Universal amplitude ratios for two-dimensional melting on smooth and periodic substrates

Subir Sachdev
Department of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 5 November 1984)

Universal amplitude ratios in the theory of continuous dislocation-mediated melting are calculated. Measurement of these ratios can serve as an additional test of the theory of melting of two-dimensional solids on smooth and periodic substrates.

I. INTRODUCTION

The production of a continuous-melting transition mediated by dislocations in two-dimensional solids by Kosterlitz and Thouless[1] has stimulated extensive theoretical[2–8] and experimental work.[9–22] The theory in its present form[5,6] gives detailed predictions about the melting of solids in triangular lattices on both smooth and periodic substrates. When extended to melting on periodic substrates, the theory is applicable only to floating or incommensurate solids in a weak periodic potential. All the analyses in this paper will be applicable only to these cases.

The continuous-melting theory predicts a phase transition as a result of the unbinding of dislocation pairs in the solid. Below the melting temperature, dislocations occur only in the form of bound pairs. The absence of unpaired dislocations at the longest wavelengths indicates the presence of quasi-long-range crystalline order. This crystalline order implies an algebraic decay of the translational correlations. If on a lattice of sites $[\mathbf{R}]$, we define the displacements of the particles by $\mathbf{u}(\mathbf{R})$, and $\rho_{\mathbf{G}}(\mathbf{R})$ by

$$\rho_{\mathbf{G}}(\mathbf{R}) \equiv e^{i\mathbf{G}\cdot[\mathbf{R}+\mathbf{u}(\mathbf{R})]} ,$$

then

$$\langle \rho_{\mathbf{G}}(\mathbf{R})\rho_{\mathbf{G}}^{*}(0)\rangle \sim R^{-\eta_G(T)} \tag{1.1}$$

at large distances for any wave vector $\mathbf{G}$. An experimentally measurable consequence of this decay of correlations is the presence of power-law singularities in the x-ray structure function at the set of the wave vectors of the reciprocal lattice $|\mathbf{G}|$:[5,7]

$$S(\mathbf{q}) = \sum_{\mathbf{R}} e^{i\mathbf{q}\cdot\mathbf{R}} \langle e^{i\mathbf{q}\cdot[\mathbf{u}(\mathbf{R})-\mathbf{u}(0)]}\rangle$$
$$\sim [\mathbf{q}-\mathbf{G}|^{-2+\eta_G(T)} . \tag{1.2}$$

In practice there are appreciable finite-size corrections to this power-law form.[7] This form of $S(\mathbf{q})$ is maintained right up to the melting temperature. On a smooth substrate the exponent η_{G_0} satisfies the following inequalities just below the melting temperature T_m,[5]

$$\tfrac{1}{4} < \eta_{G_0}(T_m^-) < \tfrac{1}{3} , \tag{1.3}$$

where $\mathbf{G}_0$ is the smallest reciprocal lattice vector. The upper limit is not affected by the presence of a periodic substrate, while the lower limit decreases only for a large substrate potential. For simplicity, in the remaining analysis we will only refer to the value of η at the reciprocal lattice vector $\mathbf{G}_0$; more generally, $\eta_G = G^2\eta_{G_0}/G_0^2$.[5] Henceforth, we will therefore drop the subscript G_0 from η_{G_0}.

Above the melting temperature, the unbinding of dislocation pairs leads to a proliferation of dislocations at scales larger than a correlation length $\xi_+(T)$. This destroys the quasi-long-range translational order, and the translational correlations now decay exponentially,

$$\langle \rho_{\mathbf{G}}(\mathbf{R})\rho_{\mathbf{G}}^{*}(0)\rangle \sim \exp\left[-\frac{R}{\xi_+(T)}\right] . \tag{1.4}$$

Nelson and Halperin[5] pointed out that the presence of unpaired dislocations is not enough to destroy orientational order in the lattice. Such a phase, which retains a memory of the orientation of the crystal it melted from, is known as the hexatic phase. The structure factor now becomes large in a ring of $\mathbf{q}$ space with a radius approximately G_0. The presence of orientational correlations shows up by an angular modulation of the structure factor in this ring. In the radial direction, the structure factor should be approximately Lorentzian with a width $\xi_+(T)$.

