Therefore, the entropy density $s \sim (L/L_p)^d \gg 1$ in this limit. In this way we explicitly realize the intuition from sections I and II above that the thermodynamics captured by classical gravity is that of a dual large $N$ theory. For instance, in the canonical duality involving $\mathcal{N} = 4$ SYM theory, we have $(L/L_p)^3 \sim N^2 \gg 1$.

The temperature and entropy are especially nice observables in holography because they are determined entirely by horizon data. The free energy is a more complicated quantity because it depends on the whole bulk solution. In particular

$$F = -T \log Z_{\text{QFT}} = -T \log Z_{\text{Grav.}} = TS[g_*]. \quad (87)$$

Here we used the equivalence of bulk and QFT partition functions (7) and also the semiclassical limit in the bulk (8). In the final expression, $S[g_*]$ is the bulk Euclidean action evaluated on the black hole solution (78). The action involves an integral over the whole bulk spacetime and diverges due to the infinite volume near the boundary as $r \to 0$. The divergences are precisely of the form
of the expected short distance divergences in the free energy of a quantum field theory. This is an instance of the general association of the near boundary region of the bulk with the UV of the dual QFT. If we believe that the scale invariant theory we are studying is a fundamental theory, then it makes sense to holographically renormalize the theory by adding appropriate boundary counterterms. However, more typically, we expect in condensed matter physics that the scaling theory is an emergent low energy description with some latticized short distance completion. The divergences we encounter in evaluating the action are then physical and simply remind us that the free energy is sensitive to (and generically dominated by) non-universal short distance degrees of freedom. For this reason we will refrain from explicitly evaluating the on shell action (87) for the moment, and note that the entropy (86) is a better characterization of the universal emergent scale-invariant degrees of freedom at temperatures that are low compared to some UV cutoff.

B. Thermal screening

Beyond equilibrium thermodynamic quantities, in §VII the Lorentzian signature version of the black hole backgrounds (78) will be the starting point for computations of linear response functions such as conductivities. At this point we can show how black holes cause thermal screening of spatial correlation. Correlators with \( \omega = 0 \) are the same in Lorentzian and Euclidean signature. To do this, consider the wave equation of a massive scalar field in the black hole background (78). For this computation, let us not consider the effects of hyperscaling violation and so set \( \theta = 0 \). The wave equation (13), at Euclidean frequency \( \omega_n \), becomes

\[
\phi'' + \left( \frac{f'}{f} - \frac{z + d - 1}{r} \right) \phi' - \frac{1}{f} \left( k^2 + \frac{(mL)^2}{r^2} + \frac{r^2 \omega_n^2}{r^2 f} \right) \phi = 0 .
\]  

(88)

This equation cannot be solved analytically in general. By rescaling the \( r \) coordinate as described above equation (67) above, and with the assumption that \( r_+ \) is the only scale in the bulk solution, so that \( f \) is a function of \( r/r_+ \), with \( r_+ \) related to \( T \) via (85), we can again conclude that the Green’s function will have the expected scaling form

\[
G_R^{\mathcal{O} \mathcal{O}}(\omega, k) = \omega^{(2\Delta - D_{\text{eff}})/z} F \left( \frac{\omega}{k^z}, \frac{T}{k^z} \right) .
\]  

(89)

Useful intuition for how thermal screening arises is obtained by solving the above equation in the WKB limit and with \( \omega_n = 0 \). This approximation holds for large \( mL \gg 1 \). The WKB analysis is especially simple for equation (88) as there are no classical turning points. Imposing an appropriate boundary condition at the horizon (which will be discussed in generality in §VII), one finds that the correlator is

\[
\langle \mathcal{O} \mathcal{O} \rangle (k) = r_+^{-2mL} \exp \left\{ -2 \int_0^{r_+} \frac{dr}{r} \left( \sqrt{\frac{(mL)^2 + (kr)^2}{f}} - mL \right) \right\} .
\]  

(90)
To see thermal screening, consider the real space correlator at large separation. This is found by taking the Fourier transform of (90) and evaluating using a stationary phase approximation for the k integral as \( x \to \infty \) to obtain:

\[
\langle \mathcal{O}\mathcal{O} \rangle (x) = \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^d} \langle \mathcal{O}\mathcal{O} \rangle (k) e^{-ik \cdot x} \sim e^{-ik_* x}.
\] (91)

It is simple to determine \( k_* \). From (90) and (91), \( k_* \) must satisfy

\[
2 \int_0^{r_+} \frac{k_* r dr}{\sqrt{f\sqrt{(mL)^2 + (k_* r)^2}}} + ix = 0.
\] (92)

In this expression we see that at large \( x \), \( k_* \) will be independent of \( x \) to leading order. Specifically: to balance the \( ix \) term, the integral must become large. This occurs if the integrand is close to developing a pole at \( r = r_+ \) (which would give a logarithmically divergent integral). For this to happen it must be that \( (mL)^2 + (k_* r)^2 \) is close to vanishing at \( r = r_+ \) (where \( f \) already vanishes). To leading order, then, we have \( k_* = \pm imL/r_+ \). Choosing the physical sign, it follows that at large \( x \)

\[
\langle \mathcal{O}\mathcal{O} \rangle (x) \sim e^{-mL x/r_+}, \quad \text{as} \quad x \to \infty.
\] (93)

We have therefore shown that the thermal screening length is

\[
L_{\text{thermal}} = \frac{r_+}{mL} \sim \frac{1}{T^{1/z}},
\] (94)

as we might have anticipated.

The screening length (94) arose because the momentum integral in (91) was dominated by momenta at the horizon scale. This can be seen even more directly by recalling that the WKB limit of the Klein-Gordon equation in a geometry describes the motion of very massive particles along geodesics. The correlation function at a separation \( x \) is then given by the length of a geodesic in the bulk that connects two points on the boundary separated by distance \( x \). The simplest quantity to consider here is the equal time correlation function. For points separated by much more than the horizon length \( r_+ \), the geodesic essentially falls straight down to the horizon, and runs along the horizon for a distance \( x \). See figure 7. Thereby one recovers (93) from

\[
\langle \mathcal{O}\mathcal{O} \rangle (x) \sim e^{-m \times \text{(Geodesic length)}} \sim e^{-mL x/r_+}.
\] (95)

Thus we see a very geometric connection between the presence of a horizon and thermal screening.

VII. QUANTUM CRITICAL TRANSPORT

A. Condensed matter systems and questions

Let us consider the simplest case of transport of a conserved U(1) “charge” density \( n(x,t) \) (we will explicitly write out the time coordinate, \( t \), separately from the spatial coordinate \( x \) in this
FIG. 7. **Thermal screening.** The geodesic runs along the horizon over a distance $x$. This contribution to its length dominates the correlation function and leads to an exponentially decaying correlation (93) in space, with scale set by the horizon radius $r_+$. Its conservation implies that there is a current $J_i(x, t)$ (where $i = 1 \ldots d$) such that

$$\partial_n + \partial_t J_i = 0.$$  \hspace{1cm} (96)

We are interested here in the consequences of this conservation law at $T > 0$ for correlators of $n$ and $J_i$ in quantum systems without quasiparticle excitations.

We begin with a simple system in $d = 1$: a narrow wire of electrons which realize a Tomonaga-Luttinger liquid [63]. For simplicity, we ignore the spin of the electron in the present discussion. In field-theoretic terms, such a liquid is a CFT2. In the CFT literature it is conventional to describe its correlators in holomorphic (and anti-holomorphic) variables constructed out of space and Euclidean time, $\tau$. The spacetime coordinate is expressed in terms of a complex number $z = x + it$, and the density and the current combine to yield the holomorphic current $J$. A well-known property of all CFT2s with a conserved $J$ is the correlator

$$\langle J(z)J(0) \rangle = \frac{K}{z^2},$$  \hspace{1cm} (97)

where $K$ is the ‘level’ on the conserved current. Here we wish to rephrase this correlator in more conventional condensed matter variables using momenta $k$ and Euclidean frequency $\omega$. Fourier transforming (97) we obtain

$$\langle |n(k, \omega)|^2 \rangle = \frac{K}{c^2k^2 + \omega^2},$$  \hspace{1cm} (98)

where $c$ is the velocity of ‘light’ of the CFT2. A curious property of CFT2s is that (98) holds also at $T > 0$. This is a consequence of the conformal mapping between the $T = 0$ planar spacetime geometry and the $T > 0$ cylindrical geometry; as shown in [40], this mapping leads to no change in the Euclidean time correlator in (98) apart from the restriction that $\omega$ is an integer multiple.
of $2\pi T$ (i.e. it is a Masturbara frequency). We can also analytically continue from the Euclidean correlation in (98), via $i\omega \to \omega + i\epsilon$, to obtain the retarded two-point correlator of the density

$$G_{nn}^R = \mathcal{K} \frac{c^2 k^2}{c^2 k^2 - (\omega + i\epsilon)^2}, \quad d = 1,$$

(99)

where $\epsilon$ is a positive infinitesimal. We emphasize that (99) holds for all CFT2s with a global U(1) symmetry at any temperature $T$.

