Universal $T$-linear resistivity in two-dimensional quantum-critical metals from spatially random interactions

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We consider two-dimensional metals of fermions coupled to critical scalars, the latter representing order parameters at zero crystal momentum or emergent gauge fields. We show that at low temperatures ($T$), such metals generically exhibit a $T$-linear resistivity arising from spatially random fluctuations in the fermion-scalar Yukawa couplings about a non-zero spatial average. We also find a $T \ln(1/T)$ specific heat in such metals. These results are obtained in the large $N$ expansion of an ensemble of critical metals.

A major theme in the study of correlated metals has been their ‘Planckian behavior’ at low temperatures i.e. a linear-in-temperature resistivity which appears to be controlled by a dissipative relaxation time of order $\hbar/(k_BT)$ (where $T$ is the absolute temperature)$^{[1-7]}$. Moreover, this anomalous resistivity is invariably accompanied by a logarithmic enhancement of the Sommerfeld metallic specific heat $\frac{1}{T^2} \ln(1/T)$ $^{[1]}$.

A popular approach towards describing such metals in two spatial dimensions is the theory of a Fermi surface coupled to a gapless scalar field $^{[8]}$. We will limit our considerations here to scalar fields near zero crystal momentum, in which case the scalar field can either represent order parameters breaking point group, time reversal, or spin rotation symmetry, or represent the transverse component of an emergent gauge field. Fortunately, all these cases are described by essentially the same low energy theory $^{[9]}$. An important early result was that the self energy of the fermion near the Fermi surface has the frequency dependence $\text{Im} \Sigma(\omega) \sim \omega^{2/3}$ $^{[10]}$, and so there are no quasiparticle excitations on the Fermi surface (although the Fermi surface remains sharp in momentum space). While this anomalous self energy can lead to singular behavior in the optical conductivity $^{[11]}$, conservation of momentum in the low energy theory implies that d.c. conductivity in not affected $^{[12-17]}$. In other words, the strong coupling between the Fermi surface and the scalar field places the system in the limit of strong ‘scalar drag’; this is in stark contrast to the electron-phonon system, where the weak electron-phonon coupling makes phonon drag a factor only in ultrapure samples $^{[18]}$.

In clean systems, umklapp scattering can lead to non-zero resistance, and its influence in quantum-critical metals has been investigated in other works $^{[16, 19]}$. Umklapp is suppressed at low $T$, the predictions for transport are not universal and depend upon specific Fermi surface details, and there is no corresponding $T \ln(1/T)$ specific heat.

This paper will focus on the effects of disorder, and finds a universal phenomenology that matches several aspects of the observations, including the $T$-linear resistivity and the $T \ln(1/T)$ specific heat. Earlier works $^{[13, 16, 20]}$ examined random potential scattering of the fermions, a central ingredient in the theory of disordered, interacting metals $^{[21]}$. The propagator of the scalar field has a form similar to those of diffusive density fluctuations, with dynamic critical exponent $z = 2$ $^{[20]}$, but there are no singular corrections to transport coefficients from random potential scattering, as we shall confirm below. Here, we examine spatially random fluctuations in the ‘Yukawa coupling’ between the fermions and the scalar, upon a background of a coupling with non-zero spatial average. While the fermion inelastic self energy corrections are dominated by the spatially uniform coupling, we show that the transport is nevertheless dominated by the spatially random coupling, and this leads to our main results. Our work follows several other recent works with random Yukawa interactions $^{[22-31]}$ inspired by the Sachdev-Ye-Kitaev (SYK) model $^{[32, 33]}$, along with studies which found linear-in-$T$ resistivity with random interactions, but with vanishing spatial average $^{[29, 31, 34, 35]}$.

Spatially uniform quantum critical metal. We recall the SYK-inspired large $N$ theory of the two-dimensional quantum critical metal $^{[29, 31]}$. The imaginary time ($\tau$) action for the fermion field $\psi_i$ and scalar field $\phi_i$ (with $i = 1 \ldots N$ a flavor index) is $^{[31]}$

$$S_g = \int d\tau \sum_{k} \sum_{i=1}^{N} \psi_i^\dagger(k) \left[ \partial_\tau + \varepsilon(k) \right] \psi_i(\tau) + \frac{1}{2} \sum_{q} \sum_{i=1}^{N} \phi_i(q) \left[ -\partial_\tau^2 + K q^2 + m_b^2 \right] \phi_{i, -q}(\tau) + \frac{g_{ijl}}{N} \int d\tau d^2r \sum_{i,j,l=1}^{N} \psi_i^\dagger(\tau, r) \psi_j(\tau, r) \phi_l(\tau, r), \quad (1)$$

