Physics Letters B 309, 285 (1993)

Polylogarithm identities in a conformal field theory in three dimensions

Subir Sachdev

Departments of Physics and Applied Physics, P.O. Box 2157, Yale University, New Haven, CT 06520

The $N = \infty$ vector $O(N)$ model is a solvable, interacting field theory in three dimensions (D). In a recent paper with A. Chubukov and J. Ye [1], we have computed a universal number, $\tilde{c}$, characterizing the size dependence of the free energy at the conformally-invariant critical point of this theory. The result [1] for $\tilde{c}$ can be expressed in terms of polylogarithms. Here, we use non-trivial polylogarithm identities to show that $\tilde{c}/N = 4/5$, a rational number; this result is curiously parallel to recent work on dilogarithm identities in $D = 2$ conformal theories. The amplitude of the stress-stress correlator of this theory, c (which is the analog of the central charge), is determined to be $c/N = 3/4$, also rational. Unitary conformal theories in $D = 2$ always have $c = \tilde{c}$; thus such a result is clearly not valid in $D = 3$.

Consider a conformally-invariant field theory in D dimensions. Place it in a slab which is infinite in $D-1$ dimensions, but of finite length L in the remaining direction. Impose periodic boundary conditions along this finite direction. An old result of Fisher and de Gennes [2] states that if hyperscaling is valid, the free energy density $\mathcal{F} = -\log Z/V$ (Z is the partition function and V is the total volume of the slab) satisfies

$$\mathcal{F} = \mathcal{F}_{\infty} - \frac{\Gamma[D/2]\zeta(D)}{\pi^{D/2}} \frac{\tilde{c}}{L^D}. \tag{1}$$

Here $\mathcal{F}_{\infty}$ is the free energy density in the infinite system and $\tilde{c}$ is a universal number. The coefficient of $1/L^D$ has been chosen such that $\tilde{c} = 1$ for a single component, massless free scalar field theory. A similar parametrization has also been discussed recently by Castro Neto and Fradkin [3].

A second universal number characterizing the conformal theory is the amplitude of the two-point correlation function of the stress tensor $T_{\mu\nu}$ in infinite flat space. Cardy has proposed a normalization convention for $T_{\mu\nu}$ in arbitrary dimensions [4], and shown that its two-point correlation is of the following form

$$\begin{aligned} \langle T_{\mu\nu}(r) T_{\lambda\sigma}(0) \rangle \;=\; & \frac{c}{r^{2D}} \left[\left(\delta_{\mu\lambda} - \frac{2 r_\mu r_\lambda}{r^2} \right) \left(\delta_{\nu\sigma} - \frac{2 r_\nu r_\sigma}{r^2} \right) + \left(\delta_{\mu\sigma} - \frac{2 r_\mu r_\sigma}{r^2} \right) \left(\delta_{\nu\lambda} - \frac{2 r_\nu r_\lambda}{r^2} \right) \right. \\ & \left. - \frac{2}{D} \delta_{\mu\nu} \delta_{\lambda\sigma} \right]. \end{aligned} \tag{2}$$

This defines a universal amplitude, c, which is the analog of the central charge in $D = 2$ conformal field theories. A key property of $D = 2$ unitary conformal field theories is $\tilde{c} = c$ [5]. The generalization of this result to arbitrary D, and in particular $D = 3$, remains an important open problem.

It would clearly be interesting to obtain results for $\tilde{c}$ and c for specific models in dimensions other than $D = 2$. In a recent paper by A. Chubukov, myself and J. Ye [1] on the critical properties of two-dimensional quantum antiferromagnets, the value of $\tilde{c}$ was computed for the vector $O(N)$ model in $D = 3$ in a $1/N$ expansion. In this note we highlight some features of the computation of $\tilde{c}$ at $N = \infty$, as we believe the results may be of interest to a broader audience of conformal field theorists. The result for $\tilde{c}$ at $N = \infty$ can be expressed in terms of di- and trilogarithm functions. Below, we use some known polylogarithm identities to simplify the result for $\tilde{c}$. The appearance of these identities is surprisingly parallel to recent work [6] establishing a connection between dilogarithmic identities and the rational central charge of $D = 2$ conformal theories. We will also compute the value of c at $N = \infty$ in the $D = 3$ $O(N)$ model.

We consider the field theory with the action

$$S = \frac{N}{2g} \int d^3x (\partial n)^2 \tag{3}$$

where n is a N-component real vector of unit length, $n^2 = 1$. The fixed length constraint is actually not crucial and identical universal properties can be obtained in a soft-spin theory with a n^4 interaction term [7]. The theory has to be suitably regulated in the ultraviolet by a momentum cutoff Λ. It becomes conformally invariant at a critical value $g = g_c = \alpha/\Lambda$ which separates the $g < g_c$ Goldstone phase with broken $O(N)$ invariance, from the $g > g_c$ massive phase. The location of the critical point, α, is of course non-universal and will depend upon the cutoff scheme.