Now we make the key observation that (99) is not the expected answer for a generic non-integrable interacting quantum system at $T > 0$ for small $\omega$ and $k$. Upon applying an external perturbation which creates a local non-uniformity in density, we expect any such system to relax towards the equilibrium maximum entropy state. This relaxation occurs via hydrodynamic diffusion of the conserved density. Namely, we expect that on long time and length scales compared to $\hbar/k_B T$ (or analogous thermalization length scale):

$$J_i \approx -D \partial_i n + O(\partial^3).$$

(100)

As described in some detail by Kadanoff and Martin [64], the (very general) assumption that hydrodynamics is the correct description of the late dynamics forces the retarded density correlator at small $k$ and $\omega$ to be

$$G_{nn}^R = \chi \frac{D k^2}{D k^2 - i\omega},$$

(101)

where $D$ is the diffusion constant, and $\chi$ is the susceptibility referred to as the compressibility.[65] The disagreement between (99) and (101) implies that all CFT2s have integrable density correlations, and do not relax to thermal equilibrium.

What is the situation for CFTs in higher dimension? We consider interacting CFT3s, such as the Wilson-Fisher fixed point describing the superfluid-insulator transition in §IV. In this case (using a relativistic notation with $c = 1$), we have a current $J_\mu (\mu = 0, \ldots d)$, and its conservation and conformal invariance imply that in $d$ spatial dimensions and at zero temperature:

$$\langle J_\mu(p)J_\nu(-p) \rangle = -\mathcal{K}|p|^{d-1} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right),$$

(102)

where $p$ is a Euclidean spacetime momentum and $\mathcal{K}$ is a dimensionless number characteristic of the CFT. The non-analytic power of $p$ above follows from the fact that the $n$ and $J_i$ have scaling dimension $d$. Taking the $\mu = \nu = 0$ component of the above, and analytically continuing to the retarded correlator, we obtain

$$G_{nn}^R = \mathcal{K} \frac{k^2}{\sqrt{k^2 - (\omega + i\epsilon)^2}}, \quad d = 2, \quad T = 0.$$

(103)

While (103) is the exact result for all CFT3s, now we do not expect it to hold at $T > 0$. Instead, we expect that for $\omega, k \ll T$ CFT3s will behave like generic non-integrable quantum systems and relax.
diffusively to thermal equilibrium. In other words, we expect a crossover from \((103)\) at \(\omega, k \gg T\) to \((101)\) at \(\omega, k \ll T\). Furthermore, using the fact that \(T\) is the only dimensionful parameter present, dimensional analysis from a comparison of these expressions gives us the \(T\) dependence of the compressibility and the diffusivity

\[
\chi = C_\chi T, \quad D = \frac{C_D}{T},
\]

where \(C_{\chi,D}\) are dimensionless numbers also characteristic of the CFT3. Note that there is no direct relationship between \(C_{\chi,D}\) and \(K\) other than the fact that they are all numbers obtained from the same CFT3.

The computation of \(C_{\chi,T}\) for CFT3s is a difficult task we shall address by several methods in the following sections. For now, we note that these expressions also yield the conductivity \(\sigma\) upon using the Kubo formula \([66]\)

\[
\sigma(\omega) = \lim_{\omega \to 0} \lim_{k \to 0} -\frac{i\omega}{k^2} G_{nn}^R(k, \omega).
\]

From \((103)\) we conclude that

\[
\sigma(\omega \gg T) = K,
\]

while from \((101)\) and \((104)\)

\[
\sigma(\omega \ll T) = C_\chi C_D.
\]

The crossover between these limiting values is determined by a function of \(\omega/T\), and understanding the structure of this function is an important aim of the following discussion.

In the application to the boson Hubbard model, we note that the above conductivity is measured in units of \((e^*)^2/\hbar\) where \(e^*\) is the charge of a boson. We also note that the above discussion has glossed over subtleties with “hydrodynamic long-time tails”: these arise from non-linearities of the equations of hydrodynamics and lead to a subtle frequency dependence of the transport coefficients at the lowest frequencies \([67]\). For the present situation, there is a logarithmic frequency dependence whose coefficient has been estimated. Such long-time tails have also been obtained by holographic methods, where they are subleading in the large \(N\) expansion \([68, 69]\).

1. Quasiparticle-based methods

We emphasized above that interacting CFT3s do not have any quasiparticle excitations. However, there are the exceptions of free CFT3s which do have infinitely long-lived quasiparticles. So we can hope that we could expand away from these free CFTs and obtain a controlled theory of interacting CFT3s. After all, this is essentially the approach of the \(\epsilon\) expansion in dimensionality by Wilson and Fisher for the critical exponents. However, we will see below that the \(\epsilon\) expansion, and the related vector \(1/M\) expansion, have difficulty in describing transport because they do not hold uniformly in \(\omega\). Formally, the \(\epsilon \to 0\) and the \(\omega \to 0\) limits do not commute.
Let us first perform an explicit computation on a free CFT3: a CFT with $M$ two-component, massless Dirac fermions, $C$, with Lagrangian

$$\mathcal{L} = \overline{C} \gamma^\mu \partial_\mu C.$$  

(108)

We consider a conserved flavor U(1) current $J_\mu = -i \overline{C} \gamma_\mu \rho C$, where $\rho$ is a traceless flavor matrix normalized as $\text{Tr} \rho^2 = 1$. The $T = 0$ Euclidean correlator of the currents is

$$\langle J_\mu(p) J_\nu(-p) \rangle = -\frac{d^3k}{8\pi^3} \frac{\text{Tr} [\gamma_\mu \gamma_\lambda k_\lambda \gamma_\nu \gamma_\delta (k_\delta + p_\delta)]}{k^2 (p + k)^2} \langle \theta(k) \rangle \int_0^1 dx \sqrt{p^2 x (1 - x)}$$

$$= -\frac{1}{2\pi} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \int_0^1 dx \sqrt{p^2 x (1 - x)}$$

$$= -\frac{1}{16} \langle |p| \rangle \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right).$$

(109)

This is clearly of the form in (102), and determines the value $\mathcal{K} = 1/16$.

The form of the current correlator of the CFT3 is considerably more subtle at $T > 0$. In principle, this evaluation only requires replacing the frequency integral in (109) by a summation over the Matsubara frequencies, which are quantized by odd multiples of $\pi T$. However, after performing this integration by standard methods, the resulting expression should be continued carefully to real frequencies. We quote the final result for $\sigma(\omega)$, obtained from the current correlator via (105)

$$\text{Re} [\sigma(\omega)] = \frac{T}{2} \ln 2 \delta(\omega) + \frac{1}{16} \text{tanh} \left( \frac{\omega}{4T} \right)$$

$$\text{Im} [\sigma(\omega)] = \int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \mathcal{P} \left( \frac{\text{Re} [\sigma(\Omega)] - 1/16}{\omega - \Omega} \right),$$

(110)

where $\mathcal{P}$ is the principal part. Note that in the limit $\omega \gg T$, (110) yields $\sigma(\omega) = K = 1/16$, as expected. However, the most important new feature of (110) is the delta function at zero frequency in the real part of the conductivity, with weight proportional to $T$. This is a consequence of the presence of quasiparticles, which have been thermally excited and can transport charge ballistically in the absence of any collisions between them.

We can now ask if the zero frequency delta function is preserved once we move to an interacting CFT. A simple way to realize an interacting CFT3 is to couple the fermions to a dynamical U(1) gauge field, and examine the theory for large $M$. It is known that the IR physics is then described by a interacting CFT3 [70–77] and its $T = 0$ properties can be computed in a systematic $1/M$ expansion (we emphasize that this is a ‘vector’ $1/M$ expansion, unlike the matrix large $N$ models considered elsewhere in this paper). The same remains true at $T > 0$ for the conductivity, provided we focus on the $\omega \gg T$ region of (110) (we will have more to say about this region in the following subsection). However, now collisions are allowed between the quasiparticles that were not present at $M = \infty$, and these collisions cannot be treated in a bare $1/M$ expansion. Rather, we have to
FIG. 8. Schematic of the real part of the conductivity of a CFT3 of $M$ Dirac fermions coupled to a U(1) gauge field in the large $M$ limit. Similar features apply to other CFT3s in the vector large $M$ limit. The peak at zero frequency is a remnant of the quasiparticles present at $M = \infty$, and the total area under this peak equals $(T \ln 2)/2$ as $M \to \infty$.

examine the effects of repeated collisions, and ask if they lead to number diffusion and a finite conductivity. This is the precise analog of the question Boltzmann asked for the classical ideal gas, and he introduced the Boltzmann equation to relate the long-time Brownian motion of the molecules to their two-particle collision cross-section. We can apply a quantum generalization of the Boltzmann equation to the CFT3 in the $1/M$ expansion, where the quasiparticles of the $M = \infty$ CFT3 undergo repeated collisions at order $1/M$ by exchanging quanta of the U(1) gauge field. The collision cross-section can be computed by Fermi’s golden rule and this then enters the collision term in the quantum Boltzmann equation [73, 78].

