Nature 405, 322 (2000)

Thermally fluctuating superconductors in two dimensions

Subir Sachdev and Oleg A. Starykh
Department of Physics, Yale University
P.O. Box 208120, New Haven, CT 06520-8120, USA
(January 1, 2000)

We describe the different regimes of finite temperature dynamics in the vicinity of a zero temperature superconductor to insulator quantum phase transition in two dimensions. New results are obtained for a low temperature phase-only hydrodynamics, and for the intermediate temperature quantum-critical region. In the latter case, we obtain a universal relationship between the frequency-dependence of the conductivity and the value of the d.c. resistance.

In many interesting two-dimensional superconducting systems[1,2,3], such as Josephson junction arrays, granular superconducting films, and the high temperature superconductors, it appears that the electrons bind into Cooper pairs below a pairing temperature (T_P) that is well above the Kosterlitz-Thouless transition, at T_{KT}, to long-range superconducting order[4,5,6,7,8]. It is natural to search for a direction in parameter space where T_{KT} vanishes at a $T = 0$ superconductor-insulator quantum phase transition[9,10], and to then expand in the deviation from the quantum critical point—it is not necessary for this point to be experimentally accessible for such an approach to be valuable, as it offers a controlled description of an intermediate coupling regime. We describe crossovers in the dynamics near the critical point: new results are obtained for a low temperature phase-only regime, and for the intermediate temperature 'quantum-critical' region. In the latter regime we describe the frequency (ω) dependent conductivity (σ) in terms of a single dimensionless parameter, $\gamma(T)$, which determines the d.c. conductivity; $\gamma(T)$ can also be determined by separate static measurements.

For clarity, we will present our results in the context of a familiar microscopic model for the superconductor-insulator transition; however, our results generalize to a much wider class of systems, and this will be discussed towards the end of the paper. We consider a array of superconducting quantum dots at the sites, i, of a regular two-dimensional lattice. The operator $\hat{n}_i$ measures the number of electron pairs on dot i, and $\hat{\varphi}_i$ is its canonically conjugate phase ($[\hat{n}_i, \hat{\varphi}_j] = \delta_{ij}$). We consider the Hamiltonian[11]

$$H = (E_C/2)\sum_i (\hat{n}_i - N_0)^2 - E_J \sum_{<ij>} \cos(\hat{\varphi}_i - \hat{\varphi}_j), \quad (1)$$

where N_0 is the mean integer number of Cooper pairs on each dot, $< ij >$ represents nearest neighbor pairs, E_C is the charging energy of a dot, and E_J is the Josephson tunnel coupling between dots. This model exhibits a superconductor to insulator transition as the dimensionless parameter $g = E_C/E_J$ is increased through a critical value g_c - the phase diagram in the T, g plane is summarized in Fig 1. We will be mainly concerned with the region $g \le g_c$. For $T < T_{KT}$, in the superconducting phase, the conductivity has the delta-function contribution $\mathrm{Re}[\sigma] = (e^{*2}\pi\rho_s(T)/\hbar^2)\delta(\omega)$, ($e^* = 2e$), which defines the superfluid stiffness $\rho_s(T)$ (in units of energy). The hyperscaling property of the quantum critical point implies that *all* dynamic properties in its vicinity are entirely characterized by a single energy, $\rho_s(0)$, which vanishes at $g = g_c$. In particular, the conductivity obeys[10]

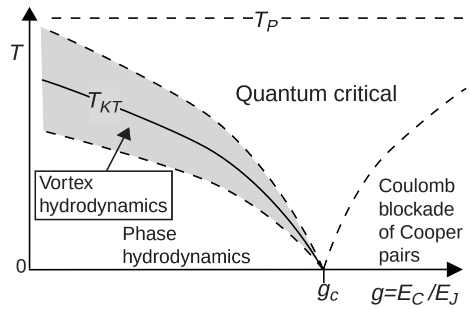

FIG. 1. Phase diagram of H in (1). Dashed lines are crossover, while the full line is a phase transition. $\rho_s(T)$ is non-zero for $g < g_c$, $T < T_{KT}$; at $T = 0$ $\rho_s(0) \sim (g_c - g)^{z\nu}$ with z the dynamic exponent and ν the correlation length exponent ($z = 1$ for H). The ground state for $g > g_c$ is an insulator. If one studies H by a tunneling probe to a good superconductor, there will be a zero voltage Josephson current in the region where $\rho_s(T)$ is non-zero, while for $g > g_c$ and low T, a Cooper pair current will only flow under a finite voltage which overcomes the Coulomb blockade.