As the system approaches the melting temperature from above, ξ_+ is predicted to diverge with the leading behavior given by[4–6]

$$\xi_+ \sim \exp\left[\frac{b}{|t|^{\bar{\nu}(\sigma)}}\right] \tag{1.5}$$

with

$$t = \frac{T-T_m}{T_m} . \tag{1.6}$$

σ is a parameter which measures the strength of the substrate potential. In the limit of a weak potential, σ lies between 0 and 1,[23] decreasing from its smooth substrate value of 1 with an increase in the strength of the potential. σ will be defined more precisely later in this paper. The exponent $\bar{\nu}(\sigma)$ satisfies the constraints

$$\bar{\nu}(+1)=0.36963\cdots ,$$
$$\bar{\nu}(0)=0.4 , \tag{1.7}$$
$$0.36963\cdots \leq \bar{\nu}(\sigma) \leq 0.4 .$$

Similar singularities also occur below T_m. It was shown by Nelson and Halperin[5] that the shear modulus μ, the bulk modulus B, and K^r ($-2\pi K^r$ is the coefficient of the logarithmic interaction term between dislocations[5]) have the following behavior just below the melting temperature:

$$\frac{1}{\mu(T \to T_m^-)} = \frac{1}{\mu^*} - b_\mu |t|^{\bar{\nu}(\sigma)} ,$$
$$\frac{1}{B(T \to T_m^-)} = \frac{1}{B^*} - b_B |t|^{\bar{\nu}(\sigma)} , \tag{1.8}$$
$$K^r(T \to T_m^-) = \frac{2}{\pi}(1 + b_{K^r} |t|^{\bar{\nu}(\sigma)}) .$$

(All elastic constants in this paper are in units of $k_B T_m / a_0^2$, a_0 being the lattice spacing.) We show that the experimentally measurable exponent η has a similar singularity below T_m:

$$\eta(T \to T_m^-) = \eta^*(1 - b_\eta |t|^{\bar{\nu}(\sigma)}) . \tag{1.9}$$

The behavior of η below T_m and that of ξ_+ above T_m is plotted in Fig. 1.

The principal result of this paper is that the following universal relationships exist between the coefficients b, b_μ, b_B, b_{K^r} and b_η:

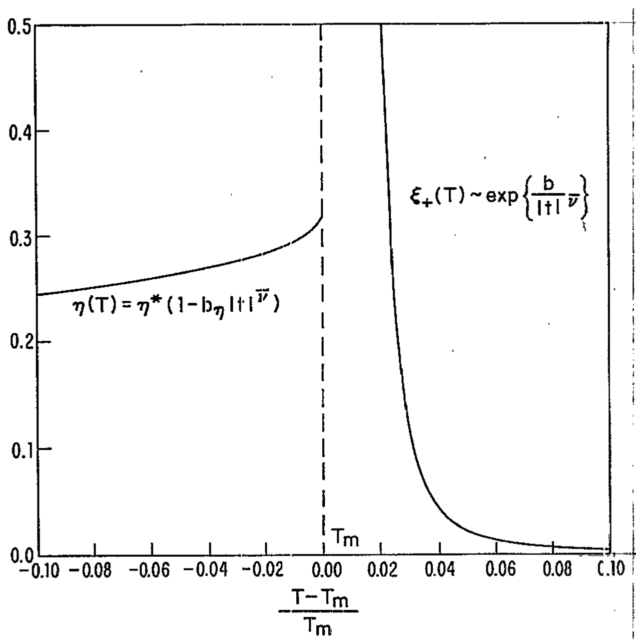

FIG. 1. Schematic drawing of the behavior of η below T_m and of ξ_+ above T_m. The product bb_η is predicted to be an exactly calculable universal function of η^* and σ.

$$bb_\mu = F_\mu(\sigma) ,$$
$$bb_B = F_B(\sigma) ,$$
$$bb_{K^r} = F_{K^r}(\sigma) , \tag{1.10}$$
$$bb_\eta = F_\eta(\sigma, \eta^*) .$$

F_μ, F_B, F_{K^r} and F_η are universal functions of their arguments and are exactly calculable. In a weak potential, the function F_η is unique. For stronger potentials F_η may be one of two possible functions—which one of these functions is chosen will depend upon the system under consideration. The values μ^*, B^*, and η^* are nonuniversal. Similar universal amplitude ratios have been noted earlier in superfluid ^{4}He films.[8]

Dislocation-mediated melting seems to have been observed in a number of experiments. Melting of the electron solid on helium[9] was observed by Grimes and Adams. The melting temperature of this solid appears to be consistent with a dislocation-mediated mechanism.[10] The measurement of the shear modulus of the electron solid,[11] the observation of propagating shear waves,[12] and the measurement of the power absorbed by dislocations moving in response to an external stress,[13] have all lent support to this interpretation. Gases physiadsorbed onto exfoliated graphite have proven to be an attractive system for observing the melting of two-dimensional solids. Xenon adsorbed on graphite has been shown to exhibit both first- and second-order transitions.[14–16] The second-order transitions occur at higher pressures and display the characteristics of a dislocation-mediated mechanism. The measured behavior of $\xi(T)$, $\bar{\nu}$, and $\eta(T)$ is consistent with equations (1.3), (1.5), (1.7), and (1.9). For the system of argon on graphite, the x-ray measurements of McTague *et al.*[17] and specific-heat measurements[18] show evidence of a dislocation-mediated transition, although recent specific-heat measurements[19] suggest a very weak first-order phase transition. X-ray measurements of methane on graphite[20] yield structure factors consistent with (1.2), (1.4), and (1.5).

