Continuum Quantum Ferromagnets at Finite Temperature and the Quantum Hall Effect

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We study finite temperature (T) properties of the continuum quantum field theory of systems with a ferromagnetic ground state. A scaling theory of the $T = 0$ system is discussed carefully, and its consequences for crossovers between different finite T regimes in dimensions 1, 2, and 3 are described. The results are compared with recent NMR measurements of the magnetization of a quantum Hall system with filling factor $\nu = 1$; we predict that the relaxation rate $1/T_1$ of this system may have a finite T "ferromagnetic coherence peak."

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The recent availability of nuclear magnetic resonance (NMR) measurements [1] of quantum Hall systems has opened a new window into the magnetic properties of a strongly correlated two-dimensional electronic system. Initially, the filling factor (ν) dependence of the zero temperature (T) magnetization in the vicinity of $\nu = 1$ attracted attention because it indicated that the low energy charged excitations of the system were spin textures (Skyrmions) [2–4]. In this paper, we examine instead the T dependence of the magnetic properties exactly at $\nu = 1$ [5]. We use a continuum quantum field theory of a ferromagnet as a model and describe its finite T properties. From a field-theoretical perspective, some features of this quantum field theory are rather unusual and lead to a noteworthy universality in the crossover functions. We will consider all values of the spatial dimension $d > 0$, although the regime of validity of the continuum limit becomes larger as d is lowered, and it is most useful for $d \leq 2$. Our theory can also be applied to other low-dimensional ferromagnets (like the ferromagnetic layer of ^{3}He on Grafoil [6]), but we will limit our discussion here to the quantum Hall system. Some limitations of the model as applied to the quantum Hall effect will also be discussed.

The required quantum field theory is obtained from the naive continuum limit of the coherent-state path integral of an insulating, lattice ferromagnet:

$$Z = \int \mathcal{D}\vec{n}\,\delta(\vec{n}^2 - 1)\exp\left(- \int d^d x \int_0^{1/T} d\tau \times (\mathcal{L}_0[\vec{n}] + \mathcal{L}_1[\vec{n}])\right),$$

$$\mathcal{L}_0[\vec{n}] = iM_0\vec{A}(\vec{n}) \cdot \partial_\tau \vec{n} + (\rho_s/2)(\nabla_x \vec{n})^2 - M_0 \vec{H} \cdot \vec{n}. \qquad (1)$$

Here $\vec{n}(x,\tau)$ is the three-component unit vector field identifying the local orientation of the ferromagnetic order (it is periodic in the Matsubara time τ), and $M_0 \geq 0$ is the magnetization per unit volume in the ferromagnetic ground state. The first term in $\mathcal{L}_0$ is the kinematical Berry phase [7], which accounts for the commutation relations between the components of the order parameter; $\vec{A}$ is the vector potential of a unit Dirac monopole at the origin of spin space with $\epsilon_{ijk}\partial A_k/\partial n_j = n_i$, ρ_s is the ground state spin stiffness, and H is the magnetic field; $\mathcal{L}_1$ contains local higher-gradient terms, with no time derivatives, which will be discussed below. (The Hopf term, which does contain a time derivative, will be discussed separately later.) We are using units in which $k_B = \hbar = 1$ and have absorbed a factor of $g\mu_B$ into H (μ_B is the Bohr magneton). For a ferromagnet on a hypercubic lattice with spacing a, spin per site S, and nearest-neighbor exchange J, $M_0 = Sa^{-d}$ and $\rho_s = JS^2a^{2-d}$. In the quantum Hall effect at $\nu = 1$, $M_0 = 1/4\pi\ell_B^2$, and (neglecting layer finite-thickness corrections which are expected to reduce ρ_s [3]) $\rho_s = e^2/16\sqrt{2\pi}\,\varepsilon\ell_B$ [3,8], where ℓ_B is the magnetic length. In the experiment of Ref. [1], $H \approx 2$ K (note that $g \approx 0.5$ in GaAs), while we estimate that $\rho_s \approx 3$ K.