$$\sigma(\omega, T) = \frac{e^{*2}}{h}\Sigma\left(\frac{\hbar\omega}{k_B T}, \frac{\rho_s(0)}{k_B T}\right) \quad (2)$$

where Σ is a completely universal scaling function. Such a two argument scaling form is not overly constraining, and permits a rich variety of behavior which we shall describe here as a function of the second argument $\lambda \equiv \rho_s(0)/k_B T$. Earlier studies of superfluid dynamics (made without reference to quantum phase transitions) emerge in limiting regimes, with (2) only placing restrictions on certain parameter values.

Demanding consistency of (2) with the definition of $\rho_s(T)$ above immediately leads to interesting conclusions[12]: (*i*) $\rho_s(T)/\rho_s(0)$ is a universal function of λ; (*ii*) the Kosterlitz Thouless transition occurs at the universal value $\lambda = \lambda_c$ where Σ is singular; (iii) $T_{KT} = \rho_s(0)/\lambda_c$. Aspects of the data on the high temperature superconductors are consistent with such trends[4,8], suggesting the proximity of an insulator of Cooper pairs (with stripe order).

We now describe the evolution of dynamic properties with decreasing λ.

Phase hydrodynamics: At very large λ ($k_B T \ll \rho_s(0)$) vortex excitations are exponentially suppressed, and we can derive an effective quantum action, S_φ for the continuum phase variable $\varphi(x, \tau)$ (where x is a two-dimensional spatial co-ordinate and τ is imaginary time) in a gradient expansion; the resulting action is valid only at the longest scales, and is in the spirit of the 'chiral lagrangians' of particle physics:

$$S_\varphi = \int d^2x d\tau \left[\frac{\rho_s(0)}{2} \left(\partial_\mu \varphi\right)^2 - \frac{A_1 (\hbar v)^2}{2\rho_s(0)} \left(\partial_\mu \varphi\right)^2 \left(\partial_\nu \varphi\right)^2 \right].$$

Here μ, ν are spacetime indices with $\partial_\mu = (\nabla_x, \partial_\tau / v)$, and v is the velocity of the 'spin-wave' excitations produced by the harmonic terms in S_φ (v remains non-singular through the quantum-critical point). The non-linear term, arises from integrating out amplitude fluctuations, and leads to spin-wave scattering; consistency of the resulting transport properties with (2) demands that the scattering cross-section be universally determined by $\rho_s(0)$–hence the dimensionless number A_1 is *universal*. In general spatial dimension, d, A_1 multiplies $(\hbar v)^{2/(d-1)} [\rho_s(0)]^{(d-3)/(d-1)}$, from which it is evident that the universality of A_1 holds only for $1 < d < 3$; we computed A_1 for H in an expansion[12] in $\epsilon = 3 - d$ and obtained

$$A_1 = \left[\frac{10(4\pi)^{-(d+1)/2}}{\Gamma((d+1)/2)\epsilon} \right]^{2/(d-1)} \left(1 - \frac{11\epsilon}{30} + O(\epsilon^2) \right).$$

Determination of the $T > 0$ transport properties of S_φ requires solution of the appropriate quantum kinetic equations–we will not do this here. The procedure is closely analogous to early work[13] on phonon transport in Galilean-invariant superfluids; an important difference is that these systems had a cubic non-linearity which is forbidden in our case by particle-hole symmetry.

Vortex hydrodynamics: In the shaded region of Fig 1, in the vicinity of T_{KT}, the vortices proliferate, and the dynamics can be described by a well-developed classical theory[14] for the vortices alone. This theory is contained within Σ for $\lambda \approx \lambda_c$, and compatibility constrains various prefactors. So *e.g.* for $T > T_{KT}$ the response of the free vortices is controlled by their diffusivity, $A_2 \hbar v^2 / \rho_s(0)$, and their screening length, $A_3(\hbar v / \rho_s(0)) e^{A_4/(\lambda_c - \lambda)^{1/2}}$; in general the A_{2-4} are arbitrary dimensionless scale factors—however, near $g = g_c$ they become universal numbers.