As noted earlier, the theory predicts that the solid melts into a phase with quasi-long-range orientational order. The experiments of Rosenbaum *et al.*[21] on monolayers of xenon on graphite have shown evidence of orientational order in the absence of translational order, although this could be a substrate effect. Free-standing liquid-crystal films exhibit genuine hexaticlike phases,[22] and the predicted hexatic—to—liquid transition.[5]

The measurements of Dimon *et al.*[14] show that xenon on graphite is a likely candidate for measuring the amplitude ratios calculated in this paper. Precise measurements of the correlation length above the melting temperature $\xi_+(T)$, and the behavior of $\eta(T)$ below the melting temperature are required. The following procedure may be adopted for testing the predicted amplitude ratios. From the measurement of $\xi_+(T)$, the values of $\bar{\nu}$ and b can be extracted. The known function $\bar{\nu}(\sigma)$ will then give the value of σ.[24] Fitting $\eta(T)$ to the form (1.9) will yield the values of η^* and b_η. Inserting the values of b, b_η, σ, and η^* into the last of Eqs. (1.10) will then give a constraint on the theory.

A potential problem with the application of the results of this paper is the small size of the critical region.[25,26] The asymptotic form of the correlation length only becomes applicable when $\xi_+ > 10^8$ lattice spacings. Since this is larger than most experimental systems, finite-size effects may also become important.

It is possible that computer simulations on the Laplacian roughening model[27] do not suffer from this problem. The Laplacian roughening model[28] is a modification of the solid-on-solid model for interfacial roughening, and is related by a duality transformation to the disclination and dislocation Hamiltonians of the melting problem on smooth substrates. Simultations of the temperature dependence of $K^r(T)$ already exist,[27] although more precise data will be necessary before a fit to Eq. (1.8) can become meaningful. The behavior of the translational correlation length $\xi_+(T)$ can be determined from the large distance behavior of the tilt-tilt correlation function, $g(r)$, just below the transition from a rough oriented interface to a rough unoriented interface.[27] The knowledge of $K^r(T)$ and $\xi_+(T)$ should make it possible to determine the amplitude ratio bb_{K^r}. On a smooth substrate, this quantity is predicted to be $F_{K^r}(1)=1.713$.

II. CALCULATION

The analysis will be carried out with a continuum elastic Hamiltonian for a floating solid on a periodic substrate. The result for a smooth substrate will then be a special case. A Hamiltonian which describes small fluctuations from the crystal in the presence of a weak incommensurate substrate potential is[5]

$$\frac{\mathscr{H}_E}{k_B T}=\frac{1}{2}\int\frac{d^2r}{a_0^2}[2\mu u_{ij}^2+\lambda u_{kk}^2+\gamma(\partial_y u_x-\partial_x u_y)^2]\,. \quad (2.1)$$

The parameters μ and λ are the usual Lame elastic constants and u_{ij} is the strain tensor. The bulk modulus B is related to μ and λ by

$$B=\mu+\lambda\,. \quad (2.2)$$

The parameter γ is an elastic constant dependent upon the strength of the substrate potential. The relation $\gamma=0$ is satisfied for a smooth substrate. Below the melting temperature, γ will also be shown to have a singularity:

$$\frac{1}{\gamma(T\rightarrow T_m^-)}=\frac{1}{\gamma^*}-b_\gamma\,|t\,|^{\bar\nu(\sigma)}\,. \quad (2.3)$$

Dislocations may now be introduced into the system. The free energy for the dislocation degrees of freedom implied by $\mathscr{H}_E$ is[5,6]

$$\frac{\mathscr{H}_D}{k_B T}=-2\pi\sum_{\langle ij\rangle}\{K^r\mathbf{b}^i\cdot\mathbf{b}^j\ln(r^{ij}/a_0) -K^\theta[(\mathbf{b}^i\cdot\mathbf{r}^{ij})(\mathbf{b}^j\cdot\mathbf{r}^{ij})/(r^{ij})^2 -\tfrac{1}{2}\mathbf{b}^i\cdot\mathbf{b}^j]\}+\frac{E_c}{k_BT}\sum_i(b^i)^2\,, \quad (2.4)$$

where $\mathbf{r}^{ij}=\mathbf{r}^i-\mathbf{r}^j$, $\mathbf{r}^i$ denotes the position of the ith dislocation with dimensionless Burger's vector $\mathbf{b}^i$, and E_c is a phenomenological core energy. The sum over $\langle ij\rangle$ occurs over all pairs of dislocations. The constants K^r and K^θ are related to the elastic constants by

$$K^r=\frac{1}{2\pi^2}\left[\frac{\mu B}{\mu+B}+\frac{\mu\gamma}{\mu+\gamma}\right],\qquad K^\theta=\frac{1}{2\pi^2}\left[\frac{\mu B}{\mu+B}-\frac{\mu\gamma}{\mu+\gamma}\right]. \quad (2.5)$$