The formal structure of the $N \to \infty$ limit is quite standard. The fixed length constraint is imposed by an auxilliary field λ. After integrating out the n field, the $N = \infty$ theory is given by the saddle point of the resulting functional integral. In this manner we find

$$\frac{\mathcal{F}}{N} = \frac{1}{2} \mathrm{Tr} \log(-\partial^2 + m^2) - \frac{m^2}{2g} \tag{4}$$

where m^2 is the saddle-point value of λ. The critical point is at $g = g_c$, where

$$\frac{1}{g_c} = \int \frac{d^3p}{8\pi^3} \frac{1}{p^2}, \tag{5}$$

and $m^2 = 0$ in the infinite volume system. In the slab with thickness L, however, we find at $g = g_c$ that

$$m = m_L = \frac{2 \log \tau}{L}, \tag{6}$$

where $\tau = (\sqrt{5} + 1)/2$ is the golden mean.

To compute $\tilde{c}$, we now need to evaluate the $\mathrm{Tr} \log$ in (4) in the slab geometry. The momentum along the finite direction is quantized in integer multiples of $2\pi/L$. The summation over these discrete modes can be accomplished with the identity

$$\lim_{M \to \infty} \left[\frac{1}{L} \sum_{n=-M}^{M} \log \left(\frac{4\pi^2 n^2}{L^2} + a^2 \right) - \int_{-2\pi M/L}^{2\pi M/L} \frac{d\omega}{2\pi} \log(\omega^2 + a^2) \right] = \frac{2}{L} \log \left(1 - e^{-L|a|} \right), \tag{7}$$

where a is any constant. The expression for $\mathcal{F}$ in the slab of width L at $g = g_c$ is then easily shown to be

$$\frac{\mathcal{F}}{N} = \frac{1}{L} \int \frac{d^2 k}{4\pi^2} \log \left(1 - e^{-L\sqrt{m_L^2 + k^2}} \right) + \frac{1}{2} \int \frac{d^3 p}{8\pi^3} \left[\log(p^2 + m_L^2) - \frac{m_L^2}{p^2} \right] \tag{8}$$

The second integral is of course badly divergent in the ultraviolet. All divergences however disappear after the infinite volume result has been subtracted, in which case

$$\frac{\mathcal{F} - \mathcal{F}_\infty}{N} = \frac{1}{L} \int \frac{d^2 k}{4\pi^2} \log \left(1 - e^{-L\sqrt{m_L^2 + k^2}} \right) + \frac{1}{2} \int \frac{d^3 p}{8\pi^3} \left[\log \left(\frac{p^2 + m_L^2}{p^2} \right) - \frac{m_L^2}{p^2} \right] \tag{9}$$

These integrals can be expressed in terms of polylogarithms. We will skip the straightforward intermediate steps and present our final result for $\tilde{c}$ obtained from (9) and (1)

$$(1/N) \mathrm{Li}_3(1) \tilde{c} = \mathrm{Li}_3(2 - \tau) - \log(2 - \tau) \mathrm{Li}_2(2 - \tau) - \frac{1}{6} \log^3(2 - \tau) \tag{10}$$

where $2 - \tau = 1/\tau^2 = (3 - \sqrt{5})/2$, and the polylogarithm function is defined by analytic continuation of the series

$$\mathrm{Li}_p(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^p}. \tag{11}$$

Note $\mathrm{Li}_p(1) = \zeta(p)$.

Remarkably, it turns out that $2-\tau$ is one of only three real, positive, z for which both $\mathrm{Li}_2(z)$ and $\mathrm{Li}_3(z)$ can be expressed in terms of elementary functions [8] (the other points are $z=1$ and $z=1/2$). As shown in the book by Lewin [8], the value of $\mathrm{Li}_2(2-\tau)$ follows from a combined analysis of the following identities

$$\begin{aligned}
\mathrm{Li}_2(z)+\mathrm{Li}_2(1-z) &= \frac{\pi^2}{6}-\log z\log(1-z) \\
\mathrm{Li}_2(z)+\mathrm{Li}_2\left(\frac{-z}{1-z}\right) &= -\frac{1}{2}\log^2(1-z) \\
\frac{1}{2}\mathrm{Li}_2(z^2)+\mathrm{Li}_2\left(\frac{-z}{1-z}\right)-\mathrm{Li}_2(-z) &= -\frac{1}{2}\log^2(1-z)
\end{aligned} \tag{12}$$

To see the special role of the golden mean in these identities, note that two of the arguments z^2 and $-z/(1-z)$ coincide when $z^2-z-1=0$. The solutions of this are $z=\tau, 1-\tau$. It is not difficult to show that the above identities evaluated at $z=2-\tau$, $\tau-1$, and $1-\tau$, can be combined to uniquely determine $\mathrm{Li}_2(2-\tau)$ [8]:

$$\mathrm{Li}_2(2-\tau)=\frac{\pi^2}{15}-\frac{1}{4}\log^2(2-\tau) \tag{13}$$

Similarly, the identities [8]

$$\begin{aligned}
\frac{1}{4}\mathrm{Li}_3(z^2) &= \mathrm{Li}_3(z)+\mathrm{Li}_3(-z) \\
\mathrm{Li}_3(z)+\mathrm{Li}_3\left(\frac{-z}{1-z}\right)+\mathrm{Li}_3(1-z) &= \mathrm{Li}_3(1)+\frac{\pi^2}{6}\log(1-z) \\
&\quad -\frac{1}{2}\log z\log^2(1-z)+\frac{1}{6}\log^3(1-z)
\end{aligned} \tag{14}$$

evaluated at $z=2-\tau$ and $\tau-1$ yield [8]

$$\mathrm{Li}_3(2-\tau)=\frac{4}{5}\mathrm{Li}_3(1)+\frac{\pi^2}{15}\log(2-\tau)-\frac{1}{12}\log^3(2-\tau) \tag{15}$$