Quantum critical: Discussion of this small λ regime will occupy the remainder of the paper. Previous work[15,16,17,10] relied upon expansions in either small $3 - d$, or small $d - 1$, or large N (the number of real order parameter components). Here we shall present a theory directly for the physical case $N = 2$, $d = 2$.

Unlike the two previous regimes discussed above, it is no longer possible to decouple the spin-wave and vortex degrees of freedom. We will therefore use the complex superconducting order parameter $\psi(x, t)$ (t is real time) which is the continuum limit of the lattice operator $e^{i\hat{\varphi}_i}$, and allow for both amplitude and phase fluctuations in ψ. Our theory follows from two hypotheses: (i) The equal-time correlations of ψ are controlled by a Gaussian effective action. Evidence for this hypothesis emerged in detailed studies of order parameter correlations in the quantum-critical region[18]–the non-Gaussian components of the ψ correlations are weak because the order parameter anomalous dimension at the quantum-critical point, $\eta \approx 0.03$, is so small. (This small η also shows why ψ is preferred over the 'dual order parameter' measuring vorticity[19]–the latter has an appreciable anomalous dimension[20].) (ii) The time evolution of $\psi(x, t)$ is described by classical equations of motion. The characteristic relaxation (phase coherence) time in the quantum-critical regime is of order[21] $\hbar / k_B T$; however the dominant spectral weight is at frequencies $\hbar\omega < k_B T$, and there is good evidence[10,12] that the errors made by focusing on this low-frequency classical regime are quite small. We note that a different classical dynamic model was considered in Ref. 12, but it does not apply to transport properties.

We now define our model for quantum-critical transport, and then present its exact (numerical) solution. In additional to the field $\psi(x, t)$, we will need the canonically conjugate variable measuring density fluctuations, $\delta n(x, t)$, which is the continuum limit of $\hat{n}_i - N_0$. The equal-time correlations of ψ and δn are described by the partition function

$$\mathcal{Z} = \int \mathcal{D}\psi(x) \mathcal{D}\delta n(x) \exp(-(\mathcal{H}_1 + \mathcal{H}_2)/k_B T), \quad (3)$$

$$\mathcal{H}_1 = \int d^2x \left[|\nabla \psi|^2 + \xi^{-2}(T) |\psi|^2 \right], \quad (4)$$

$$\mathcal{H}_2 = (1/[2\chi_u(T)]) \int d^2x (\delta n(x))^2. \quad (5)$$

The overall scale of ψ is arbitrary, and has been adjusted to obtain a unit coefficient for the gradient term in (4). The correlation length, $\xi(T)$, (assumed to be larger than all microscopic scales), and the compressibility $\chi_u(T)$, are determined by the underlying quantum physics, and will generally be regarded as unknown input parameters in our dynamic theory; proximity to the quantum-critical point does (weakly) restrict their allowed T dependencies, but we can evisage a more general application of our dynamic theory, without reference to quantum criticality, in which case $\xi(T)$, $\chi_u(T)$ are arbitrary. The time evolution of ψ, δn, is defined by a minimal model—the Hamilton-Jacobi equations of $\mathcal{H}_1 + \mathcal{H}_2$, and the only non-zero Poisson bracket

$$\{\delta n(x), \psi(x')\}_{\text{P.B.}} = (i/\hbar)\psi(x)\delta^2(x-x'), \qquad (6)$$

which is the continuum, classical limit of the commutatator between $\hat{\varphi}_i$ and $\hat{n}_i$. Notice that $\hbar$ appears on the r.h.s. even though we are considering classical equations. The equations of motion implied by (4)-(6) are simple and familiar. They are the continuity equation

$$\partial \delta n(x,t)/\partial t + \nabla \cdot J(x,t) = 0, \qquad (7)$$

where $J = -(i/\hbar)(\psi^* \nabla \psi - \psi \nabla \psi^*)$, and

$$\partial \psi(x,t)/\partial t = -(i/\hbar)\Phi(x,t)\psi(x,t), \qquad (8)$$

which is the Josephson equation, with the electrochemical potential $\Phi(x,t) = \delta n(x,t)/\chi_u(T)$. We restate our dynamical model for quantum critical transport: choose a set of equal-time initial conditions for ψ and δn from the thermal ensemble defined by $\mathcal{Z}$, and evolve them deterministically by (7) and (8). Correlation functions are determined by the average over initial conditions.