One subtlety should be kept in mind while considering the analogy with the classical ideal gas. The free CFT3 has both particle and anti-particle quasiparticles, and these move in opposite directions in the presence of an applied electric field. So the collision term must also consider collisions between particles and anti-particles. Such collisions have the feature that they can degrade the electrical current while conserving total momentum. Consequently, a solution of the Boltzmann equation shows that the zero-frequency conductivity is \textit{finite} even at $T > 0$, after particle-anti-particle collisions have been accounted for [32, 78]. As in the usual Drude expression for the conductivity, the zero frequency conductivity is inversely proportional to the collision rate. As the latter is proportional to $1/M$, we have $\sigma(0) \sim M$. Similarly, we can consider the frequency dependence of the conductivity at $\omega \ll T$, and find that the width of the peak in the conductivity extends up to frequencies of order $T/M$. So the zero frequency delta function in (110) has broadened into peak of height $M$ and width $T/M$, as sketched in Fig. 8. It required
a (formally uncontrolled) resummation of the $1/M$ expansion using the Boltzmann equation to obtain this result. And it should now also be clear from a glance at Fig. 8 that the $\omega \to 0$ and $M \to \infty$ limits do not commute at $T > 0$.

A similar analysis can also be applied to the Wilson-Fisher CFT3 described by (46), using a model with a global $O(M)$ symmetry. The Boltzmann equation result for the zero frequency conductivity is $\sigma(0) = 0.523M$, where the boson superfluid-insulator transition corresponds to the case $M = 2$ [73, 79].

Our main conclusion here is that the vector $1/M$ expansion of CFT3s, which expands away from the quasiparticles present at $M = \infty$, yields fairly convincing evidence that $\sigma(\omega)$ is a non-trivial universal function of $\omega/T$. However, it does not appear to be a reliable way of computing $\sigma(\omega)$ for $\omega < T$ and smaller values of $M$.

B. Holographic spectral functions

In the next few sections we will cover holographic computations of the retarded Green’s functions for operators in quantum critical systems at nonzero temperature. In general we will be interested in multiple coupled operators and so we write, in frequency space,

$$\delta \langle O_A \rangle (\omega, k) = G_{O_A \partial B}^R (\omega, k) \delta h_B (\omega, k).$$

(111)

Here $\delta \langle O_A \rangle$ is the change in the expectation value of the operator $O_A$ due to the (time and space dependent) change in the sources $\delta h_B$. From the holographic dictionary discussed in previous sections, we know that expectation values and sources are given by the near boundary behavior of the bulk fields $\{\phi_A\}$ dual to the operators $\{O_A\}$. Therefore

$$G_{O_A \partial B}^R = \frac{\delta \langle O_A \rangle}{\delta h_B} = \frac{2\Delta - D_{\text{eff}}}{L^2} \frac{\delta \phi_A^{(1)}}{\delta \phi_B^{(0)}}.$$

(112)

Here $\phi^{(0)}$ and $\phi^{(1)}$ are as defined in (64) above. The near boundary behavior of the fields has the same form at zero and nonzero temperature because, as we saw in §VI above, putting a black hole in the spacetime changes the geometry in the interior (IR), but not near the boundary (UV).

To compute the Green’s function (112) in a given background, one first perturbs the background fields

$$\phi_A (r) \rightarrow \phi_A (r) + \delta \phi_A (r) e^{-i\omega t + ik \cdot x}.$$  

(113)

Linearizing the bulk equations of motion leads to coupled, linear, ordinary differential equations for the perturbations $\delta \phi_A (r)$. Once these linearized equations are solved, the asymptotic behavior of fields can be read off from the solutions and the Green’s function obtained from (112). A crucial part of solving the equations for the perturbations is to impose the correct boundary conditions at the black hole horizon. One way to understand these boundary conditions is to recall that the
retarded Green’s function is obtained from the Euclidean (imaginary time) Green’s function by analytic continuation from the upper half frequency plane (i.e. $\omega_n > 0$). An important virtue of the holographic approach is that this analytic continuation can be performed in a very efficient way, as we now describe.

1. Infalling boundary conditions at the horizon

Starting with the Euclidean black hole spacetime (78), we described how to zoom in on the spacetime near the horizon in the manipulations following equation (79). To see how fields behave near the horizon, we can consider the massive wave equation in the black hole background (88) at $r \sim r_+$. The equation becomes

$$\phi'' + \frac{1}{r - r_+} \phi' - \frac{\omega_n^2}{(4\pi T)^2(r - r_+)^2} \phi = 0.$$  \hspace{1cm} (114)

The temperature $T$ is as given in (82). The two solutions to this equation are

$$\phi_\pm = (r - r_+)^{\pm \omega_n/(4\pi T)}.$$  \hspace{1cm} (115)

For $\omega_n > 0$, it is clear that the regular solution is the one that decays as $r \to r_+$ (that is, the positive sign in the above equation). Once this condition is imposed on all fields, the solution is determined up to overall an overall constant, which will drop out upon taking the ratios in (112) to compute the Green’s function.

The real time equations of motion are the same as the Euclidean equations, with the substitution $i\omega_n \to \omega$. In particular, the regular boundary condition at the horizon analytically continues to the “infalling” boundary condition

$$\phi_{\text{infalling}} = (r - r_+)^{-i\omega/(4\pi T)}.$$  \hspace{1cm} (116)

This behavior at the horizon can also be derived directly from the real time, Lorentzian, equations of motion. Infalling modes are required for regularity in the Kruskal coordinates that extend the black hole spacetime across the horizon (see e.g. [43]). The derivation via the Euclidean modes makes clear, however, that infalling boundary conditions correspond to computing the retarded Green’s function.

The infalling boundary condition (116) allows the retarded Green’s function to be computed directly at real frequencies. This is a major advantage relative to e.g. Monte Carlo methods that work in Euclidean time. The infalling boundary condition is also responsible for the appearance of dissipation at leading order in the classical large $N$ limit, at any frequencies. This is also a key virtue relative to e.g. vector large $N$ expansions, in which $\omega \to 0$ and $N \to \infty$ do not commute at nonzero temperatures, as we explained above. Infalling modes have an energy flux into the horizon.
that is lost to an exterior observer. The black hole horizon has geometrized the irreversibility of entropy production. Physically, energy crossing the horizon has dissipated into the ‘order $N^2$’ degrees of freedom of the deconfined gauge theory.

Infalling boundary conditions were first connected to retarded Green’s function in [80]. We derived the infalling boundary condition from the Euclidean Green’s function. An alternative derivation of this boundary condition starts from the fact that the fully extended Penrose diagram of (Lorentzian) black holes has two boundaries, and that these geometrically realize the thermofield double or Schwinger-Keldysh approach to real time thermal physics [81, 82].

2. Example: spectral weight $\text{Im} G^{\text{IR}}_{\mathcal{O}}(\omega)$ of a large dimension operator

We will now illustrate the above with a concrete example. We will spell out here in some detail steps that we will go over more quickly in later cases. To start with we will put $k = 0$ and obtain the dependence of the Green’s function on frequency. The scalar wave equation (88) in the Lorentzian signature black hole background becomes

\[ (Lm)^2 \frac{r}{f} \phi' + \frac{1}{f} \left( \frac{(mL)^2}{r^2} - \frac{r^2 \omega^2}{f} \right) \phi = 0. \]  

(117)

To develop an intuition for an equation like the above, it is often useful to put the equation into a Schrödinger form. To do this we set $\phi = r^{d/2} \psi$, so that (117) becomes

\[ - \frac{f}{r} \frac{d}{dr} \left( \frac{f}{r^{z-1}} \frac{d}{dr} \psi \right) + V \psi = \omega^2 \psi. \]  

(118)

If we define $r_*$ such that $\partial_{r_*} = r^{1-z} f \partial_r$, we immediately recognize the above equation as the time-independent Schrödinger equation in one spatial dimension. All the physics is now contained in the Schrödinger potential

\[ V(r) = \frac{f}{r^{2z}} \left( (Lm)^2 + \frac{d(d+2z)f}{4} - \frac{d r f'}{2} \right). \]  

(119)

The Schrödinger equation (118) is to be solved subject to the following boundary conditions. We will discuss first the near-horizon region, $r \to r_+$. The Schrödinger coordinate $r_* = \int dr \left( r^{z-1}/f \right) \to \infty$ as $r \to r_+$, so this is a genuine asymptotic region of the Schrödinger equation. The potential (119) vanishes on the horizon because $f(r_+) = 1$. Therefore for any $\omega^2 > 0$, recall that $\omega^2$ plays the role of the energy in the Schrödinger equation (118), the solution oscillates near the horizon. The infalling boundary condition is thus seen to translate into the boundary condition that modes are purely outgoing as $r_* \to \infty$. This is a scattering boundary condition of exactly the sort one finds in elementary one dimensional scattering problems.

