

Multicritical Crossovers near the Dilute Bose Gas Quantum Critical Point

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Many zero temperature transitions, involving the deviation in the value of a U(1) conserved charge from a quantized value, are described by the dilute Bose gas quantum critical point. On such transitions, we study the consequences of perturbations which break the symmetry down to Z_N in d spatial dimensions. For the case $d = 1$, $N = 2$, we obtain exact, finite temperature, multicritical crossover functions by a mapping to an integrable lattice model. [S0031-9007(96)00399-7]

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The zero temperature (T), quantum phase transition in a dilute Bose gas has recently attracted some interest [1–5] because of its importance in a variety of different physical situations. For bosons with repulsive interactions in a chemical potential, μ , the quantum-critical point is at $\mu = 0$, where the $T = 0$ density of bosons has a nonanalytic dependence on μ . This quantum-critical point controls the finite T quantum-classical crossovers in the dilute Bose gas [1–4]; in addition, it is the critical theory for (i) the Mott-insulator to superfluid transition in a lattice boson system at a generic μ [2], (ii) the onset of uniform magnetization in a gapped quantum antiferromagnetic in a magnetic field [4], (iii) the deviation from a saturated polarization in a quantum ferromagnet [5], and possibly other physical systems [5]. A common feature of all these systems is that the Hamiltonian always has at least a global U(1) symmetry, and the transition involves a deviation in the

expectation value of the U(1) conserved charge from a quantized (possibly zero) value.

In this paper we will study the consequences of perturbations which break the U(1) symmetry down to Z_N (the cyclic group of N elements), and compute associated exponents and crossover functions. This is of interest, e.g., in the system (ii), where crystal-field effects reduce the symmetry down to Z_2 in the experimental realization of such a quantum-critical point in NENP [6]. Our most detailed results will be for $N = 2$ and spatial dimension $d = 1$, which is also the most interesting and nontrivial case: we shall obtain explicit results for multicritical crossovers as a function of μ , T , and the strength of the U(1)-breaking perturbation. These are the first nontrivial, exact results for universal finite T crossovers near a quantum multicritical point in any system.

We begin by reviewing the U(1) symmetric Bose gas theory. The theory is described by the action

$$S_0 = \int d^d x \int_0^{1/T} d\tau \left(\Psi_B^\dagger \frac{\partial \Psi_B}{\partial \tau} + \frac{1}{2m} |\nabla_x \Psi_B|^2 - \mu |\Psi_B|^2 + \frac{u}{2} |\Psi_B|^4 \right), \quad (1)$$

where Ψ_B is a complex scalar field, m is the boson mass, and we use $\hbar = k_B = 1$. A $T = 0$ renormalization group analysis shows that the interaction u is irrelevant for $d > 2$ [2]. Near $d = 2$, we define the dimensionless bare coupling g_0 by $u = (2m)\kappa^\varepsilon g_0/S_d$ where κ is a renormalization momentum scale, $S_d = 2\pi^{d/2}/(2\pi)^d \Gamma(d/2)$ is a phase space factor, and $\varepsilon = 2 - d$. The renormalized theory is defined by a single coupling constant renormalization of g_0 to g , and no other renormalizations are necessary. This renormalization leads to the β function

$$dg/d \ln \kappa = -\varepsilon g + A_d g^2, \quad (2)$$

where $A_d = 1/2 + \mathcal{O}(\varepsilon)$ is a known constant independent of g , with the value of the $\mathcal{O}(\varepsilon)$ terms depending upon the precise renormalization condition. The result (2) is valid to all orders in g . The quantum-critical point of S_0 is at $\mu = 0$, and has dynamic exponent $z = 2$, correlation length exponent $\nu = 1/2$ for all d [2]. At $T = 0$ and μ small, the boson density $n = \langle |\Psi_B|^2 \rangle$ obeys [4] $n = \Lambda \theta(\mu) 2m\mu$ for $d > 2$ (Λ is a nonuniversal, cutoff dependent, constant), while for $d < 2$ we have $n = C_d \theta(\mu) (2m\mu)^{d/2}$, with

C_d a universal number which has nontrivial contributions at each order in ε [5] due to the fixed-point interaction $g = g^* = \varepsilon/A_d$.