We will shortly present convincing numerical evidence that (3)-(8) define a sensible continuum theory free of short distance or short time ('ultraviolet') divergences; this is supported by perturbative and renormalization group arguments, which we do not describe here. So unlike S_φ, the present theory describes the couplings of vortex and spin-wave fluctuations at different length scales. It is helpful to visualize the continuum theory by coarse-graining to a lattice spacing, a: the shorter distance degrees of freedom lead to fluctuations in the phase and amplitude of ψ, but the long distance transport properties are insensitive to the value of a. The absence of factors of a in the final results allows us to deduce their functional dependence on $\xi(T)$, $\chi_u(T)$ by simple engineering dimensional analysis. In this manner we conclude that

$$\sigma(\omega, T) = \frac{e^{*2}}{h}\gamma(T)\Sigma_c\left(\gamma(T)\frac{\hbar\omega}{k_B T}\right), \qquad (9)$$

where Σ_c is a universal function ('exact' numerical results are below), and the dimensionless $\gamma(T)$ is defined by

$$\gamma(T) \equiv [k_B T \chi_u(T) \xi^2(T)]^{1/2}. \qquad (10)$$

Consistency of (9) with (2) only requires that $\gamma(T)$ is a universal function of λ (also recall that the present quantum-critical theory is valid only for small λ). The result (9) has a clear experimental signature: it implies a correlation between the value of the d.c. conductivity and the inverse frequency-width of $\sigma(\omega)$, as they are both determined by $\gamma(T)$. More stringent comparisons can be made by using the measured d.c. conductivity to determine the unknown $\gamma(T)$, and then using our numerical results below for Σ_c to determine the ω dependence of σ. It should be noted that our present computations for Σ_c will not apply for very large $\hbar\omega/k_B T$, for then a full quantum theory is necessary, and we crossover to the phase-coherent regime discussed earlier[6,15].

Exactly at $g = g_c$, hyperscaling arguments imply that $\xi(T) \sim T^{-1/z}$ and $\chi_u(T) \sim T^{(2-z)/z}$; so $\gamma(T)$ is T-independent for any z, and is expected to be a universal number. For the model H in (1), this universal number can be computed[18] in a $1/N$ expansion:

$$\gamma(T)|_{g=g_c} = \left[\frac{\sqrt{5}}{4\pi \ln((\sqrt{5}+1)/2)}\right]^{1/2}\left(1 - \frac{0.5468}{N}\right). \qquad (11)$$

We turn to our numerical results for Σ_c. The simulations were carried out on an $N \times N$ square lattice of spacing a with periodic boundary conditions, and $a \ll \xi(T) \ll Na$. For each a, successively larger values of N were used, until the results became N-independent, and the continuum limit was then approached as $a \to 0$. The lattice form of $\mathcal{H}_1 + \mathcal{H}_2$ was obtained by mapping the momentum (k_x, k_y) dependence of the couplings of the continuum Hamiltonian under $k_x^2 \to (4/a^2)[K_1 \sin^2(k_x a/2) + K_2 \sin^2(k_x a) + K_3 \sin^2(3k_x a/2)]$, and similarly for k_y (for $|\nabla\psi|^2$, this amounts to including first, second, and third neighbor couplings between the ψ). We chose two different sets of values for $K_{1,2,3}$ to test the independence of the continuum theory on the lattice realization—(A) the familiar $K_1 = 1$, $K_2 = K_3 = 0$, and (B) $K_1 = 3/2$, $K_2 = -3/20$, $K_3 = 1/90$ for which $k_x^2 \to k_x^2(1 + O(k_x a)^6)$. The initial conditions specified by $\mathcal{Z}$ form a Gaussian ensemble, and are easily generated in a single sweep. The time evolution was carried out by a fourth order predictor-corrector algorithm with a time step determined to conserve total energy to a relative accuracy better than 10^{-5}. We measured the autocorrelation function of the total current, C_J, by averaging over 3000 initial conditions; C_J is normalized such that

$$\Sigma_c(\overline{\omega}) = \int_0^\infty d\overline{t}\, C_J(\overline{t})\, e^{i\overline{\omega}\,\overline{t}}, \qquad (12)$$

where $\overline{\omega} = \gamma(T)\hbar\omega/(k_B T)$ and $\overline{t} = k_B T t/(\hbar\gamma(T))$.