Since $\mathcal{L}_1$ contains no time derivatives, the Hilbert space is fully determined (through canonical arguments) by M_0, and the remainder of the action describes the Hamiltonian acting in this space. For $M_0 = 0$, there are no degrees of freedom in the system (the Hilbert space is one dimensional), so the Hamiltonian is immaterial. For $M_0 \neq 0$, the explicit quantization of the continuum quantum ferromagnet (CQFM) defined by Eq. (1) is difficult, but it is not hard to establish the quantization condition that $2M_0L^d$ must be integral; M_0L^d is the total spin of the fully polarized state (L^d is the volume of the system). Since all states must have half-integral spin, we can associate a length ξ_0 with M_0, $2M_0 = \xi_0^{-d}$. We expect that the degrees of freedom of the CQFM correspond roughly to independent spins 1/2 per volume ξ_0^d, and the scale ξ_0 shows up naturally in the quantum theory [9]; for example, it is likely that the commutation relations for the spin density operators are smeared over the scale ξ_0 (a similar effect occurs in the quantum Hall system as a result of restriction to the lowest Landau level) and that the correlation length is never less than ξ_0.

In determining the applicability of the CQFM to a real system, we must consider a renormalization group (RG) analysis. There is a fixed point at $M_0 = 0$, and the terms in $\mathcal{L}_0$ are the most relevant perturbations. This can be seen by simple power counting: Under a rescaling $x \to e^{-\ell}x$, $\tau \to e^{-z\ell}\tau$ (with $z = 2$ so that the long wavelength spin wave dispersion is invariant), we find that M_0 has dimension d (corresponding to its $-d$ powers of ξ_0), ρ_s has dimension $d + z - 2 = d$, and T and H have dimension $z = 2$. Terms in $\mathcal{L}_1$ have $k \geq 4$ gradients, and their coefficients have dimension $d + z - k$; for $d < 2$, all such terms are irrelevant. To go beyond power counting requires a diagrammatic RG which will be described elsewhere; the results include an RG reinterpretation of earlier spin wave calculations in $d = 3$ [10] and $d = 2$ [11]. At $T = 0$, the power-counting flows for the couplings in $\mathcal{L}_0$ are exact, but these couplings do generate a term in $\mathcal{L}_1$, $\lambda(\partial_a n_i \partial_a n_i \partial_b n_j \partial_b n_j - 2\partial_a n_i \partial_b n_i \partial_a n_j \partial_b n_j)$, associated with spin wave scattering; this is described by the RG flow $d\lambda/d\ell = (d - 2)\lambda + c\rho_s/M_0$ (with c a positive constant), which sets in at scales $>\xi_0$. For $d < 2$, λ flows to a fixed point value $c\rho_s/(2 - d)M_0$. Similar phenomena are expected for other, even less relevant, interactions. Thus all the irrelevant couplings actually flow either to zero or to nonzero fixed point values, and approach these values with eigenvalues given by their dimensions established above, $y_k = d + 2 - k$. The simple form of these results, compared with more familiar field theories, is due to the fluctuationless fully polarized ground state, so that contributions come only from scattering of already existing spin waves (similar to the dilute Bose gas [12]), and to the rotational symmetry requirements.

For $d < 2$, these considerations imply that all observables should be universal functions of the bare couplings M_0, ρ_s, and H, realizing a *no-scale-factor universality* similar to that discussed in Ref. [12] for the dilute Bose gas in $d < 2$. Scaling forms can therefore be deduced from a naive dimensional analysis of the length and time scales in the CQFM; for the free energy density F we obtain