where the fermion dispersion $\varepsilon(k)$ determines the Fermi surface, the scalar $m_b$ is needed for infrared regularization but does not appear in final results, and $g_{ijl}$ is space.
We now add a spatially random fermion potential

\[ S_v = \frac{1}{\sqrt{N}} \int d^2 r d\tau \, v_{ij}(r) \psi^\dagger_i(r,\tau) \psi_j(r,\tau) \]

and here the overline is an average over spatial coordinates and flavor space. The present large \( N \) limit yields results similar to earlier studies [13, 16, 20]. The low frequency boson propagator is now characterized by \( z = 2 \), while the fermion self energy has an elastic scattering term, along with a marginal Fermi liquid [36] inelastic term at low frequencies

\[ \Pi(i\omega, q) = -\frac{N g^2 |\omega|}{T}, \quad \Gamma = 2\pi v_F^2 N, \]

\[ \Sigma(i\omega, k = k_F \hat{k}) = -\frac{\Gamma}{2} \text{sgn}(\omega) - \frac{i g^2 \omega}{2\pi T} \ln \left( \frac{e^{\frac{1}{2} |\omega|^2 N g^2 v_F^2 T}}{N g^2 v_F^2 |\omega|} \right), \]

where the overline represents average over flavor space

\[ \bar{g}_{jkl} = 0, \quad \bar{g}_{ijkl} g_{abc} = g^2 \delta_{ia} \delta_{jb} \delta_{kc}, \]

(2)

The hypothesis is that a large domain of flavor couplings all flow to the same universal low energy theory (as in the SYK model), so we can safely examine the average of an ensemble of theories. Momentum is conserved in each member of the ensemble, and the flavor-space randomness does not lead to any essential difference from non-random theories. This is in contrast to position-space randomness which we consider later, which does relax momentum and modify physical properties.

The large \( N \) saddle point of (1) has singular fermion self energy \( (\Sigma) \) and boson \( (\Pi) \) self-energy terms at high frequencies \( \omega \gg 1 \) and \( \omega \ll 1 \) at low frequencies \( \omega = 2 \), while the fermion self energy has an elastic scattering term

\[ \Sigma(i\omega, k) = -i c f \text{sgn}(\omega) |\omega|^{2/3} \]

\[ c_f = \frac{g^2}{2\pi v_F \sqrt{\delta}} \left( \frac{2\pi v_F k}{K^2 g^2} \right)^{1/3} \]

These are evaluated in the vicinity of antipodal points \( \pm k_0 \) on the Fermi surface to which \( q \) is tangent, with axes chosen so that \( q = (0, q) \) and fermionic dispersion \( \epsilon(\pm k_0 + k) = \pm k_F k_x + \kappa k_y^2/2 \). We assumed the whole Fermi surface is circular, and therefore \( \kappa = 1/m \) where \( m \) is the effective mass of the fermions.

The large \( N \) computation of the optical conductivity yields only the clean Drude result \( \text{Re}[\sigma(\omega)]/N = \pi N v_F^2 |\delta(\omega)/2| \), where \( N = m/(2\pi) \) is fermion density of states at the Fermi level. This holds in the patch theory, and also for a circular Fermi surface. The absence of a \( \omega \neq 0 \) contribution is tied to an exact cancellation between self-energy and vertex diagrams arising from momentum conservation. More general Fermi surfaces are expected to have an additional \( \sim |\omega|^{-2/3} \) contribution [11].

Potential disorder. We now add a spatially random fermion potential

\[ S_v = \frac{1}{\sqrt{N}} \int d^2 r d\tau \, v_{ij}(r) \psi^\dagger_i(r,\tau) \psi_j(r,\tau) \]

and here the overline is an average over spatial coordinates and flavor space. The present large \( N \) limit yields results similar to earlier studies [13, 16, 20]. The low frequency boson propagator is now characterized by \( z = 2 \), while the fermion self energy has an elastic scattering term, along with a marginal Fermi liquid [36] inelastic term at low frequencies

\[ \Pi(i\omega, q) = -\frac{N g^2 |\omega|}{T}, \quad \Gamma = 2\pi v_F^2 N, \]