Our results for C_J and Σ_c are in Fig 2. Notice the a dependence at $\overline{t} = 0$—this is expected as an analytic calculation of equal-time correlations shows that $C_J(0) = (1/\pi)\ln(1/a) + \ldots$. However, the a dependence quickly disappears at small non-zero $\overline{t}$, and then a universal continuum value obtains; this is strong evidence for the existence of the continuum theory (3)-(8). We obtained $\Sigma_c(0) \approx 1.85$; for the model H in (1), this combines with (11) to yield $\sigma(0,T) = 0.82(e^{*2}/h)$ at $g = g_c$.

The experiments of Rimberg *et al.*[2] appear to be a convenient testing ground for our theory. The metallic gate screens the long-range Coulomb interactions and so justifies[22] the short-range coupling in $\mathcal{H}_2$. However, it also introduces dissipation, which will induce additional terms in the equations of motion *e.g.* (8) is modified to

$$\partial\psi/\partial t = -\Gamma\delta\mathcal{H}_1/\delta\psi^* - (i/\hbar)\Phi\psi + \zeta, \qquad (13)$$

where Γ is a damping co-efficient and ζ is the associated random noise. Fortunately, it can be shown, as in

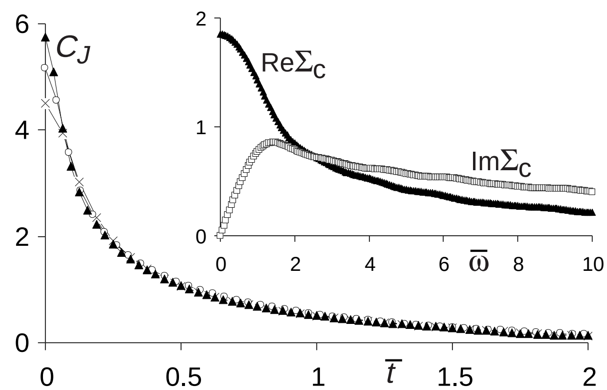

FIG. 2. Current autocorrelation, C_J, of (3)-(8), with lattice realization A, $N = 64$, and $a/\xi(T) = 1/8$ (crosses), $\sqrt{2}/16$ (open circles), 1/16 (triangles). Inset: real and imaginary parts of Σ_c, obtained from (12) for $a/\xi(T) = 1/16$; for large $|\overline{\omega}|$ we have $\text{Re}\Sigma_c \sim 1/|\overline{\omega}|$ and $\text{Im}\Sigma_c \sim \ln(|\overline{\omega}|)/\overline{\omega}$.

Refs 23, that for large $\xi(T)$, such modifications leave the dynamic function Σ_c *unchanged*. The damping can be a relevant perturbation on the $T = 0$ quantum critical point[24], but this manifested only via changes in $\xi(T)$, $\chi_u(T)$, and $\gamma(T)$. (Outside the quantum critical region, the change in quantum universality leaves the form (2) unchanged, but with a new Σ; the structure of the phase hydrodynamics, S_φ can also be modified. Similar comments apply to the following paragraph.)

As a second modification, we consider the inclusion of long-range Coulomb interactions. This will modify $\mathcal{H}_2$ to $e^{*2} \int d^2x d^2x' \delta n(x) \delta n(x')/2|x - x'|$, and leave (7,8) unchanged, but with $\Phi(x,t) = e^{*2} \int d^2x' \delta n(x',t)/|x - x'|$. Assuming the existence of the continuum limit of the classical dynamic theory, we again obtain (9), but with $\gamma(T) = [k_B T \xi(T)/e^{*2}]^{1/2}$; the functional form of Σ_c will of course be changed and requires a separate numerical simulation. Note that at $g = g_c$, $\gamma(T)$ is a T-independent universal number only if $z = 1$, the value expected for long-range interactions[6].