$$F = TM_0\Phi_F(\overline{\rho}_s/T, H/T), \qquad (2)$$

where $\overline{\rho}_s \equiv \rho_s/M_0^{(d-2)/d}$ is a rescaled stiffness and $\Phi_F(r,h)$ is a universal scaling function with no arbitrary scale factors and dependent only on d and the symmetry group [O(3)] of the ferromagnet (in particular, for the lattice ferromagnet, S enters, only indirectly, through ρ_s and M_0). Scaling forms for other thermodynamic observables can be obtained by taking derivatives of F. Because the scaling form is obtained by setting the irrelevant couplings to their fixed point values, it is valid only when the deviation of those couplings from their fixed points are negligible. If the bare values of the irrelevant couplings (defined at the scale ξ_0) are set to their fixed point values in (1), then the free energy is given by (2) at all values of T, ρ_s, and H. For a real system, this tuning of parameters does not occur, and the behavior approaches the universal form only for $T < T_{\max}(H)$ or $H < H_{\max}(T)$; we expect $T_{\max} \sim \overline{\rho}_s$ as $H \to 0$ for systems with small S. For $d > 2$ additional scaling variables, associated with other relevant couplings, will be necessary in a generalization of (2).

We now consider the different T regimes of the CQFM, ignoring $\mathcal{L}_1$, so this will be universal for $d < 2$ and marginally so for $d = 2$. Figure 1 shows a phase diagram as a function of the three dimensionless ratios of the energy scales T, H, and $\overline{\rho}_s$ plotted in the projective plane. All boundaries are smooth crossovers with the exception (for $d > 2$) of the ferromagnetic phase transition at the single point $H = 0$, $T = T_c \sim \overline{\rho}_s$. The regimes in Fig. 1 are as follows.

(i) Quantum activated (QA), $T < H$.—Most spins are aligned as in the ground state along H, with thermal corrections associated with a thermal activation factor $e^{-H/T}$. There is also a crossover (indicated by the dotted line) between $T < \overline{\rho}_s < H$ and $\overline{\rho}_s < T < H$, but it only affects the prefactor of $e^{-H/T}$.

(i) Renormalized classical (RC), $H < T < \overline{\rho}_s$.—The behavior is dominated by fluctuations of classical Goldstone modes with energies smaller than T.

(iii) Quantum critical (QC), $T > H, \overline{\rho}_s$.—This regime was proposed recently in $d = 2$ in Ref. [13]. It is the high T limit of the CQFM. One may interpret the behavior here as the response of the $M_0 \neq 0$ system with zero Hamiltonian to a finite size, $1/T$, along the time direction. Reference [13] also suggested that, for $S = 1/2$, $T_{\max}$ is large enough for the square lattice ferromagnet to exhibit QC behavior.

In $d = 1$, Nakamura and Takahashi [14] have studied the magnetization of the spin S chain in the RC region, and their results are described by the CQFM. The expected scaling form is $M = M_0\Phi_M$, where Φ_M is a function similar to Φ_F; they find a scaling function, ϕ_M, to which our function Φ_M reduces in a limit appropriate for the RC region: $\Phi_M(r \to \infty, h \to 0) = \phi_M(rh)$ and they computed $\phi_M(y) = 2y/3 - 44y^3/135 + \cdots$ for small y.

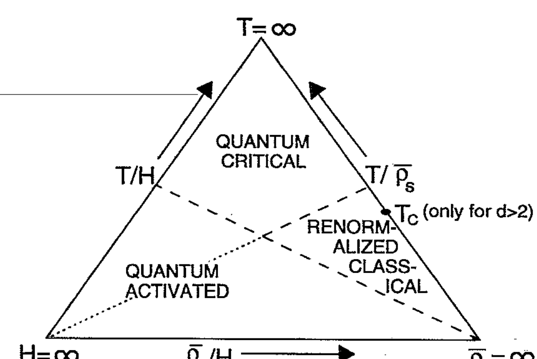

FIG. 1. Phase diagram of the CQFM as a function of dimensionless ratios of the three energies T, H, and $\overline{\rho}_s \equiv \rho_s/M_0^{(d-2)/d}$. Each energy becomes infinite at one of the vertices, and equals zero on the opposite side. Dashed and dotted lines are crossovers. Increasing T from 0 to ∞ at fixed $\overline{\rho}_s/H$ corresponds to moving along a straight line from the base to the apex.