At low temperatures dislocations will occur only as bound pairs. The pairs at the shortest wavelengths may be integrated out leading to a renormalization of K^r, K^θ, and the elastic constants. Introducing a vortex fugacity $y=\exp(-E_c/k_BT)$, we obtain the recursion relations[2,6]

$$\frac{dK^r}{dl}=-6\pi^3y^2\{[(K^r)^2+(K^\theta)^2]I_0(\pi K^\theta)-K^rK^\theta I_1(\pi K^\theta)\}\,, \quad (2.6a)$$

$$\frac{dK^\theta}{dl}=-6\pi^3y^2\{2K^rK^\theta I_0(\pi K^\theta)-\tfrac{1}{2}[(K^r)^2+(K^\theta)^2]I_1(\pi K^\theta)\}\,, \quad (2.6b)$$

$$\frac{dy}{dl}=(2-\pi K^r)y+2\pi I_0(\pi K^\theta)y^2\,, \quad (2.6c)$$

$$\frac{d\mu^{-1}}{dl}=3\pi y^2 I_0(\pi K^\theta)\,, \quad (2.6d)$$

$$\frac{dB^{-1}}{dl}=3\pi y^2[I_0(\pi K^\theta)-I_1(\pi K^\theta)]\,, \quad (2.6e)$$

$$\frac{d\gamma^{-1}}{dl}=3\pi y^2[I_0(\pi K^\theta)+I_1(\pi K^\theta)]\,. \quad (2.6f)$$

At the melting temperature, the renormalized value of K^r is universal:

$$K^{r*}=\frac{2}{\pi}\,. \quad (2.7)$$

We will analyze these equations only to second order in y and $2-\pi K^r$. This will be sufficient to obtain the critical exponents and amplitude ratios. It is therefore permissible to replace K^r and K^θ on the right-hand side (rhs) of Eqs. (2.6) by their renormalized values K^{r*} and $K^{\theta*}$ [except in the first term on the rhs of (2.6c)]:

$$K^{\theta*}=\frac{1}{2\pi^2}\left[\frac{\mu^*B^*}{\mu^*+B^*}-\frac{\mu^*\gamma^*}{\mu^*+\gamma^*}\right]. \quad (2.8)$$

Following the notation of Young,[6] we now introduce variables to simplify the appearance of Eq. (2.6). We define

$$\sigma=\frac{K^{\theta*}}{K^{r*}}\,, \quad (2.9a)$$

$$x=2-\pi K^r\,, \quad (2.9b)$$

$$\alpha^2=\frac{\frac{1}{24}I_0(2\sigma)}{1+\sigma^2-\sigma[I_1(2\sigma)/I_0(2\sigma)]}, \tag{2.9c}$$

$$Y=\frac{\pi I_0(2\sigma)}{\alpha}y . \tag{2.9d}$$

Equations (2.6a) and (2.6c) now take the simple form

$$\frac{dx}{dl}=Y^2 , \tag{2.10a}$$

$$\frac{dY}{dl}=xY+2\alpha Y^2 . \tag{2.10b}$$

The equations for the elastic constants may be streamlined in a similar manner. We obtain

$$\frac{d\mu^{-1}}{dl}=C_\mu Y^2 , \tag{2.11a}$$

$$\frac{dB^{-1}}{dl}=C_B Y^2 , \tag{2.11b}$$

$$\frac{d\gamma^{-1}}{dl}=C_\gamma Y^2 . \tag{2.11c}$$

C_μ, C_B, and C_γ are defined by

$$C_\mu=\frac{1}{8\pi}\frac{1}{1+\sigma^2-[\sigma I_1(2\sigma)/I_0(2\sigma)]}, \tag{2.12a}$$

$$C_B=\left[1-\frac{I_1(2\sigma)}{I_0(2\sigma)}\right]C_\mu , \tag{2.12b}$$

$$C_\gamma=\left[1+\frac{I_1(2\sigma)}{I_0(2\sigma)}\right]C_\mu . \tag{2.12c}$$

We now analyze the renormalization-group flows of the variables x and Y. Equations (2.10) may be used to yield

$$Y\frac{dY}{dx}=x+2\alpha Y . \tag{2.13}$$

This equation is homogeneous in x and Y. It can therefore be integrated to yield

$$|Y+[(1+\alpha^2)^{1/2}-\alpha]x|^{\bar\nu} \times |Y-[(1+\alpha^2)^{1/2}+\alpha]x|^{1-\bar\nu}=c\,|t|^{\bar\nu}, \tag{2.14}$$