The boundary at $r = 0$ is not an asymptotic region of the Schrödinger equation. There will be normalizable and non-normalizable behaviors of $\phi$ as $r \to 0$. To compute the Green’s function we
will need to fix the coefficient of the non-normalizable mode and solve for the coefficient of the normalizable mode.

A key physical quantity is the imaginary part of the retarded Green’s function of the operator $O$ dual to $\phi$ (also called the spectral weight). The imaginary part of the retarded Green’s function is a direct measure of the entropy generated when the system is subjected to the sources $\delta h_B$ of equation (111). Specifically, with an assumption of time reversal invariance (see e.g. [43]), the time-averaged rate of work done on the system per unit volume by a spatially homogeneous source driven at frequency $\omega$ is given by

$$\frac{d\omega}{dt} = \omega \delta h_A^* \text{Im} G_{AB}^R(\omega) \delta h_B \geq 0.$$  

(120)

We now illustrate how this spectral weight is given by the amplitude with which the field can tunnel from the boundary $r = 0$ through to the horizon. Thus, we will exhibit the direct connection between dissipation in the dual QFT and rate of absorption by the horizon.

The analysis is simplest in the limit of large $mL \gg 1$. Recall that this corresponds to an operator $O$ with large scaling dimension. In this limit the equation (117) can be solved in a WKB approximation. In the large mass limit, the potential (119) typically decreases monotonically from the boundary towards the horizon. There is therefore a unique turning point $r_o$ at which $V(r_o) = \omega^2$. The infalling boundary condition together with standard WKB matching formulae give the solution

$$\psi = \begin{cases} 
\exp \left\{ i \int_{r}^{r_o} \frac{r^{z-1} dr}{f} \sqrt{\omega^2 - V(r)} - \frac{i\pi}{4} \right\} & r_o > r > r_+, \\
\exp \left\{ \int_{r}^{r_o} \frac{r^{z-1} dr}{f} \sqrt{V(r) - \omega^2} \right\} + \frac{i}{2} \exp \left\{ - \int_{r}^{r_o} \frac{r^{z-1} dr}{f} \sqrt{V(r) - \omega^2} \right\} & r_0 > r.
\end{cases}$$

(121)

In this limit $V = (mL)^2 f/r^{2z}$. Expanding the above solution near the $r \to 0$ boundary, the retarded Green’s function is obtained from the general formula (112). In particular, the imaginary part is given by

$$\chi''(\omega) \equiv \text{Im} G_{\phi\phi}^R(\omega)$$

$$\propto r_o^{-2mL} \exp \left\{ -2 \int_{0}^{r_o} \frac{dr}{r} \left( \sqrt{\frac{(mL)^2}{f} - \frac{r^{2z}\omega^2}{f^2}} - mL \right) \right\}$$

(123)

$$= \text{Probability for quanta of } \phi \text{ to tunnel from boundary into the near horizon region ('transmission probability')}.$$
On general grounds \( \chi'' \) must be an odd function of \( \omega \). This can be seen in a more careful WKB computation that includes the order one prefactor. Relatedly, to obtain (124) we used the fact that the scaling dimension (66) is given by \( \Delta = mL \) in the limit \( mL \gg 1 \). We also used the expression (85) for the temperature. The full result (123) gives an explicit and relatively simple crossover between these two limits.

More generally, the equation (117) can easily be solved numerically. Later we will give an example of the type of Mathematica code one uses to solve such equations. We should note, however, that for general values of \( \Delta \) (e.g. not integer, etc.) the boundary conditions often need to be treated to a very high accuracy to get stable results, as discussed in e.g [83]. Also, one should be aware that when the fast and slow falloffs differ by an integer, one can find logarithmic terms in the near boundary expansion. These must also be treated with care and, of course, correspond to the short distance logarithmic running of couplings in the dual QFT that one expects in these cases.

3. Infalling boundary conditions at zero temperature

In equation (116) above we obtained the infalling boundary condition for finite temperature horizons. Zero temperature geometries that are not gapped will also have horizons in the far interior (or, more generally, as we noted in our discussion of Lifshitz geometries in §V above, mild null singularities). To compute the retarded Green’s function from such spacetimes we will need to know how to impose infalling boundary conditions in these cases. This is done as before, by analytic continuation of the Euclidean mode that is regular in the upper half complex frequency plane. It is not always completely straightforward to find the leading behavior of solutions to the wave equation near a zero temperature horizon, especially when there are many coupled equations. The following three examples illustrate common possibilities. In each case we give the leading solution to \( \nabla^2 \phi = m^2 \phi \) near the horizon.

1. Poincaré horizon. Near horizon metric (as \( r \to \infty \)):

\[
ds^2 = L^2 \left( \frac{-dt^2 + dr^2 + d\vec{x}_d^2}{r^2} \right). \tag{125}\]

Infalling mode as \( r \to \infty \) and with \( \omega^2 > k^2 \)

\[
\phi_{\text{infalling}} = r^{d/2} e^{i r \sqrt{\omega^2 - k^2}}. \tag{126}\]

2. Extremal \( AdS_2 \) charged horizon. Near horizon metric (as \( r \to \infty \)):

\[
ds^2 = L^2 \left( \frac{-dt^2 + dr^2 + d\vec{x}_d^2}{r^2} \right). \tag{127}\]
Infalling mode as \( r \to \infty \)

\[
\phi_{\text{infalling}} = e^{i\omega r}. \tag{128}
\]

3. Lifshitz 'horizon'. Near horizon metric (as \( r \to \infty \)):

\[
ds^2 = L^2 \left( \frac{-dt^2}{r^{2z}} + \frac{dr^2 + d\vec{x}_2^2}{r^2} \right). \tag{129}
\]

Infalling mode as \( r \to \infty \), with \( z > 1 \),

\[
\phi_{\text{infalling}} = r^{d/2} e^{i\omega r^z / z}. \tag{130}
\]

Finally, occasionally one wants to find the Green's function directly at \( \omega = 0 \) (for instance, in our discussion of thermal screening around (90) above). The equations at \( \omega = 0 \) typically exhibit two possible behaviors near the horizon, one divergent and the other regular. At finite temperature the divergence will be logarithmic. Clearly one should use the regular solution to compute the correlator.

C. Quantum critical charge dynamics

So far we have discussed scalar operators \( O \) as illustrative examples. However, a very important set of operators are instead currents \( J^\mu \) associated to conserved charges. The retarded Green's functions of current operators capture the physics of charge transport in the quantum critical theory. We explained around equation (30) above that a conserved \( U(1) \) current \( J^\mu \) in the boundary QFT is dually described by a Maxwell field \( A_a \) in the bulk. Therefore, to compute the correlators of \( J^\mu \), we need an action that determines the bulk dynamics of \( A_a \).

1. Conductivity from the dynamics of a bulk Maxwell field

As we are considering zero density quantum critical matter in this section, we restrict to situations where the Maxwell field is not turned in the background. Therefore, in order to obtain linear response functions in a zero density theory, it is sufficient to consider a quadratic action for the Maxwell field about a fixed background. The simplest such action is the Maxwell action (with \( F = dA \), as usual)

\[
S[A] = -\frac{1}{4e^2} \int d^{d+2}x \sqrt{-g} F^2, \tag{131}
\]

so we will start with this.

Before obtaining the Maxwell equations we should pick a gauge. A very useful choice for many purposes in holography is the 'radial gauge', in which we put \( A_r = 0 \). Among other things, the bulk Maxwell field in this gauge has the same nonzero components as the boundary current operator
$J^\mu$. The Maxwell equations about a black hole geometry of the form \((78)\) can now be written out explicitly. Firstly, write the Maxwell field as

$$A_\mu = a_\mu(r)e^{-i\omega t+ikx},$$

where without loss of generality we have taken the momentum to be in the \(x\) direction. The equations of motion, \(\nabla_a F^{ab} = 0\), become coupled ordinary differential equations for the \(a_\mu(r)\).