The experiments of Corson *et al.*[3] on an underdoped high temperature superconductor observe scaling closely related to (9), with the frequency scale proportional to the resistivity scale, and a scaling function with large ω behavior similar to that in Fig 2. However their fits use a prefactor T_θ^0 whose weak T-dependence disagrees with our continuum two-dimensional theory. We speculate that this can be explained by corrections to scaling which are appreciable because both modifications discussed above are present here – damping from fermionic excitations and long-range Coulomb interactions. Corson *et al.*[3] motivated the scaling using the vortex theory[14], but did not consider bound vortex pairs, whose contributions do not obey their scaling assumptions.

In summary, this paper has delineated the distinct dynamic regimes of thermally fluctuating superconductors: the low T phase-only hydrodynamics, the classical vortex hydrodynamics in the vicinity of T_{KT}, and the quantum-critical region where phase and vortex fluctuations strongly coupled—here we proposed a dynamical model in which non-linear 'mode-coupling' terms demanded by the Poisson bracket (6) dominate the universal, low-frequency, dissipative dynamics. Our approach could be extended to other two-dimensional systems: (*i*) the magnetic field-tuned superconductor-insulator transition, where a coupling to the external field would be required in (4), and (*ii*) quantum Hall transitions.

We thank J. Orenstein, D. Grempel, and A. Sudbø for useful discussions. This research was supported by NSF Grant No DMR 96–23181.

1. Y. Liu and A. M. Goldman, Mod. Phys. Lett. B **8**, 277 (1994) and references therein; J. A. Chervenak and J. M. Valles, Phys. Rev. B **59**, 11209 (1999); N. Markovic et al, Phys. Rev. B **60**, 4320 (1999).
2. A. J. Rimberg et al., Phys. Rev. Lett. **78**, 2632 (1997).
3. J. Corson et al, Nature, **398**, 221 (1999).
4. Y. J. Uemura et al., Phys. Rev. Lett. **62**, 2317 (1992).
5. S. Doniach and M. Inui, Phys. Rev. B **41**, 6668 (1990).
6. M. P. A. Fisher, G. Grinstein and S. M. Girvin, Phys. Rev. Lett. **64**, 587 (1990).
7. N. Trivedi and M. Randeria, Phys. Rev. Lett. **75**, 312 (1995); A. Ghosal et al., Phys. Rev. Lett. **81**, 3940 (1998).
8. V. J. Emery and S. A. Kivelson, Nature **374**, 434 (1995); J. Phys. Chem. Solids, **59**, 1705 (1998).
9. S. L. Sondhi, S. M. Girvin, J. P. Carini, and D. Shahar, Rev. Mod. Phys. **69**, 315 (1997).
10. S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1999).
11. S. Doniach, Phys. Rev. B **24**, 5063 (1981).
12. S. Sachdev, Phys. Rev. B **55** 142 (1997); *ibid* **59**, 14054 (1999).
13. I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (W. A. Benjamin, New York, 1965); V. N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics* (D. Reidel, Boston, 1983).
14. V. Ambegaokar et al., Phys. Rev. Lett. **40**, 783 (1978).
15. K. Damle and S. Sachdev, Phys. Rev. B **56**, 8714 (1997).
16. I. F. Herbut, Phys. Rev. Lett. **81**, 3916 (1998).
17. S. Sachdev, Phys. Rev. B **57**, 7157 (1998).
18. A. V. Chubukov, S. Sachdev, and J. Ye, Phys. Rev. B **49**, 11919 (1994).
19. M. P. A. Fisher, Phys. Rev. Lett. **65**, 923 (1990).
20. A. Sudbø, A. K. Nguyen, and J. Hove, cond-mat/9907386.
21. S. Sachdev and J. Ye, Phys. Rev. Lett. **69**, 2411 (1992).
22. K.-H. Wagenblast et al., Phys. Rev. Lett. **79**, 2730 (1997).
23. K. Kawasaki, Ann. Phys. (N.Y.) **61**, 1 (1970); P. C. Hohenberg, E. D. Siggia, and B. I. Halperin, Phys. Rev. B **14**, 2865 (1976); D. Grempel, Phys. Rev. Lett. **61**, 1041 (1988).
24. K.-H. Wagenblast et al., Phys. Rev. Lett. **78**, 1779 (1997); D. Dalidovich and P. Phillips, cond-mat/9907123.