where $\bar\nu$ is defined by

$$\bar\nu=\frac{(1+\alpha^2)^{1/2}-\alpha}{2(1+\alpha^2)^{1/2}}, \tag{2.15}$$

and c is a constant of integration. A few renormalization-group trajectories as given by Eq. (2.14) are plotted in Fig. 2. Figure 3 shows a plot of $\bar\nu$ as function of σ. When c is 0 the system is on the critical manifold. A small change in temperature will change the initial values of Y and x by an amount that is proportional to the change in temperature. This gives us the temperature dependence of the rhs of Eq. (2.14). At a temperature t above the critical temperature, let the initial values of x and Y be $(-x_i,Y_i)$. If one integrates the renormalization-group equations to a value of $l=l^*$ such that (x,Y) reach a value (x_f,Y_f) well away from the critical point, then

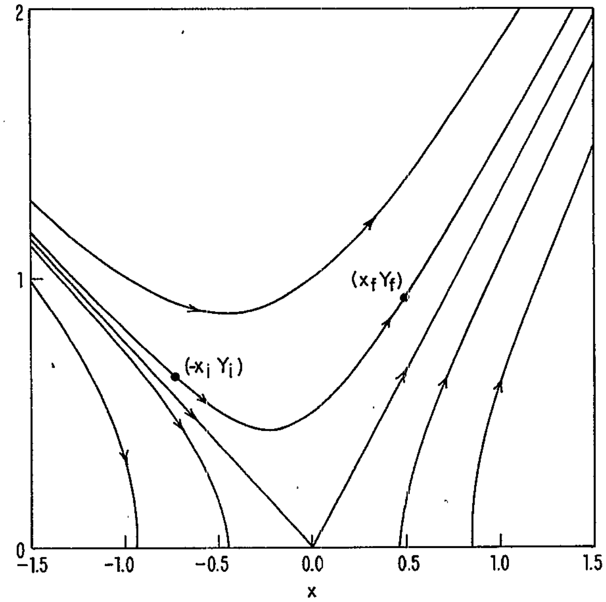

FIG. 2. Sample renormalization-group flows in the $x-Y$ plane. To determine the correlation length one integrates the flow equations from $(-x_i,Y_i)$ to (x_f,Y_f).

$$\xi_+=a_0\exp(l^*) , \tag{2.16}$$

where a_0 is a microscopic length. The parameter l^* is given by

$$l^*=\int_{-x_i}^{x_f}\frac{dx}{Y^2} . \tag{2.17}$$

Rescaling x and Y by

$$\bar x=\frac{x}{c\,|t|^{\bar\nu}}, \qquad \bar Y=\frac{Y}{c\,|t|^{\bar\nu}}, \tag{2.18}$$

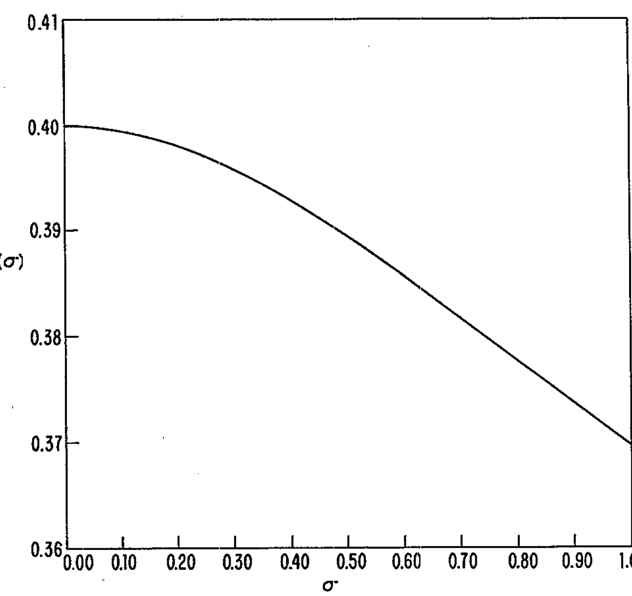

FIG. 3. Exponent $\bar\nu$ as function of σ.

we obtain

$$l^* = \frac{1}{c\,|t|^{\bar\nu}} \int_{-x_i/(c|t|^{\bar\nu})}^{x_f/(c|t|^{\bar\nu})} \frac{d\bar{x}}{\bar{Y}^2} . \tag{2.19}$$

The leading temperature dependence of l^* has now been brought to the front of the integral. The limits of the integral may be replaced by $-\infty$ and $+\infty$ without affecting the coefficient of the $1/|t|^{\bar\nu}$ term. So we obtain for the value of b defined in Eq. (1.5),

$$b = \frac{A}{c} , \tag{2.20a}$$

where

$$A = \int_{-\infty}^{+\infty} \frac{d\bar{x}}{\bar{Y}^2} , \tag{2.20b}$$

with

$$|\bar{Y} + [(1+\alpha^2)^{1/2} - \alpha]\bar{x}|^{\bar\nu} \times |\bar{Y} - [(1+\alpha^2)^{1/2} + \alpha]\bar{x}|^{1-\bar\nu} = 1 . \tag{2.20c}$$

Note that A is a function of σ only.