We now break the symmetry down to Z_N . The most relevant perturbation which accomplishes this is

$$S_N = \int d^d x \int_0^{1/T} d\tau (\lambda_N \Psi_B^N + \text{c.c.}), \quad (3)$$

where the coupling λ_N can be taken to be real, without loss of generality. For $d > 2$, the scaling dimension of λ_N is simply its canonical dimension, $d + 2 - Nd/2$; so λ_2 is relevant for all $d > 2$ [$\dim(\lambda_2) = 2$], λ_3 is relevant for $2 < d < 4$, while $\lambda_{N>3}$ are irrelevant for all $d > 2$. For $d < 2$, we need to consider the renormalization of Ψ_B^N insertions; these are nontrivial because S_N creates or annihilates N bosons, and these preexisting bosons can then interact. The two-loop computation of this renormalization yields the scaling dimensions

$$\dim(\lambda_N) = 4 - N - (N^2/2 - N + 1)\varepsilon + (N/2) \times (N - 1)(N - 2) \ln(4/3)\varepsilon^2 + \mathcal{O}(\varepsilon^3). \quad (4)$$

Notice that the $\mathcal{O}(\varepsilon^2)$ term vanishes for $N = 2$. This is also true for all subsequent terms, as the only renormalization of Ψ_B^2 comes from a single series of ladder diagrams, and we have $\dim(\lambda_2) = 2 - \varepsilon$ to all orders in ε ; there is no similar simplification for $N \geq 3$. Later, we will verify the value of $\dim(\lambda_2)$ in $d = 1$ by an entirely different method. So λ_2 is relevant in $d = 1$. Evaluation of the series (4) at $\varepsilon = 1$ predicts that $\lambda_{3 \leq N \leq 5}$ are irrelevant in $d = 1$, while the $\lambda_{N \geq 6}$ are again relevant. This result for $N \geq 6$ is surely an artifact of the poorer accuracy of the series at larger N , and it is likely that all $\lambda_{N \geq 3}$ are irrelevant in $d = 1$.

The λ_2 perturbation is expected to drive the quantum phase transition in $S_0 + S_2$ (accessed by the control parameter μ) into the universality class of the transverse-field Ising model. This latter class has $z = 1$ and upper critical dimension $d = 3$. We will not discuss the details of the crossover between the U(1) symmetric, dilute Bose gas fixed point and the transverse-field Ising fixed point for general d here: we shall confine our attention in

the remainder of the paper to the $d = 1$ case where *both* fixed points are below their respective upper critical dimensions.

We now present a detailed analysis for $d = 1, N = 2$. It has been argued that the $d = 1$ critical properties of the action S_0 are identical to those of the $\mu = 0$ critical point in a dilute, spinless, noninteracting, Fermi gas [4]. The Bose field, Ψ_B , is related to the Fermi field, Ψ_F , by a continuum Jordan-Wigner transformation,

$$\Psi_B(x) = \exp\left(-i\pi \int_{-\infty}^x dx' \Psi_F^\dagger(x') \Psi_F(x')\right) \Psi_F(x) \quad (5)$$

(we are momentarily interpreting Ψ_B, Ψ_F as operators). We can use this mapping to deduce the mapping of S_2 into the fermionic theory. We point split $\Psi_B^2(x)$ into $\Psi_B(x)\Psi_B(x+a)$, rewrite in terms of Ψ_F using (5), and expand in powers of a . Retaining only low order terms, we obtain the following fermionic form for $S_0 + S_2$ in $d = 1$:

$$S_F = \int dx \int_0^{1/T} d\tau \left(\Psi_F^\dagger \frac{\partial \Psi_F}{\partial \tau} + \frac{1}{2m} |\partial_x \Psi_F|^2 - \mu |\Psi_F|^2 + \frac{\tilde{\lambda}_2}{2} (\Psi_F^\dagger \partial_x \Psi_F^\dagger - \Psi_F \partial_x \Psi_F) \right), \quad (6)$$

where $\tilde{\lambda}_2 \propto \lambda_2$, and Ψ_F is a complex Grassmanian field. The action S_F is quadratic in fermionic fields, and correlations of the Ψ_F can be computed. In particular, it can be shown that all other terms derivable from $S_0 + S_2$ are irrelevant at the $\mu = 0, \tilde{\lambda}_2 = 0$ multicritical point. So we expect the universal, multicritical crossover functions to emerge from an analysis of S_F . Note also, from simple power counting, that $\dim(\tilde{\lambda}_2) = 1$, which agrees with our earlier result for $\dim(\lambda_2)$, obtained by expanding to all orders in ε .

The $T = 0$ phase diagram of S_F is shown in Fig. 1. The $z = 2$, dilute Bose gas critical point is M , and it is the point of intersection of three second-order phase transition lines; there is a gap to all excitations everywhere, except at M and along these three lines. (i) The line along $\tilde{\lambda}_2 = 0$ has U(1) symmetry, and describes the $d = 1$ Bose gas with quasi-long-range order. At sufficiently long scales and for $\mu > 0$, this line is described by its own critical theory which has $z = 1$, is conformally invariant, and has central charge $c = 1$. There is an operator $(|\Psi_F^\dagger \partial_x \Psi_F|^2)$ which is marginal along this line, and is responsible for the continuously varying exponents at $c = 1$; however, this operator is *irrelevant* at the critical end point M [4] and can be ignored while computing the multicritical crossovers of M . Under this condition, the $c = 1$ theory can be written as two copies of the $c = 1/2$ Ising field theory. (ii) The lines $\mu = 0, \tilde{\lambda}_2 > 0$ and $\mu = 0, \tilde{\lambda}_2 < 0$ are also conformally invariant at long scales, and are then described by a single $z = 1, c = 1/2$ Ising field theory. The nonzero expectation values for $\langle \Psi_B \rangle$ (Fig. 1) appear only at $T = 0$, and we always have $\langle \Psi_B \rangle = 0$ for $T > 0$; this will become clear from our computations below.

We now wish to describe the finite T crossovers in the vicinity of M . We will study the two-point correlators of Ψ_B . It is useful to define $\Psi_X \equiv \Psi_B + \Psi_B^\dagger$ and

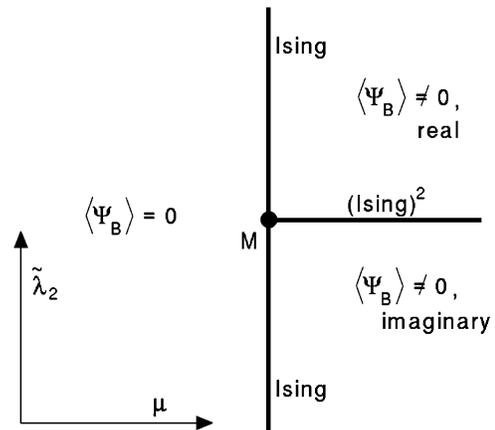


FIG. 1. Phase diagram of S_F (equivalent to $S_0 + S_2$ in $d = 1$) at $T = 0$. The lines denote second-order quantum phase transitions with gapless excitations, and there is gap to all excitations elsewhere. The multicritical point M is at $\mu = 0, \tilde{\lambda}_2 = 0$, and has $z = 2$. The action has U(1) invariance only for $\tilde{\lambda}_2 = 0$, and the conserved U(1) charge, $\langle \Psi_B^\dagger \Psi_B \rangle = \langle \Psi_F^\dagger \Psi_F \rangle$, is quantized at zero for $\tilde{\lambda}_2 = 0, \mu < 0$. On the critical line $\mu > 0, \tilde{\lambda}_2 = 0$, this charge varies continuously and the system is described at sufficiently long scales by the conformally and U(1) invariant $(\text{Ising})^2$ theory with $z = 1$, central charge $c = 1$. Similarly, the critical lines $\mu = 0, \tilde{\lambda}_2 > 0$ and $\mu = 0, \tilde{\lambda}_2 < 0$ eventually map on to the conformally invariant ($z = 1$) Ising field theory with $c = 1/2$. The point M is the only truly scale invariant point, but it is not conformally (or even Lorentz) invariant.