By considering the discrete symmetry \(y \to -y\) of the background, where \(y\) is a boundary spatial dimension orthogonal to \(x\), we immediately see that the perturbation will decouple into longitudinal \((a_t\) and \(a_x\)) and transverse \((a_y)\) modes. It is then useful to introduce the following ‘gauge invariant’ variables \([84]\), that are invariant under a residual gauge symmetry

$$a_\perp(r) = a_y(r), \quad a_\parallel(r) = \omega a_x(r) + ka_t(r).$$

In terms of these variables, the Maxwell action becomes:

$$S = -\frac{L^{d-2}}{2e^2} \int dr \left[ f r^{3-d-z} a_\perp'^2 + a_\perp^2 \left(k^2 r^{3-d-z} - \frac{\omega^2 r z + 1 - d}{f}\right) + \frac{f r^{3-d-z}}{\omega^2 - k^2 f r^{2-2z}} a_\parallel'^2 - \frac{r z + 1 - d}{f} a_\parallel^2 \right]$$

leading to the following two decoupled second order differential equations

$$\left(\frac{f r^{3-d-z}}{\omega^2 - k^2 f r^{2-2z}} a_\parallel'\right)' = -\frac{r z + 1 - d}{f} a_\parallel,$$

$$\left(f r^{3-d-z} a_\perp'\right)' = k^2 r^{3-d-z} a_\perp - \frac{\omega^2 r z + 1 - d}{f} a_\perp.$$

When \(k = 0\) the transverse and longitudinal equations are the same, as we should expect. These equations with \(z = 1\) have been studied in several important papers \([40, 84–86]\), sometimes using different notation.

Near the boundary at \(r \to 0\), the asymptotic behavior of \(a_\mu\) is given by:

$$a_t = a_t^{(0)} + a_t^{(1)} r^{d-z} + \cdots,$$

$$a_i = a_i^{(0)} + a_i^{(1)} r^{d+z-2} + \cdots.$$ (\ref{eq:asymptotic_behavior})

Note that when \(z \neq 1\), as expected, the asymptotic falloffs are different for the timelike and spacelike components of \(a_\mu\). So long as \(d > z\), it is straightforward to add boundary counterterms to make the bulk action finite. We can then, using the general formalism developed previously, obtain the expectation values for the charge and current densities:

$$\langle J^t \rangle = -\frac{L^{d-2}}{e^2} (d - z) a_t^{(1)},$$

$$\langle J^i \rangle = \frac{L^{d-2}}{e^2} (d + z - 2) a_i^{(1)}.$$ (\ref{eq:expectation_values})
If instead \(d < z\), the ‘subleading’ term in (137a) is in fact the largest term near the boundary. This leads to many observables having a strong dependences on short distance physics, we will see an example shortly when we compute the charge diffusivity. A closely related fact is that when \(d < z\) there is a relevant (double trace) deformation to the QFT given by \(\int d^{d+1}x \ n^2\). As first emphasized in [87], this deformation will generically drive a flow to a new fixed point in which the role of \(a^{(0)}_t\) and \(a^{(1)}_t\) are exchanged. The physics of such a theory is worth further study, as there are known examples (nematic or ferromagnetic critical points in metals, at least when treated at the ‘Hertz-Millis’ mean field level [88, 89]) of \(d = 2\) and \(z = 3\).

A quantity of particular interest is the frequency dependent conductivity
\[
\sigma(\omega) = \frac{\langle J_x(\omega) \rangle}{E_x(\omega)} = \frac{\langle J_x(\omega) \rangle}{i\omega a^{(0)}_x(\omega)} = \frac{G^R_{J_x,J_x}(\omega)}{i\omega}.
\]

As we have noted above, the dissipative (real) part of the conductivity will control entropy generation due to Joule heating when a current is driven through the system. We shall compute the optical conductivity (139) shortly, but first consider the physics of certain low energy limits.

2. The dc conductivity

The dc conductivity
\[
\sigma = \lim_{\omega \to 0} \sigma(\omega),
\]
determines the dissipation (120) due to an arbitrarily low frequency current being driven through the system. It should therefore be computable in an effective low energy description of the physics. This means that we might hope to obtain the dc conductivity from a computation in purely the near-horizon, far interior, part of the spacetime. Indeed this is the case. We will follow a slightly modernized (in the spirit of [90]) version of the logic in [91], which in turn built on [92]. The upshot is the formula (147) below for the dc conductivity which is expressed purely in terms of data at the horizon itself.

The dc conductivity can be obtained by directly applying a uniform electric field, rather than taking the \(k,\omega \to 0\) limit of (136). Consider the bulk Maxwell potential
\[
A = (-Et + ax(r))dx.
\]

Given the near boundary expansion (137b), the QFT source term is given by the non-normalizable constant term \(a^{(0)}_x\) of the bulk field near the boundary \(r \to 0\). The boundary electric field is then obtained from (141) as \(E = -\partial_r a^{(0)}_x = -F_{tx}(r \to 0)\), and is uniform. To obtain the dc conductivity we must now determine the uniform current response.

The bulk Maxwell equations can be written as \(\partial_a (\sqrt{-g}F^{ab}) = 0\). [93] For the field (141) one immediately has that
\[
\partial_r (\sqrt{-g}F^{tx}) = 0.
\]
We have thereby identified a radially conserved quantity. Furthermore, near the boundary
\[
\lim_{r \to 0} \sqrt{-g} F^{rx} = L^{d-2}(d + z - 2) a_x^{(1)} = e^2 \langle J_x \rangle. \tag{143}
\]
Here we used the near boundary expansion (137), as well as (138b) to relate \( \langle J_x \rangle \) to \( a_x^{(1)} \).

From (142) and (143) it follows that we can obtain the field theory current response \( \langle J_x \rangle \) if we are able to evaluate \( \sqrt{-g} F^{rx} \) at any radius. It turns out that we can compute it at the horizon. It is a standard trick that physics near black hole horizons is often elucidated by going to infalling coordinates; so we replace \( t \) in favor of the ‘infalling Eddington-Finkelstein coordinate’ \( v \) by
\[
v = t + \int \frac{\sqrt{g_{rr}} dr}{\sqrt{-g_{tt}}}. \tag{144}
\]
It is easily seen that the (Lorentzian version of the) black hole spacetime (78) is regular at \( r = r_+ \) in the coordinates \( \{ v, r, \bar{x} \} \). Therefore, a regular mode should only depend on \( t \) and \( r \) through the combination \( v \). It follows that, at the horizon,
\[
\partial_r A_x \bigg|_{r=r_+} = - \sqrt{g_{rr}} \partial_t A_x \bigg|_{r=r_+}. \tag{145}
\]
We find that
\[
a_x'(r_+) = \frac{r_+^{z-1}}{f} E. \tag{146}
\]
We are now able to compute the conductivity by evaluating
\[
\sigma = \frac{\langle J_x \rangle}{E} = \frac{1}{e^2} \lim_{r \to r_+} \sqrt{-g} g^{rr} g^{xx} a_x'(r) = \frac{L^{d-2}}{e^2} \left( f r_+^{3-d-z} \right) \frac{r_+^{z-1}}{f} = \frac{L^{d-2}}{e^2} r_+^{2-d}. \tag{147}
\]
Using the relation (85) between \( r_+ \) and \( T \), we obtain
\[
\sigma \sim T^{(d-2)/z}. \tag{148}
\]
Various comments are in order:

1. Equation (147) is an exact, closed form expression for the dc conductivity that comes from evaluating a certain radially conserved quantity at the horizon.

2. The temperature scaling of the conductivity is precisely that expected for a quantum critical system (without hyperscaling violation or an anomalous dimension for the charge density operator) [94].

3. The derivation above generalizes easily to more complicated quadratic actions for the Maxwell field, such as with a nonminimal coupling to a dilaton [91].

4. Infalling boundary conditions played a crucial role in the derivation. This boundary condition is the origin of nontrivial dissipation in holography.
The same argument given above, applied to certain perturbations of the bulk metric rather than a bulk Maxwell field, gives a direct proof of a famous result for the shear viscosity $\eta$ over entropy density $s$ in a large class of theories with classical gravity duals [91]:

$$\frac{\eta}{s} = \frac{1}{4\pi}.$$  

(149)

Here $\eta$ plays an analogous role to the dc conductivity $\sigma$ in the argument above. The shear viscosity was first computed for holographic theories in [95] and the ratio above emphasized in [92, 96].

3. Diffusive limit

The longitudinal channel includes fluctuations of the charge density. Because the total charge is conserved, this channel is expected to include a collective diffusive mode [64]. This fact is why the longitudinal equation (135) is more complicated than the transverse equation (136). It is instructive to see how the diffusive mode can be explicitly isolated from (135). We will adapt the argument in [97].

Diffusion is a process that will occur at late times and long wavelengths if we apply a source to the system to set up a nontrivial profile for the charge density, turn off the source, and then let the system evolve. Therefore diffusion should appear as a mode in the system that (i) satisfies infalling boundary conditions at the horizon and (ii) has no source at the asymptotic boundary. These are the conditions that define a so-called quasinormal mode, which correspond precisely to the poles of retarded Green’s functions in the complex frequency plane. More immediately, we must solve the longitudinal equation (135) with these boundary conditions, and in the limit of small frequency and wavevector.