For $d < 2$ the function $\Phi_M(r,h)$ can also be computed in the usual spin wave expansion [10,11], which yields a universal series containing integral powers of $r^{-d/2}$ times functions of h.

In $d = 2$, the flow of λ is logarithmic, and universality at low T is violated by logarithms, unlike the situation in antiferromagnets [15]. As a result, the QC regime lies at the edge of where quasiuniversality holds. However, the logarithmic terms contain prefactors of powers of T at low T, and are absent in the leading low T behavior. The flow of λ, in particular, has been overlooked in previous analyses of $d = 2$ ferromagnets [16].

Before turning to calculations of the scaling functions, we discuss other aspects of the $d = 2$ case more fully. For the CQFM in general in $d = 2$, a conserved topological current, defined as $j_\mu = \epsilon_{\mu\nu\lambda}\epsilon_{ijk}n_i\partial_\nu n_j\partial_\lambda n_k/8\pi$ ($\mu = x, y, \tau$), exists and represents the number density and current of Skyrmions. The Skyrmions experience an effective orbital magnetic field of strength $4\pi M_0$, produced by the Berry phase term, as can be seen from the identity $M_0\int\vec{A}(\vec{n})\cdot\partial_\tau\vec{n} = \int\mathbf{j}\cdot\mathcal{A}$, where $\nabla\times\mathcal{A} = 4\pi M_0$ represents the uniform field. Thus there is an effective magnetic length for the Skyrmions that is related to ξ_0; moreover, Skyrmions come in quantized sizes that are multiples of ξ_0^2. In the quantum Hall system, Skyrmions carry real electric charge [2,3], so the use of the CQFM does not exclude charged excitations. However, in this case, there are also other terms that are known to appear in the long-wavelength description [2,3] but are not included in the CQFM thus far. The extra terms are (i) the Hopf term $2\pi i\int j_\mu a_\mu$, where a_μ obeys $\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda = j_\mu$, which endows the Skyrmions with Fermi statistics and half-integral spin [17]; (ii) the Coulomb interaction $\int\int j_\tau(x)j_\tau(x')e^2/2\varepsilon|x-x'|$. The Hopf term is marginal, but since it contains a time derivative it affects the quantization directly, and may change the dimension of the Hilbert space in a finite system. It will not affect the discussion of universality and its violation by logarithms, but it will change the precise scaling functions in general. The Coulomb interaction has dimension 1, so it is relevant, though less so than ρ_s, and, in principle, requires that an additional scaling variable appear in the scaling functions. However, both terms enter only through Skyrmions which, in the large ρ_s region, always have an energy $>\rho_s$, and their contributions are exponentially small at low T.

Finally, we present our large N results for the CQFM. These are valid over the entire phase diagram of Fig. 1, and exhibit all the crossovers. We discuss two different large N limits; the first generalizes the symmetry group from SU(2) $\cong$ O(3) to SU(N) and the second to O(N). To obtain the SU(N) theory we write $\vec{n} = z_\alpha^*\vec{\sigma}_{\alpha\beta}z_\beta$, where $\vec{\sigma}$ are the Pauli matrices, z_α is a two-component complex field, and $\sum_\alpha|z_\alpha|^2 = 1$. The Berry phase in $\mathcal{L}$ now becomes $2M_0\sum_\alpha z_\alpha^*\partial_\tau z_\alpha$. We can now obtain SU($N$) symmetry by allowing α to run from 1 to N [for N even, the field H is taken to couple to the generator diag $(1_{N/2}, -1_{N/2})$]; the gradient terms are as in the CP^{N-1} model [2,18]. For the O(N) generalization we parametrize $n_i = i\epsilon_{ijk}w_j^*w_k$, where w_i is a three-component complex field obeying $\sum_i|w_i|^2 = 1$ and $\sum_i w_i^2 = 0$. The Berry phase is now $M_0\sum_i w_i^*\partial_\tau w_i$, and O($N$) symmetry is achieved by allowing i to run from 1 to N [for N divisible by 3, H couples to a generator which contains $N/3$ copies of the O(3) generator]. The $1/N$ expansion of both theories is standard, and we omit all details: the constraints are imposed by Lagrange multipliers, and ρ_s and M_0 should be of order N as $N\to\infty$. We present below $N = \infty$ results from both theories for some observables in $d = 2$ (although results can be obtained for arbitrary d); the results are universal as the logarithmic violations of universality appear only at higher orders in $1/N$.