As noted earlier, the value of x at the melting temperature is 0. Away from the melting temperature, we expect x to have a behavior similar to the elastic constants, i.e.,

$$x(T \to T_m^-) = -b_x |t|^{\bar\nu} . \tag{2.21}$$

The value of x at long wavelengths will be obtained as l tends toward ∞ in the renormalization-group equations. Below T_m, Y tends toward 0 in this limit. From Eq. (2.14) we can therefore obtain

$$b_x = \frac{c}{[(1+\alpha^2)^{1/2} - \alpha]^{\bar\nu}[(1+\alpha^2)^{1/2} + \alpha]^{1-\bar\nu}} . \tag{2.22a}$$

Using Eqs. (2.20) and (2.22a), we get

$$bb_x = \frac{A}{[(1+\alpha^2)^{1/2} - \alpha]^{\bar\nu}[(1+\alpha^2)^{1/2} + \alpha]^{1-\bar\nu}} . \tag{2.22b}$$

This is the first of our universal amplitude ratios. Note that the nonuniversal quantity c has dropped out.

We now turn to a determination of the universal amplitude ratios for the elastic constants. From Eqs. (2.10a) and (2.11a) we obtain

$$\frac{d\mu^{-1}}{dx} = C_\mu . \tag{2.23}$$

This indicates that near the critical temperature, the changes in x and μ^{-1} are proportional. An immediate consequence is that

$$bb_\mu \equiv F_\mu(\sigma) = C_\mu bb_x . \tag{2.24}$$

The amplitude ratios for B^{-1} and γ^{-1} can be obtained in a similar manner:

$$bb_B \equiv F_B(\sigma) = C_B bb_x , \tag{2.25a}$$

$$bb_\gamma = C_\gamma bb_x . \tag{2.25b}$$

The functions $bb_\mu(\sigma)$, $bb_B(\sigma)$, and $bb_\gamma(\sigma)$ are plotted in Fig. 4. For a smooth substrate, σ takes the value 1 and the amplitude ratios are then equal to

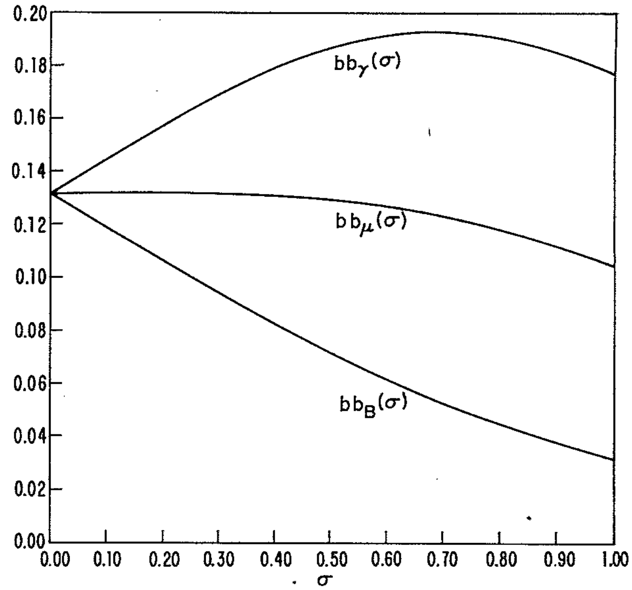

FIG. 4. Universal amplitude ratios bb_μ, bb_B, and bb_γ as a function of σ.

$$\begin{aligned} bb_x &= 3.425 , \\ bb_\mu &= 0.105 , \\ bb_B &= 0.032 , \\ bb_{K'} &= 1.713 , \\ bb_\gamma &= 0.178 . \end{aligned} \tag{2.26}$$

Note however that γ is 0 on a smooth substrate and

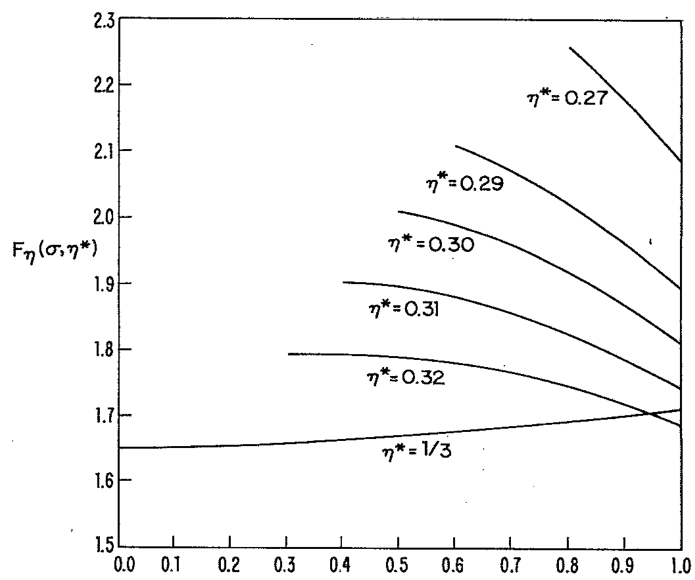

FIG. 5. Universal amplitude ratio bb_η as a function of σ for various values of η^*. These values are correct for weak substrate potentials. The lines end abruptly because only a limited set of values of η^* and σ give physically sensible elastic constants.