We cannot simply take the limit $\omega \to 0$ of the longitudinal equation (135). This is because taking $\omega \to 0$ in this equation does not commute with the near horizon limit $r \to r_+$, at which $f \to 0$. As we take the low frequency limit, we need to ensure that the infalling boundary condition as $r \to r_+$ is correctly imposed. This can be achieved by writing

$$a_\parallel(r) = f(r)^{-i\omega/(4\pi T)}S(r).$$  

(150)

By extracting the infalling behavior (116) in this way, the equation satisfied by $S$ will no longer have a singular point at the horizon. $S$ must tend to a constant at the horizon. We can therefore safely expand $S$ in $\omega, k \to 0$. We do this by setting $\omega = \epsilon \hat{\omega}$ and $k = \epsilon \hat{k}$ and then expanding in small $\epsilon$. With a little benefit of hindsight [97], we look for a solution of the form

$$S(r) = \epsilon \hat{\omega} + \epsilon^2 s(r) + \cdots.$$  

(151)

The resulting differential equation for $s$, to leading order as $\epsilon \to 0$, can be integrated explicitly.
The solution that is regular at the horizon is
\[ s(r) = -\int_{r}^{r^+} \frac{dr'}{f(r')} \left[ \frac{i\tilde{\omega}^2 f'(r')}{4\pi T} - \frac{i r^d-d-z f'(r_+)}{4\pi T r_+^{d-3-z}} \left( \tilde{\omega}^2 r'^2 z - \hat{k}^2 f(r') r'^2 \right) \right]. \] (152)

To isolate the diffusive regime, consider \( \omega \sim k^2 \). This corresponds to taking \( \tilde{\omega} \) small. In this regime we can ignore the \( \tilde{\omega}^2 \) terms in (152). We had to keep these terms in the first instance in order to impose the regularity condition at the horizon and fix a constant of integration in the solution (152). The full solution to the order we are working can now be written
\[ S(r) = \omega + \frac{ik^2}{r^{d-2}} \int_{r}^{r^+} r'^{d-z-1} dr'. \] (153)

Here we used (82) to write \( f'(r_+) = -4\pi T r_+^{z-1} \). The remaining boundary condition to impose is the absence of a source \( a^{(0)}_\parallel \) in the near boundary expansion (137) of the Maxwell field. Thus we require \( S(0) = 0 \). Imposing this condition on (153) we obtain a diffusive relationship between frequency and wavevector:
\[ \omega = -iDk^2, \quad D = \frac{r_+^{2-z}}{d-z}. \] (154)

This is the anticipated diffusive mode. Some comments on this result:

1. Beyond finding the diffusive mode (154), one can also find the full diffusive part of the longitudinal channel retarded Green’s functions, giving a density Green’s function of the form (101), as was originally done in [85, 86].

2. The diffusion constant in (154), unlike the dc conductivity (147), is not given purely in terms of horizon data. To find the diffusive mode we had to explicitly solve the Maxwell equations everywhere. Because there is only one diffusive mode compared to the many (order ‘\( N^2 \)’) gapless modes of the IR theory, the diffusive mode is ‘not powerful’ enough to backreact on the dynamics of the IR fixed point theory. To some extent this is an artifact of the large \( N \) limit of holography. The consequence is that the diffusive mode is not fully described by the dynamics of the event horizon, but is rather a quasinormal mode in its own right, with support throughout the spacetime. A holographic Wilsonian discussion of the diffusive mode can be found in [91, 98, 99].

3. When \( d < z \), the diffusivity is UV divergent and (154) no longer holds [100]. This gives an extreme illustration of the previous comment: in this case the diffusivity is dominated by non-universal short distance physics, even while the dc conductivity (147) is captured by the universal low energy dynamics of the horizon.

4. A basic property of diffusive processes is the Einstein relation \( \sigma = \chi D \), where \( \chi \) is the charge susceptibility (compressibility). The susceptibility is obtained from the static, homogeneous two point function of \( a_t \). The easiest way to do this is to show that the following \( a_t \)
perturbation solves the linearized Maxwell equations (with \( \omega = 0 \)):

\[
a_t = \mu \left( 1 - \frac{r^{d-z}}{r_+^{d-z}} \right),
\]

(155)

where \( \mu \) is a small chemical potential. The susceptibility is given by \( \partial_\mu n(\mu, T) \) as \( \mu \to 0 \). Combining (155) and (138a), and using \( \langle J^t \rangle = n \) we find

\[
\chi = \frac{L^{d-2} d - z}{e^2 r_+^{d-z}}.
\]

(156)

It is immediately verified from (147), (154) and (156) that the Einstein relation holds.

4. \( \sigma(\omega) \) part I: Critical phases

We have emphasized in section VIIA and in equation (120) above that the real part of the frequency-dependent conductivity at zero momentum \( (k = 0) \)

\[
\text{Re} \sigma(\omega) = \frac{\text{Im} G_{j_x j_x}^R(\omega)}{\omega},
\]

(157)

is a direct probe of charged excitations in the system as a function of energy scale. We also emphasized that this quantity is difficult to compute in strongly interacting theories using conventional methods, whereas it is readily accessible holographically. Two key aspects of the holographic computation are firstly the possibility of working directly with real-time frequencies, via the infalling boundary conditions discussed in section VII B 1, and secondly the fact that dissipation occurs at leading, classical, order in the ‘t-Hooft large \( N \) expansion, and that in particular the \( N \to \infty \) and \( \omega \to 0 \) limits commute for the observable (157).

Consider first a bulk Maxwell field in a scaling geometry with exponent \( z \) as discussed in section VII C 1 above. Putting \( k = 0 \) in the equations of motion (135) or (136) for the Maxwell potential leads to

\[
\left( f r^{3-d-z} a'_x \right)' = -\frac{\omega^2 r^{2+z+1-d}}{f} a_x.
\]

(158)

We must solve this equation subject to infalling boundary conditions at the horizon. Given the solution, equations (137b) and (138b) for the near-boundary behavior of the field imply that the conductivity (139) will be given by

\[
\sigma(\omega) = \frac{L^{d-2}}{e^2} \lim_{r \to 0} \frac{1}{r^{d+z-3}} a'_x a_x.
\]

(159)

The equation (158) can be solved explicitly in two boundary space dimensions, \( d = 2 \). This is the most interesting case, in which the conductivity is dimensionless. The solution that satisfies infalling boundary conditions is

\[
a(r) = \exp \left( i \omega \int_0^r \frac{s^{z-1} ds}{f(s)} \right).
\]

(160)
The overall normalization is unimportant. It follows immediately from the formula for the conductivity (159), using only the fact that at the boundary $f(0) = 1$, that

$$\sigma(\omega) = \frac{1}{e^2}. \quad (161)$$

In particular, there is no dependence on $\omega/T!$ In the language of section VII A, this means that the, a priori distinct, constants characterizing the diffusive ($\omega \to 0$) and zero temperature ($\omega \to \infty$) limits are equal in this case: $K = C L C_D$, first noted in [40]. We have obtained this result for all $z$ in $d = 2$. We will see in section VII C6 that the lack of $\omega$ dependence is due to the electromagnetic duality enjoyed by the equations of motion of the bulk 3+1 dimensional Maxwell field, which translates into a self-duality of the boundary QFT under particle-vortex duality. Meanwhile, however, this means that in order to obtain more generic results for the frequency-dependent conductivity, we will need to depart from pure Maxwell theory in the bulk.

In the remainder of this subsection, we will restrict ourselves to the important case of CFT3s. That is, we put $d = 2$ and $z = 1$. The existence of a relevant deformation of the quantum critical theory plays an important role in the structure of $\sigma(\omega)$, because it determines the leading correction to the constant $\omega/T \to \infty$ limit. Such an operator will always be present if the critical theory is obtained by tuning to a quantum critical point. However, a quantum critical phase, by definition, does not admit relevant perturbations. In such cases one may continue to focus on the universal sector described in the bulk by the metric and Maxwell field. It is an intriguing fact that the simplest holographic theories lead naturally to critical phases rather than critical points.