Inserting (13) and (15) into (10), we get one of our main results

$$\frac{\tilde{c}}{N}=\frac{4}{5} \tag{16}$$

Surprisingly, $\tilde{c}/N$ has turned out to be a rational number, although none of the intermediate steps suggested that this might be the case. Interestingly, this phenomenon is similar to that in recent determinations of $\tilde{c}$ from the size dependence of $\mathcal{F}$ in $D = 2$ conformal theories [5, 6]. There, the free energy was determined from integrable lattice models, or by evaluating the characters of a representation of the Virasoro algebra; in both cases the result was obtained in terms of dilogarithm sums, which thus must equal the rational central charge.

We turn next to the determination of c for the $D = 3$, $N = \infty$ vector $O(N)$ model. The stress tensor $T_{\mu\nu}$ for (3) is

$$T_{\mu\nu} = \frac{4\pi}{g} \left(\partial_\mu n \partial_\nu n - \frac{\delta_{\mu\nu}}{2} (\partial n)^2 \right) - \delta_{\mu\nu} t \tag{17}$$

where t is a cutoff-dependent subtraction needed to make $T_{\mu\nu}$ a proper scaling variable at the critical point. The general structure of these subtractions in the $1/N$ expansion for arbitrary operators in the $O(N)$ model with a hard momentum cutoff has been discussed by Ma [9]. Here, we simply note that dimensional regularization of the loop integrals in the vicinity of $D = 3$ leads to $t = 0$ at $N = \infty$. We evaluated $\langle T_{\mu\nu}(r) T_{\lambda\sigma}(0) \rangle$ at $N = \infty$ in the infinite system using dimensional regularization. There are two Feynman graphs which contribute at this order [9], including one involving fluctuation of the auxilliary field, λ, which imposed the fixed length constraint. The loop integrals are quite tedious, but straightforward. We found that our final result was indeed consistent with (2) with

$$\frac{c}{N} = \frac{3}{4} \tag{18}$$

Note that $c \neq \tilde{c}$, unlike $D = 2$. Instead, we have $c/\tilde{c} = 15/16$ in this theory.

We emphasize that all of the results of this paper are special to $D = 3$; $\tilde{c}$ can also be computed for general D, but the results simplify only in $D = 3$. The major question raised by this work is, of course, whether $\tilde{c}$ and c have any of these special properties at finite N in

$D = 3$. It would also be interesting to obtain the simple $D = 3$ results for c and $\tilde{c}$ at $N = \infty$ by algebraic methods.

I thank A. Chubukov, G. Moore, N. Read and R. Shankar for useful discussions. This research was supported by NSF Grant No. DMR 8857228.

References

[1] A.V. Chubukov, S. Sachdev and J. Ye, paper 9304046 on cond-mat@babbage.sissa.it.

[2] M.E. Fisher and P.-G. de Gennes, C.R. Acad. Sci. Ser. B **287**, 207 (1978); V. Privman and M.E. Fisher, Phys. Rev. B **30**, 322 (1984).

[3] A.H. Castro Neto and E. Fradkin, paper 9301009 on cond-mat@babbage.sissa.it

[4] J.L. Cardy, Nucl. Phys. **B290**, 355 (1987).

[5] H.W.J. Blote, J.L. Cardy and M.P. Nightingale, Phys. Rev. Lett. **56**, 742 (1986); I. Affleck, Phys. Rev. Lett. **56**, 746 (1986).

[6] V.V. Bazhanov and N. Yu. Reshetikhin, Int. J. Mod. Phys. A **4**, 115 (1989); A.N. Kirillov, J. Sov. Math. **47** 2450 (1989); T.R. Klassen and E. Melzer, Nucl. Phys. B **338**, 485 (1990) and **370**, 511 (1992); A. Klumper and P.A. Pearce, J. Stat. Phys. **64**, 13 (1991); F. Ravanini, Phys. Lett. B **282**, 73 (1992); W. Nahm, A. Recknagel, and M. Terhoeven, paper 9211034 on hep-th@xxx.lanl.gov.

[7] E. Brezin and J. Zinn-Justin, Phys. Rev. B **14**, 3110 (1976).

[8] *Polylogarithms and Associated Functions*, by L. Lewin, North Holland, New York (1981).

[9] S.-k. Ma, Phys. Rev. **10**, 1818 (1974).