$\Psi_Y \equiv -i(\Psi_B - \Psi_B^\dagger)$. Then, elementary considerations show that, for $\tilde{\lambda}_2$ real, $\langle \Psi_X \Psi_Y \rangle = 0$ and $\langle \Psi_X \Psi_X \rangle_{\tilde{\lambda}_2} = \langle \Psi_Y \Psi_Y \rangle_{-\tilde{\lambda}_2}$. So it is sufficient to compute $\langle \Psi_X \Psi_X \rangle$ for both signs of $\tilde{\lambda}_2$. We will describe the long distance behavior of its equal time correlation, where we expect

$$\lim_{|x| \rightarrow \infty} \langle \Psi_X(x, \tau) \Psi_X(0, \tau) \rangle \sim A e^{-|x|/\xi} \quad \text{for } T > 0. \quad (7)$$

The correlation length, ξ , and the amplitude, A , obey the multicritical scaling forms

$$\xi^{-1} = (2mT)^{1/2} F(x, y), \quad A = 2(2mT)^{1/2} G(x, y), \quad (8)$$

where F and G are fully universal scaling functions of the dimensionless variables

$$x = \mu/T, \quad y = (2m)^{1/2} \tilde{\lambda}_2/T^{1/2}. \quad (9)$$

The powers of T in (8) and (9) follow from the exponents and scaling dimensions at M : $z = 2$, $\nu = 1/2$, $\dim(\mu) = 2$, $\dim(\tilde{\lambda}_2) = 1$, and $\dim(\Psi_B) = 1/2$ (the mass, m , is not to be interpreted as a scaling variable; it converts between the engineering dimensions of space and time, and is analogous to the velocity of light in a Lorentz invariant theory).

We will now provide an exact, closed-form, computation of the functions F and G . Our strategy is to perform the computation in an integrable lattice model with a multicritical point in the universality class of M . It turns out that the well-known Lieb-Schultz-Mattis [7] spin chain has two such critical points. This spin chain is described by the Hamiltonian

$$H = - \sum_i \{ J[(1 + \gamma) \sigma_i^x \sigma_{i+1}^x + (1 - \gamma) \sigma_i^y \sigma_{i+1}^y] + \Delta \sigma_i^z \}, \quad (10)$$

where $\sigma^{x,y,z}$ are Pauli matrices on the sites i of an infinite chain, $J > 0$. The Jordan-Wigner transformation maps H into a model of free, spinless, lattice fermions, and its phase diagram can then be computed exactly [7–9]; the result is shown in Fig. 2. The points M_1 , M_2 are in both the universality class of M and the continuum limit of the Jordan-Wigner transform of H yields precisely S_F ; near M_1 we find $m = \hbar^2/4Ja^2$, $\tilde{\lambda}_2 = 4J\gamma a$, $\mu = 2\Delta + 4J$ (a is the lattice spacing), and the operator correspondences $\Psi_X \sim \sigma^x$, $\Psi_Y \sim \sigma^y$. We note that although continuum limits of H have been studied earlier [9], the identification of the universality class of M_1 , M_2 has not been made.