\[ \Sigma(i\omega, k = k_F \hat{k}) = -\frac{\Gamma}{2} \text{sgn}(\omega) - \frac{i g^2 \omega}{2\pi T} \ln \left( \frac{e^{\frac{1}{2} |\omega|^2 N g^2 v_F^2 T}}{N g^2 v_F^2 |\omega|} \right), \]

at \( T = 0 \). However, the marginal Fermi liquid self energy, while leading to a \( T \ln(1/T) \) specific heat, does not lead to a corresponding [36] linear-\( T \) term in the DC resistivity, as it arises from forward scattering of electrons off the \( q \sim 0 \) bosons. These forward scattering processes are unable to relax either current or momentum due to the small wavevector of the bosons involved and the momentum conservation of the \( g \) interactions.

As a result, even a perturbative computation of the conductivity at \( O(g^2) \) (Fig. 1) shows a cancellation between the interaction-induced self energy contributions and the interaction-induced vertex correction, leading to a DC conductivity that is just a constant, set by the elastic potential disorder scattering rate \( \Gamma \). A full summation of all diagrams at large \( N \) shows that the \( g \) interactions only renormalize the frequency term in the Drude formula

\[ \frac{1}{N} \text{Re}[\sigma(\omega \gg T)] = \frac{1}{2} \frac{N v_F^2 \Gamma}{\omega^2 + \Gamma^2}, \]

where \( \omega \) and \( \Lambda \) is a UV momentum cutoff. In the limit of large \( k_F \) (and hence large \( v_F \)), this renormalization is negligible and \( \omega \approx \omega \). The leading frequency dependence of the optical conductivity at frequencies \( \omega \ll \Gamma \) is therefore just a constant, and there is no linear in frequency correction. Correspondingly, in the DC limit, there is no linear in \( T \) correction, and a conventional \( T^2 \) correction is expected.

Interaction disorder. Our main results are obtained with additional spatially random interactions. In principle, such terms will be generated under renormalization from \( S_v \). However, this does not happen in our large \( N \) limit, and so we have to account for the renormalization by adding an explicit term:

\[ S_{g'} = \frac{1}{N} \int d^2 r d\tau \, g'_{ij}(r) \psi^\dagger_i(r,\tau) \psi_j(r,\tau) \phi(r,\tau) \]

\[ g'_{ij}(r) = 0, \quad \text{sgn}(\omega) g'_{ijkl}(r) g'_{abc}(r) = g'^2 \delta(r - r') \delta_{ia} \delta_{jb} \delta_{kc}. \]

Note that \( v, g, g' \) are all independent flavor-random variables. Earlier work has considered the limiting case \( g = 0, v = 0, g' \neq 0 \) [29, 31]. We will instead describe the more physically relevant regime where spatial disorder is a weaker perturbation to a clean quantum critical system, with \( g \) the largest interaction coupling. We therefore now have all of \( g, v, g' \) nonzero. As in the discussion above on potential disorder, we find that the low frequency boson propagator is characterized by \( z = 2 \), and the low frequency fermion self energy again has an elastic scattering term, along with a marginal Fermi liquid inelastic term

\[ \Pi(i\omega, q) = -\frac{N g^2 |\omega|}{T} - \pi N^2 g^2 |\omega| \equiv -c_d |\omega|, \]
\[ \Sigma(i\omega, k = k_F \hat{k}) = -i\frac{\Gamma}{2}\text{sgn}(\omega) - \frac{i g^2 \omega}{2\pi^2 T} \ln \left( \frac{e\Gamma^2}{v_F^2 c_d|\omega|} \right) \]
\[ - \frac{i N g^2 \omega}{4\pi} \ln \left( \frac{e\Lambda^2}{c_d|\omega|} \right) \quad (T = 0). \quad (9) \]

This self-energy leads to a \( T \ln(1/T) \) specific heat, as for the large \( g' \) case \[31\]. However, there is now an important difference with respect to the previous case where \( g' = 0 \), which leads to markedly different charge transport properties: the marginal Fermi liquid self energy now contains a term (last line of (9)), that does not arise solely from forward scattering of electrons. This term is produced by the \textit{disordered} part of the interactions (8). Therefore, this part of the self energy represents scattering that relaxes \textit{both} current and momentum carried by the electron fluid, and therefore its imaginary part on the real frequency axis determines the inelastic transport scattering rate.