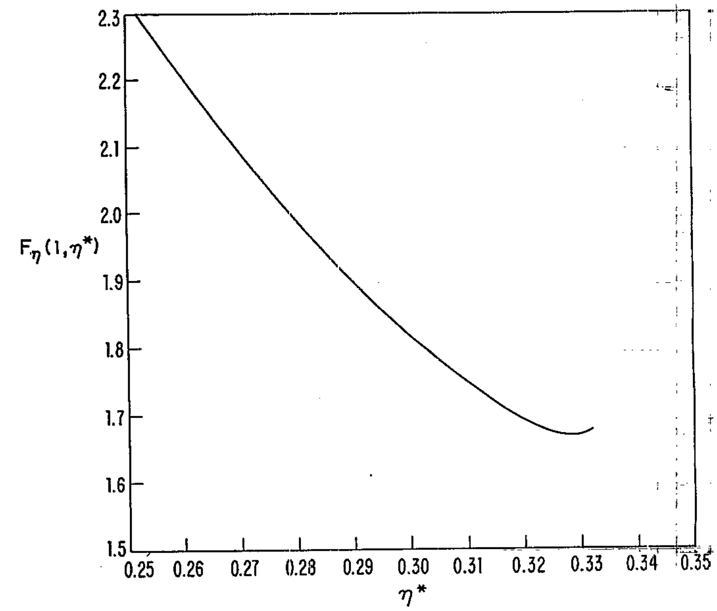

FIG. 6. Universal amplitude ratio bb_η for a smooth substrate as a function of η^*.

remains so under a renormalization-group transformation. It is however still possible to formally evaluate bb_γ as the limit of the coefficient of the cusplike singularity in γ^{-1} as γ tends toward 0 on a smooth substrate. On superfluid ^{4}He films, the quantity analogous to bb_{K^r} takes on the value $\pi/2$.[8]

It now remains to determine the amplitude ratio associated with η. This ratio turns out to be a straightforward algebraic consequence of Eqs. (2.24) and (2.25) and the relation[5]

$$\eta=\frac{4\pi}{3}\frac{(2\mu+B+\gamma)}{(\mu+\gamma)(\mu+B)} . \tag{2.27}$$

The details of the calculation are presented in the Appendix. The function $F_\eta(\sigma,\eta^*)$ is plotted as a function of σ for various values of η^* in Fig. 5. This form of the function is good for weak substrate potentials. On a smooth substrate, where $\sigma=1$, the amplitude ratio turns out to be

$$bb_\eta=\frac{1.142-0.265(1-3\eta^*)^{1/2}}{\eta^*}-1.712 . \tag{2.28}$$

This is plotted as a function of η^* in Fig. 6. There is no ambiguity in this function and it should be valid for melting on any smooth substrate.

III. CONCLUSION

We have calculated several amplitude ratios for dislocation-mediated melting. These amplitude ratios should be useful in testing this theory of melting on smooth and periodic substrates. In particular, for melting on periodic substrates, these amplitude ratios should eliminate the ambiguity associated with having a range of allowed values for the critical exponents.

ACKNOWLEDGMENTS

It is a pleasure to thank David R. Nelson for suggesting the possible existence of these amplitude ratios and for a critical reading of the manuscript. This research was supported by the National Science Foundation, through the Harvard Materials Science Laboratory and through Grant No. DMR-82-07431.

APPENDIX

In this appendix we present the details of the calculation of bb_η. Inserting the expansions (1.8) and (2.3) into (2.27), and comparing the result with (1.9), we get

$$b_\eta=\frac{b_\mu(\mu^*)^2+b_\gamma(\gamma^*)^2}{\mu^*+\gamma^*}+\frac{b_\mu(\mu^*)^2+b_B(B^*)^2}{\mu^*+B^*}-\frac{2b_\mu(\mu^*)^2+b_B(B^*)^2+b_\gamma(\gamma^*)^2}{2\mu^*+B^*+\gamma^*} . \tag{A1}$$