(a) Magnetization.—From the SU(∞) theory we obtain the scaling function $\Phi_M(r,h) = \ln[(q_1 - e^{-h/2})/(q_1 - e^{h/2})]/8\pi r$, where $q_1 > 1$ is the solution of $(q_1 - e^{-h/2})(q_1 - e^{h/2}) = q_1^2 e^{-8\pi r}$. Similarly, we obtain from the O(∞) theory $\Phi_M(r,h) = \ln[(q_2 - e^{-h})/(q_2 - e^h)]/4\pi r$, where $q_2 > 1$ is the solution of $(q_2 - e^{-h})(q_2 - 1)(q_2 - e^h) = q_2^3 e^{-4\pi r}$. We show in Fig. 2 a plot of these results for M/M_0 as a function of T/H for a few values of ρ_s/H, including $\rho_s/H = 0$. For $\rho_s \gg H$ it is possible, in principle, to use simpler functions characteristic of the different regions of Fig. 1, punctuated by crossovers between them. At the lowest T we have QA behavior with $\Phi_M - 1 \propto e^{-h}$. At larger T we have RC behavior described by the scaling function of Ref. [19]; in our SU(∞) theory, Φ_M reduces in this limit to $\Phi_M = 1 + \ln\{h/2 + [(h/2)^2 + e^{-8\pi r}]^{1/2}\}/4\pi r$. At the largest T we have QC behavior in which we expect $\Phi_M \propto h$. Although the analytic forms are rather different

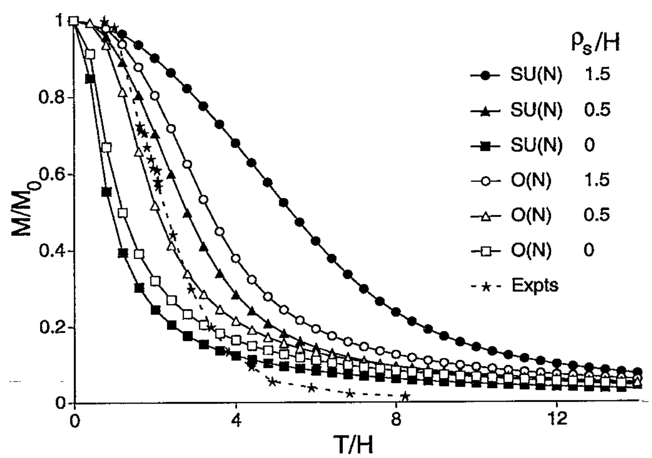

FIG. 2. Magnetization of the CQFM in $d = 2$, computed in the $N = \infty$ limit of SU(N) and O(N) theories for a number of values of ρ_s/H, and compared with the experiments of Ref. [1]. The $\rho_s = 0$ limit of the large N results yield the spin S Brillouin function, with $S = 1/2$ in the SU(N) model, and $S = 1$ for O(N).

in the three regimes, the qualitative trends in an M vs T plot are similar; this makes picking out the regimes from experimental data rather difficult.