An especially simple deformation of Maxwell theory that does not introduce any additional bulk fields is given by the action

$$S[A] = \frac{1}{e^2} \int d^4x \sqrt{-g} \left( -\frac{1}{4} F^2 + \gamma L^2 C_{abcd} F^{ab} F^{cd} \right). \quad (162)$$

Here $C_{abcd} = R_{abcd} - (g_{a[c} R_{d]b} - g_{b[c} R_{d]a}) + \frac{1}{3} g_{a[c} g_{d]b}$ is the Weyl curvature tensor. There is a new bulk coupling constant $\gamma$. Aspects of transport in this theory have been studied in [101–105]. The theory (162) has several appealing features. In particular, it is still second order in derivatives of the Maxwell field. Therefore the formalism we have described in the last few sections for computing conductivities does not need to be modified. As we are interested in linear response around a zero density background (that is, with no Maxwell field turned on), for the present purposes it is sufficient to consider actions quadratic in the field strength and evaluated in geometric backgrounds that solve the vacuum Einstein equations (10). The action (162) is the unique such deformation of Maxwell theory that is of fourth order in derivatives of the metric and field strength, see e.g. [102, 104]. The reason to limit the number of total derivatives is that we can then imagine that the new term in (162) is the leading correction to Einstein-Maxwell theory in a bulk derivative expansion. Indeed, such terms will be generated by stringy or quantum effects in the bulk. See the references just cited for entry points into the relevant literature on higher derivative corrections.
in string theory. The effects on the optical conductivity of terms that are higher order yet in derivatives than those in (162) were studied e.g. [106, 107].

A word of caution is necessary before proceeding to compute in the theory (162). Generically, the bulk derivative expansion will only be controlled if the coupling constant $\gamma$ is parametrically small. Considering a finite nonzero $\gamma$ while neglecting other higher derivative terms requires fine tuning that may not be possible in a fully consistent bulk theory. Remarkably, it can be shown, both from the bulk and also from general QFT arguments, that consistency requires [102, 104, 108–110]

$$|\gamma| \leq \frac{1}{12}. \quad (163)$$

In the concrete model (162) we will see shortly that this bound has the effect of bounding the dc conductivity. However, physically speaking, and thinking of (162) as representative of a broader class of models, the bound (163) is a statement about short rather than long time physics [108–110]. Specifically, for the class of theories (162) one can show that as $\omega \to \infty$ [106]

$$\sigma(\omega) = \frac{1}{e^2} \left( 1 - \frac{i}{9} \left( \frac{4\pi T}{\omega} \right)^3 + \frac{\gamma(15 + 38\gamma)}{324} \left( \frac{4\pi T}{\omega} \right)^6 + \cdots \right). \quad (164)$$

This expansion is obtained by solving (165) below in a WKB expansion. A slight correction to the $1/\omega^6$ term has been made.

Satisfying the bound (163) by no means guarantees that the bulk theory is a classical limit of a well defined theory of quantum gravity. However, it does mean that no pathology will arise at the level of computing the retarded Green’s function for the current operator in linear response theory.

The equations of motion for the Maxwell potential $A$ following from (162) are easily derived. The background geometry will be the AdS-Schwarzschild spacetime (i.e. the metric (78) with emblackening factor (83), with $z = 1$, $\theta = 0$ and $d = 2$). As above, the perturbation takes the form $A = a_x(r)e^{-i\omega t} dx$. The Weyl tensor term in the action (162) changes the previous equation of motion (158) to [103]

$$a''_x + \left( \frac{\omega^2}{f^2} a_x + \frac{12r^2}{r^3} a'_x \right) a'_x + \frac{\omega^2}{f^2} a_x = 0. \quad (165)$$

The only effect of the coupling $\gamma$ is to introduce the second term in brackets. To obtain this equation of motion, one should use a Mathematica package that enables simple computation of curvature tensors (such as the Weyl tensor). Examples are the RGTC package or the diffgeo package, both easily found online.

Near the boundary, the spacetime approaches pure AdS$_4$, which has vanishing Weyl curvature tensor, and hence the new term in the action (162) does not alter the near boundary expansions of the fields. Therefore, from a solution of the equation of motion (165), with infalling boundary
conditions at the horizon, the conductivity is again given by (159). The following Mathematica code solves equation (165) numerically. The output is plots of the conductivity $\sigma(\omega)$ for different values of $\gamma$. These plots are shown in figure 9 below. In performing the numerics, it is best to rescale the coordinates and frequencies by setting $r = r_+ \tilde{r}$ and $\omega = \tilde{\omega}/r_+$. In this way one can put the horizon radius at $r_+ = 1$ in doing calculations. Recall that $r_+ = 3/(4\pi T)$ according to (84). We must remember to restore the factor of $r_+$ in presenting the final result.

\begin{verbatim}
% emblackening factor of background metric
fs[r_] = 1 - r^3;
% equation of motion
+ (12r^-2 γ A'[r])/(1+4r^-3 γ) /. f → fs;
% series expansion of Ax near the horizon. Infalling boundary conditions
Axnh[r_,ω_,γ_] = (1-r)^(-iω/3) (1+((3+8(6-iω)-2iω)ω)/(3(1+4γ)(3i+2ω))(1-r));
% series expansion of Ax'
Axnhp[r_,ω_,γ_] = D[Axnh[r,ω,γ],r];
% small number for setting boundary conditions just off the horizon
e = 0.00001;
% numerical solution
Asol[ω_,γ_] := NDSolve[{eq[ω,γ] == 0, A[1-e] == Axnh[1-e,ω,γ], A'[1-e] == Axnhp[1-e,ω,γ]}, A, {r,0,1-e}][[1]];
% conductivity
σsol[ω_,γ_] := 1/i/ω Ax'[0]/Ax[0] /. Asol[ω,γ];
% data points for $\sigma(\omega)$
tabb[γ_] := Table[{ω, Re[σsol[ω, γ]]}, {ω, 0.001, 4, 0.05}];
% make table with different values of $\gamma$
tabs = Table[tabb[γ], {γ, -1/12, 1/12, 1/12/3}];
% make plot
ListPlot[tabbs, Axes → False, Frame → True, PlotRange → {0, 1.5}, Joined → True, FrameLabel → {3ω/(4πT), e^2 Re σ}, RotateLabel → False, BaseStyle → {FontSize → 12}]
\end{verbatim}

Figure 9 shows that for $\gamma$ nonzero, the conductivity acquires a nontrivial frequency dependence. This was the main reason to consider the coupling to the Weyl tensor in (162), which breaks the electromagnetic self-duality of the equations of motion. To leading order in small $\gamma$, electromagnetic duality now maps $\gamma → -\gamma$ [102, 103]. Furthermore we see that $\gamma > 0$ is associated to a ‘particle-like’ peak at low frequencies whereas $\gamma < 0$ is associated with a ‘vortex-like’ dip. These features
FIG. 9. **Frequency-dependent conductivity** computed from the bulk theory (162). From bottom to top, $\gamma$ is increased from $-1/12$ to $+1/12$.

can be related to the location of a ‘quasinormal pole’ or zero in the complex frequency plane. We see in the plot that larger $|\gamma|$ results in larger deviations from the constant $\gamma = 0$ result at low frequencies. The magnitude of this deviation is limited by the bound (163) on $\gamma$. Therefore, within this simple class of strongly interacting theories, the magnitude of the ‘Damle-Sachdev’ [94] low frequency peak is bounded.

5. $\sigma(\omega)$ part II: Critical points and holographic analytic continuation

It was emphasized in [111] that in order to describe quantum critical points rather than critical phases, it is essential to include the scalar relevant operator $\mathcal{O}$ that drives the quantum phase transition. A minimal holographic framework that captures the necessary physics was developed in [112]. The action is

$$S[g, A, \phi] = -\int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R + \frac{6}{L^2} + (\nabla \phi)^2 + m^2 \phi^2 - 2\alpha_1 L^2 \phi C_{abcd} C^{abcd} \right) \right. $$
$$\left. + \frac{1}{4e^2} (1 + \alpha_2 \phi) F^2 \right]. \quad (166)$$

Here the new bulk field $\phi$ is dual to the relevant operator $\mathcal{O}$. Relevant means that the operator scaling dimension $\Delta < 3$. The scaling dimension is determined by the mass squared $m^2$ according to (17) above.
The two couplings $\alpha_1$ and $\alpha_2$ in the theory (166) implement two important physical effects. The first, $\alpha_1$, ensures that $\mathcal{O}$ acquires a nonzero expectation value at $T > 0$. This is of course generically expected for a scalar operator, but will not happen unless there is a bulk coupling that sources $\phi$. Because $C_{abcd} = 0$ in the $T = 0$ pure AdS$_4$ background, this term does not source the scalar field until temperature is turned on. While the Weyl tensor thus is a very natural way to implement a finite temperature expectation value, the $C_{abcd}C^{abcd}$ term is higher order in derivatives and therefore should be treated with caution in a full theory. The second, $\alpha_2$, term ensures that there is a nonvanishing $C_{\mathcal{J}\mathcal{J}\mathcal{O}}$ OPE coefficient. Both $\alpha_1$ and $\alpha_2$ must be nonzero for the anticipated term to appear in the large frequency expansion of the conductivity. Plots of the resulting frequency-dependent conductivities can be found in [112].