The values of μ^*, B^*, and γ^* are needed. Using the values of $K^{r*}=2/\pi$, σ, and η^*, and Eqs. (2.5), (2.8), and (2.27) we obtain

$$\mu^*=\frac{4\pi}{1\pm(1-3\eta^*)^{1/2}} , \tag{A2}$$

$$B^*=\frac{4\pi}{2/(1+\sigma)-[1\pm(1-3\eta^*)^{1/2}]} , \tag{A3}$$

$$\gamma^*=\frac{4\pi}{2/(1-\sigma)-[1\pm(1-3\eta^*)^{1/2}]} . \tag{A4}$$

Note that given the values of σ, η^*, and K^{r*}, there are two possible values of the elastic constants. Not all values of σ and η^* will give physically sensible elastic moduli. We need in addition to impose the constraints that μ, B, and γ are positive, and for a positive Poisson's ratio it is required that $B>\mu$. On a smooth substrate, only the negative sign can satisfy these constraints. This remains true for small values of γ also. The results we have presented are obtained by choosing the negative sign, inserting Eqs. (A2)–(A4) into (A1), and using (2.24) and (2.25). Note also that for this choice of sign, σ can only take positive values to yield a positive Poisson's ratio. For stronger substrate potentials, the positive sign may also give physically sensible results. The values of the elastic constants for the two signs, given the values of η^* and σ, are widely separated. It should therefore not be difficult in any experiment to pick the correct sign, using other estimates of the sizes of the elastic constants.

---

[1]J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973).
[2]D. R. Nelson, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, New York, 1983), Vol. 7.
[3]D. R. Nelson, Phys. Rev. B 18, 2318 (1978).
[4]B. I. Halperin and D. R. Nelson, Phys. Rev. Lett. 41, 121 (1978); 41, 519 (1978).
[5]D. R. Nelson and B. I. Halperin, Phys. Rev. B 19, 2456 (1979).

[6]A. P. Young, Phys. Rev. B **19**, 1855 (1979).
[7]P. Dutta and S. K. Sinha, Phys. Rev. Lett. **47**, 50 (1981).
[8]V. A. Ambegaokar, B. I. Halperin, D. R. Nelson, and E. D. Siggia, Phys. Rev. B **21**, 1806 (1980).
[9]C. C. Grimes and G. Adams, Phys. Rev. Lett. **42**, 795 (1979).
[10]R. H. Morf, Phys. Rev. Lett. **43**, 931 (1979).
[11]F. Gallet, G. Deville, A. Valdes, and F. I. B. Williams, Phys. Rev. Lett. **49**, 212 (1982).
[12]G. Deville, A. Valdes, E. Y. Andrei, and F. I. B. Williams, Phys. Rev. Lett. **53**, 588 (1984).
[13]C. J. Guo, D. B. Mast, R. Mehrotra, Y.-Z. Ruan, M. A. Stan, and A. J. Dahm, Phys. Rev. Lett. **51**, 1461 (1983).
[14]P. Dimon, P. M. Horn, M. Sutton, R. J. Birgeneau, and D. E. Moncton (unpublished).
[15]P. A. Heiney, R. J. Birgeneau, G. S. Brown, P. M. Horn, D. E. Moncton, and P. W. Stephens, Phys. Rev. Lett. **48**, 104 (1982).
[16]P. A. Heiney, P. W. Stephens, R. J. Birgeneau, P. M. Horn, and D. E. Moncton, Phys. Rev. B **28**, 6416 (1983).
[17]J. P. McTague, J. Als-Nielsen, J. Bohr, and M. Nielsen, Phys. Rev. B **25**, 7765 (1982).
[18]T. T. Chung, Surf. Sci. **87**, 348 (1979).
[19]A. D. Migone, Z. R. Li, and M. H. W. Chan, Phys. Rev. Lett. **53**, 810 (1984).
[20]P. Dutta, S. K. Sinha, P. Vora, M. Nielsen, L. Passell, and M. Bretz, in *Ordering in Two Dimensions,* edited by S. Sinha (North-Holland, New York, 1980).
[21]T. F. Rosenbaum, S. E. Nagler, P. M. Horn, and Roy Clarke, Phys. Rev. Lett. **50**, 1791 (1983).
[22]W. F. Brinkman, D. S. Fisher, and D. E. Moncton, Science **217**, 693 (1982), and references therein.
[23]The reason for negative values of σ is discussed in the Appendix.
[24]$\overline{\nu}$ is not a monotonic function of σ, but, for weak substrate potentials, σ is expected to be positive. Imposing this constraint is sufficient to determine σ as a unique function of $\overline{\nu}$.
[25]J. L. Cardy, Phys. Rev. B **26**, 6311 (1982).
[26]A. J. Dahm, Phys. Rev. B **29**, 484 (1984).
[27]K. J. Strandburg, S. A. Solla, and G. V. Chester, Phys. Rev. B **28**, 2717 (1983).
[28]D. R. Nelson, Phys. Rev. B **26**, 269 (1982).