The required Ψ_X correlators can be obtained from a scaling analysis of results of Barouch and McCoy [8] on finite T correlators. Express σ^x in terms of the Jordan-Wigner fermions, and evaluate the resulting fermion correlators; this yields an expression in the form of a Toeplitz determinant [7,8]

$$\langle \sigma_i^x \sigma_{i+n}^x \rangle = \det(D_{\ell-m})_{\ell=1\dots n, m=1\dots n}, \quad (11)$$

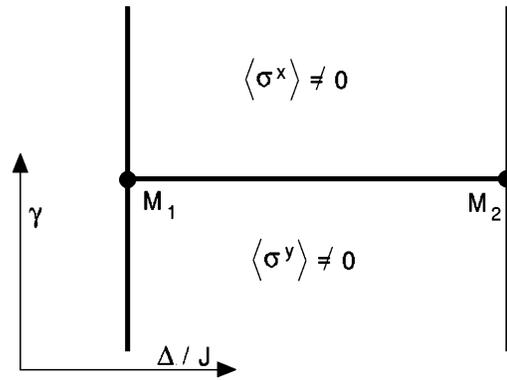


FIG. 2. Phase diagram of the lattice spin-1/2 model H at $T = 0$ (the classical model, with $\sigma^{x,y,z}$ c -numbers, has a $T = 0$ phase diagram with the same phases and topology). The expectation values of σ_x and σ_y vanish unless otherwise noted. The point M_1 is at $\gamma = 0$, $\Delta/J = -2$, while the point M_2 is at $\gamma = 0$, $\Delta/J = 2$. The vicinities of both M_1 and M_2 map independently onto the continuum model of Fig. 1.

where $D_\ell = \int_0^{2\pi} (d\phi/2\pi) e^{-i\ell\phi} \tilde{D}(\phi)$ with $\tilde{D}(\phi) = e^{i\phi} \times \tanh[|E(\phi)|/k_B T] E(\phi)/|E(\phi)|$ and $E(\phi) = 2J \cos \phi + \Delta - i2J\gamma \sin \phi$. It now remains to take the large n limit of (11); in general, this is quite difficult, and leads to a computation of considerable complexity [8]. However, in a limited portion of the phase diagram ($|\Delta/2J| < 1$, $\gamma > 0$ which corresponds to $x > 0$, $y > 0$ in the scaling limit associated with M_1) the computation is simpler because it is possible to directly apply Szego's lemma [10]. This limited result is all we shall need as it is possible to deduce the scaling functions elsewhere by the powerful requirement that both F and G are analytic for all real, finite x and y [11]. This analyticity is a consequence of the absence of thermodynamic singularities in one dimensional quantum systems at any finite temperature. The use of analyticity was essential in our being able to express the final results in a compact form, and it would have been practically impossible to see the hidden structure in the very lengthy results of Ref. [8] otherwise.

In its region of applicability, Szego's lemma [10] tells us

$$\lim_{n \rightarrow \infty} \langle \sigma_i^x \sigma_{i+n}^x \rangle \sim e^{n\alpha_0} \exp\left(\sum_{p=1}^{\infty} p \alpha_p \alpha_{-p} \right), \quad (12)$$

where $\ln \tilde{D}(\phi) = \sum_{p=-\infty}^{\infty} \alpha_p e^{ip\phi}$. We can read off results for ξ and A by comparing (12) with (7). We obtained for the scaling function of the correlation length

$$F(x, y) = \int_0^\infty \frac{dq}{\pi} \ln \coth \frac{[q^2 y^2 + (x - q^2)^2]^{1/2}}{2} + \theta(-y) |y| + \theta(-x) \frac{(y^2 - 4x)^{1/2} - |y|}{2}. \quad (13)$$

Only the $x > 0$, $y > 0$ portion of (13) was obtained from (12); the remainder was deduced by the requirement of

analyticity. Indeed, even though it appears otherwise, the result (13) is, in fact, a smooth, differentiable function of x, y for all real x, y including along the lines $x = 0$ and $y = 0$.