An exciting use of the theory (166) is as a ‘machine’ to analytically continue Euclidean Monte Carlo data [105, 111, 112]. The conductivity of realistic quantum critical theories such as the $O(2)$ Wilson-Fisher fixed point can be computed reliably along the imaginary frequency axis, using Monte Carlo techniques. However, analytic continuation of this data to real frequencies poses a serious challenge, as errors are greatly amplified. One way to understand this fact is that the analytic structure of the conductivity in the complex frequency plane in strongly interacting CFTs at finite temperature is very rich [103]. In particular there are an infinite number of ‘quasinormal poles’ extending down into the lower half frequency plane.

The conductivity of holographic models, in contrast, can be directly computed with both Euclidean signature (imposing regularity at the tip of the Euclidean ‘cigar’) and Lorentzian signature (with infalling boundary conditions at the horizon). Solving the holographic model thereby provides a method to perform the analytic continuation, which is furthermore guaranteed to satisfy sum rules and all other required formal properties of the conductivity. A general discussion of sum rules in holography can be found in [113]. It is found that the model (166) allows an excellent fit to the imaginary frequency conductivity of the $O(2)$ Wilson-Fisher fixed point. The fit fixes the two parameters $\alpha_1$ and $\alpha_2$ in the action. The dimension $\Delta$ of the relevant operator at the Wilson-Fisher fixed point is already known and is therefore not a free parameter. Once the parameters in the action are determined, the real frequency conductivity $\sigma(\omega)$ is easily calculated numerically using the same type of Mathematica code as described in the previous subsection.

Further discussion of the use of sum rules and asymptotic expansions to constrain the frequency-dependent conductivity of realistic quantum critical points can be found in [114].

6. Particle-vortex duality and Maxwell duality

All CFT3s with a global $U(1)$ symmetry have a ‘particle-vortex dual’ or ‘S-dual’ CFT3, see e.g. [115]. The name ‘particle-vortex’ duality can be illustrated with the following simple example.
Consider a free compact boson $\theta \sim \theta + 2\pi$, with Lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial_{\mu} \theta \partial^{\mu} \theta.$$  \hspace{1cm} (167)

Vortices are (spatially) point-like topological defects where $\theta$ winds by $2\pi \times$ an integer, and so we write

$$\theta = \tilde{\theta} + \theta_{\text{vortex}} = \tilde{\theta} + \sum w_n \arctan \frac{y - y_n}{x - x_n}. \hspace{1cm} (168)$$

Here $w_n \in \mathbb{Z}$ denotes the winding number of a vortex located at $(x_n, y_n)$, and $\tilde{\theta}$ smooth. We now perform a series of exact manipulations to the Lagrangian (167), understood to be inside a path integral. We begin by adding a Lagrange multiplier $J^\mu$:

$$\mathcal{L} = -\frac{1}{2} J_\mu J^\mu - J^\mu \partial_\mu (\tilde{\theta} + \theta_{\text{vortex}}). \hspace{1cm} (169)$$

Now, we integrate out the smooth field $\tilde{\theta}$, which enforces a constraint $\partial_\mu J^\mu = 0$. The constraint is solved by writing $2J^\mu = \epsilon^{\mu\nu\rho} \partial_\nu A_\rho$, and after basic manipulations we obtain

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A^\rho \epsilon^{\rho\mu\nu} \partial_\nu \partial_\mu \theta_{\text{vortex}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\rho(x) \sum_n w_n \delta^{(2)}(x - x_n) n_\rho, \hspace{1cm} (170)$$

with $F = dA$ and $n_\rho$ a normal vector parallel to the worldline of the $n$th ‘vortex’. Evidently, the compact boson is the same as a free gauge theory interacting with charged particles. The vortices have become the charged particles, which leads to the name of the duality.

Above, we see that (167) and (170) are not the same. In some cases, such as non-compact $\mathbb{CP}^1$ models [116] or supersymmetric QED3 [117], the theory is identical after the particle-vortex transformation. These theories can be called particle-vortex self-dual. We will see that the bulk Einstein-Maxwell theory (131) is also in this class, at least for the purposes of computing current-current correlators. We firstly describe some special features of transport in particle-vortex self-dual theories.

In any system with current conservation and rotational invariance, the correlation functions of the current operators may be expressed as [40]

$$G_{\mu\nu}^{R}(\omega, k) = \sqrt{k^2 - \omega^2} \left[ (\eta_{\mu\nu} - \frac{p^\mu p^\nu}{p^2}) K^L(\omega, k) - \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) (K^L(\omega, k) - K^T(\omega, k)) \right] \hspace{1cm} (171)$$

with $p^\mu = (\omega, k)$. No Lorentz invariance is assumed. $K^L$ and $K^T$ determine the longitudinal and transverse parts of the correlator. The second term in the above equation is defined to vanish if $\mu = t$ or $\nu = t$. In [40] it was argued that particle-vortex self-duality implies

$$K^L K^T = K^2, \hspace{1cm} (172)$$

at all values of $\omega$, $k$ and temperature $T$. $K$ is the constant $\omega \rightarrow \infty$ value of the conductivity, as defined in section VII A above, and must be determined for the specific CFT. The key step in the
argument of [40] is to note that particle-vortex duality is a type of Legendre transformation in which source and response are exchanged. This occurred in the simple example above when we introduced $2J^\mu = \epsilon^{\mu\nu\rho\sigma} \partial_\nu A_\rho$. When sources and responses are exchanged, the Green’s functions relating them are then inverted. Self-duality then essentially requires a Green’s function to be equal to its inverse, and this is what is expressed in equation (172).

Setting $k \to 0$ in the above discussion, spatial isotropy demands that $K_L(\omega) = K_T(\omega) = \sigma(\omega)$. Hence, (172) in fact implies that for the self-dual theories:

$$\sigma\left(\frac{\omega}{T}\right) = \mathcal{K},$$

(173)

for any value of $\omega/T$!

The absence of a frequency dependence in (173) mirrors what we found for Einstein-Maxwell theory in (161). Let us rederive that result from the perspective of self-duality [40]. Recall that in any background four dimensional spacetime, the Maxwell equations of motion are invariant under the exchange

$$F_{ab} \leftrightarrow G_{ab} = \frac{1}{2} \epsilon_{abcd} F^{cd}.$$  

(174)

This is not a symmetry of the full quantum mechanical Maxwell theory partition function, as electromagnetic duality inverts the Maxwell coupling. However, all we are interested in here are the equations of motion. To see how (174) acts on the function $\sigma(\omega)$ we can write

$$e^2 \sigma(\omega) = \frac{1}{i\omega} \lim_{r \to 0} \frac{a^d}{a_x} = \lim_{r \to 0} \frac{F_{zx}}{F_{xt}} = \lim_{r \to 0} \frac{G_{yt}}{G_{ry}} = \lim_{r \to 0} \frac{G_{xt}}{G_{rx}} = \frac{1}{e^2 \tilde{\sigma}(\omega)}. $$

(175)

We have used isotropy in the second-to-last equality. We have written $\tilde{\sigma}(\omega)$ to denote the conductivity of the dual theory, defined in terms of $G$ rather than $F$. However, electromagnetic duality (174) means that $G$ satisfies the same equations of motion as $F$. Therefore, we must have

$$\sigma(\omega) = \tilde{\sigma}(\omega) = \frac{1}{e^2}. $$

(176)

To paraphrase the above argument: from the bulk point of view, the conductivity is a magnetic field divided by an electric field. Electromagnetic duality means we have to get the same answer when we exchange electric and magnetic fields. This fixes the conductivity to be constant.

The argument of the previous paragraph can be extended to include the spatial momentum $k$ dependence and recover (172) from bulk electromagnetic duality (174), allowing for the duality map to exchange the longitudinal and transverse modes satisfying (135) and (136) above [40]. Furthermore, an explicit mapping between electromagnetic duality in the bulk and particle-vortex duality in the boundary theory can be shown at the level of the path integrals for each theory. See for instance [40, 115, 118].

The inversion of conductivities (175) under the duality map (174) is useful even when the theory is not self-dual. We noted above that for small deformations $\gamma$, the theory (162) is mapped back
to itself with $\gamma \to -\gamma$ under (174). We can see the corresponding inversion of the conductivity clearly in figure 9.


[10] With the factors of $L$ appearing as in (16), the leading coefficient explicitly has the mass dimension $[\phi_0] = [h] = d + 1 - \Delta$, as expected for the source in the dual field theory.


[33] We will see shortly that Lorentz invariance is by no means essential for holography.


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[65] In terms of thermodynamic quantities, $\chi = \partial n / \partial \mu$, with $\mu$ the chemical potential.


For vectors and antisymmetric tensors, covariant derivatives
\[ \nabla_a X^{a\cdots} = (-g)^{-1/2} \partial_a (\sqrt{-g} X^{a\cdots}). \]