(b) NMR relaxation rate $1/T_1$.—Unlike the static magnetization, the dynamic susceptibility has significantly different behavior in the regions of Fig. 1, and this leads to clear signatures of them in $1/T_1$. We model the nuclear-electron contact coupling by $AM_0\vec{I}(\tau) \cdot \vec{n}(0, \tau)$. Then the relaxation rate is given by $1/T_1 = A^2T \lim_{\omega\to 0} \text{Im}\, \chi_{L+-}(\omega)/\omega$, where χ_{L+-} is the local transverse susceptibility. Dimensional analysis shows that $1/T_1$ satisfies the scaling form $1/T_1 = (A^2M_0^2/T)\Phi_{T_1}$, where Φ_{T_1} is a universal function like Φ_F. We have determined Φ_{T_1} in both large N limits, and a plot of $1/T_1$ is shown in Fig. 3 as a function of T/ρ_s for some values of ρ_s/H. The most notable feature is the "ferromagnetic coherence peak," which signals a crossover between the QA and RC regimes. This becomes clear from the asymptotic behavior for $\rho_s \gg H$. In the low T QA regime we have activated behavior $1/T_1 \sim e^{-H/T}$ [$\Phi_{T_1} = e^{-h}/16\pi r^2$ for SU(∞) and $\Phi_{T_1} = e^{-h}/8\pi r^2$ for O(∞)]. In contrast, in the RC regime, $1/T_1$ decreases exponentially fast with increasing T [$\Phi_{T_1} = e^{4\pi r}/16\pi r^2$ for SU(∞) and $\Phi_{T_1} = e^{4\pi r/3}/4\pi r^2$ for O(∞)] due to the rapid decrease in the ferromagnetic correlation length; this behavior of $1/T_1$ is similar to that observed in the RC region of $d = 2$ quantum antiferromagnets [20]. Finally in the large T QC region we find $1/T_1 \sim$ const [$\Phi_{T_1} = 1/4r$ for SU(∞) and $\Phi_{T_1} = 1/3r$ for O(∞)]. Notice from Fig. 3 that the coherence peak survives even for moderate values of ρ_s/H, though it may be absent for ρ_s/H sufficiently small. We emphasize that for ρ_s/H large, this peak is not dependent upon a large value of $T_{\max}$, as it occurs at the crossover between the low T QA and RC regimes.

Nontrivial textures (Skyrmions) exist, and the Hopf and Coulomb interaction terms can be included, for all N in both the SU(N) and O(N) models, but have no effect in the $N \to \infty$ limit. While agreement with the quantum Hall experiments [1] is fair, at this point it is not clear whether

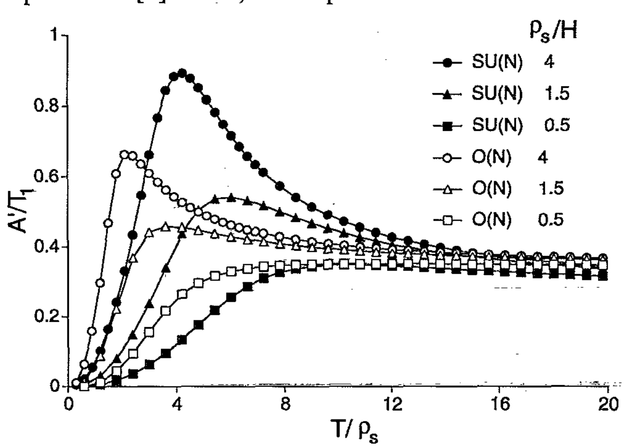

FIG. 3. As in Fig. 2 but for $1/T_1$ vs T/ρ_s. The constant $A' = \rho_s/A^2M_0^2$.

the differences between theory and experiment are due to these effects, other differences between $N = \infty$ and $N = 2$ or 3, the possibility that $T_{\max}$ is small, or the uncertainty in the value of ρ_s. More complete measurements of the T dependence of $1/T_1$, particularly in samples with a larger ρ_s/H, could help answer these questions.

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