It is quite interesting to see how the Ising and (Ising)² behavior of the critical lines in Fig. 1 emerges from (13). First, we observe that along the line $\tilde{\lambda}_2 = 0$

$$f_B(x) = F(x, 0), \quad (14)$$

where f_B is precisely the correlation length crossover function for the U(1) invariant dilute Bose gas in the form presented in [4,11], where it was obtained from the results of Ref. [12]. The Ising transition realized by crossing the $\mu = 0$ axis for finite $\tilde{\lambda}_2 > 0$ is obtained as follows:

$$f_I(x) = \lim_{y \rightarrow \infty} yF(x, y), \quad (15)$$

where f_I is now the correlation length crossover function of the transverse-field Ising model obtained in Ref. [11].

$$\ln[G(x, y)] = - \int_x^\infty dx' \left\{ 2 \left(\frac{\partial F(x', y)}{\partial x'} \right)^2 - \frac{1}{x'} \left(y \frac{\partial F(x', y)}{\partial x'} + \frac{\partial F(x', y)}{\partial y} \right)^2 \right\} - \int_x^{4/y^2} \frac{dx'}{4x'}. \quad (17)$$

Both integrands above are singular at $x' = 0$, but the singularities cancel in the sum; it can be verified that the expression (17) defines G as a function which is smooth for all real x, y , as required. The limiting behavior of G near the critical lines of Fig. 1 is similar to that of F ; we have $g_B(x) = G(x, 0)$ which is the analog of (14) [with g_B now the crossover function of the amplitude of the U(1) invariant dilute Bose gas [11,12]], $\ln[g_I(x)] = \lim_{y \rightarrow \infty} \ln[(2/y)^{1/2} G(x, y)]$ as the analog of (15) for Ising transition line (with g_I the crossover functions of the amplitude of the Ising model [11]), and $\ln[g_I(s)] = (1/2) \lim_{x \rightarrow \infty} \ln[2^{1/2} G(x, y = s/x^{1/2})]$ as the analog of (16) for the (Ising)² transition. Establishing these limits required the use of the recently introduced [11] identities obeyed by Glaisher's constant.

From the above finite T results for the amplitude, we can also deduce the value of the spontaneous magnetization: $\lim_{T \rightarrow 0} A = \langle \Psi_X \rangle^2$ in a regime where $\lim_{T \rightarrow 0} \xi^{-1} = 0$. This method gives us a $T = 0$ universal function Φ ,

$$\langle \Psi_X \rangle = \theta(\mu) \theta(\tilde{\lambda}_2) (2m\mu)^{1/4} \Phi \left(\frac{(2m)^{1/2} \tilde{\lambda}_2}{\mu^{1/2}} \right), \quad (18)$$

with $\Phi(s) = (2s)^{1/4}$, which describes the crossover of the spontaneous magnetization between the Ising and (Ising)² lines of Fig. 1.

Finally, we note that exact, finite temperature, crossover functions of spin correlators near bulk quantum-critical points, below their upper critical dimension, had previously been computed only for the $d = 1$ dilute Bose gas [4,11,12] and the $d = 1$ transverse-field Ising model [11]; our results (13) and (17) for F, G contain these earlier re-

The prefactor of y on the right-hand side of (15) ensures that f_I is multiplied by a power of T appropriate to the $z = 1$ Ising transition. Crossing the $\mu = 0$ axis for $\tilde{\lambda}_2 < 0$ requires one to compute $\langle \Psi_Y \Psi_Y \rangle$ to obtain f_I . Finally, the (Ising)² transition realized by crossing the $\tilde{\lambda}_2$ axis for $\mu > 0$ is characterized by the limit

$$f_I(s) = \lim_{x \rightarrow \infty} x^{1/2} F(x, y = s/x^{1/2}). \quad (16)$$

Precisely the same function f_I emerges in the very distinct limits in (15) and (16).

For the amplitude of the correlation function, the computation of $G(x > 0, y > 0)$ from (12) [8] leads to a very lengthy and complicated result. However, it was found that the analogous result for the Ising model [11] simplified considerably when expressed in terms of derivatives of the correlation length crossover function. We found the same remarkable simplification here, and the analytic continuation to all x, y was then straightforward; we obtained

results as limiting cases, and also (universally) interpolate between them.

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