We can show this as follows by computing the conductivity using the Kubo formula. If we work perturbatively in \( g \) and \( g' \), then the conductivity at \( \mathcal{O}(g^2) \) and \( \mathcal{O}(g'^2) \) in the large \( N \) limit is given by the sum of self energy contributions and vertex corrections (Fig. 1). However, due to the isotropy of the scattering processes arising from the \( g' \) interactions, only the vertex correction due to the \( g \) interactions survives. The conductivity up to the first sub-leading frequency dependent correction is then given by

\[ \frac{1}{N} \text{Re}[\sigma(\omega \gg T)] = \sigma_v + \sigma_{\Sigma,g} + \sigma_{V,g} + \sigma_{\Sigma,g'}; \]
\[ \sigma_v(\omega) = \frac{N v_F^2}{2\Gamma}, \quad \sigma_{\Sigma,g}(\omega) = -\frac{N v_F^2 g^2 |\omega|}{8\pi \Gamma^3}, \]
\[ \sigma_{V,g}(\omega) = \frac{N v_F^2 g^2 |\omega|}{8\pi \Gamma^3} - \frac{4\pi N v_F^2 g^2 c_1}{\alpha \Gamma}, \]
\[ \sigma_{\Sigma,g'}(\omega) = -\frac{N^2 v_F^2 g^2 |\omega|}{16\Gamma^2}, \quad (10) \]

where \( c_1 \) is a dimensionless constant that depends upon the UV cutoff of the theory. Eq. (10) can then be suitably re-expressed as

\[ \frac{1}{N} \text{Re}[\sigma(\omega \gg T)] = \frac{N v_F^2 \left( 1 - \frac{8\pi g^2 c_1}{v_F c_d} - \frac{N g^2 |\omega|}{8\Gamma} \right)}{2\Gamma} \approx \frac{N v_F^2}{2\Gamma} \left( 1 + \frac{8\pi g^2 c_1}{v_F c_d} + \frac{N g^2 |\omega|}{8\Gamma} \right) \approx \frac{N v_F^2}{2\Gamma + N g^2 |\omega|}, \quad k_F \to \infty. \quad (11) \]

The incomplete cancellation of the self-energy contributions \( \sigma_{\Sigma,g}(\omega) + \sigma_{\Sigma,g'}(\omega) \) by the vertex correction \( \sigma_{V,g}(\omega) \) now leads to a linear in frequency correction to the constant transport scattering rate \( \Gamma \). In the opposite limit \( |\omega| \ll T \), this translates into a \( T \)-linear correction to the resistivity in the DC limit; computing the co-efficient of the linear-\( T \) resistivity requires a self-consistent numerical analysis, which has been carried out in the large \( g' \) limit \[31\]. Remarkably, the slope of this scattering rate with respect to \( |\omega| \) (and therefore \( T \)) does not depend on \( \Gamma \) and hence on the residual \( (\omega = T = 0) \) resistivity. The perturbative result described here continues to be valid under a full resummation of all diagrams at large \( N \) in the Kubo formula, as all higher order contributions are merely repetitions of the interaction insertions in Fig. 1b,c.

We can also consider the case where \( v = 0 \) but \( g \neq 0 \) and \( g' \neq 0 \). In this case we have \( (T = 0) \)

\[ \Pi(i\omega, q) = -c_b \frac{|\omega|}{|q|} - \pi N^2 g^2 |\omega|, \quad (12) \]
\[ \Sigma(i\omega, k = k_F \hat{k}) = -ic_F\text{sgn}(\omega)|\omega|^{2/3} - \frac{iN g^2 \omega}{6\pi} \ln \left( \frac{e\Lambda^3}{c_b|\omega|} \right). \]

Interestingly, the disordered interactions induce a marginal Fermi liquid term in \( \Sigma \), which manifests as the first higher order correction to the translationally invariant result (3). It is sufficient in this case to compute the conductivity using the theory of antipodal patches. \[?\].

We then find, as before, that \( \sigma_{\Sigma,g} \) and \( \sigma_{V,g} \) cancel, and

\[ \frac{1}{N} \sigma(\omega \gg T) = \frac{N v_F^2}{2\omega} - \frac{N^2 v_F^2 g^2}{24\pi \omega} \ln \left( \frac{e\Lambda^3}{c_b|\omega|} \right) \approx \frac{N v_F^2}{2\omega} + \frac{N g^2 \omega}{6\pi} \ln \left( \frac{e\Lambda^3}{c_b|\omega|} \right); \quad (13) \]

\[ \frac{1}{N} \frac{\text{Re}[\sigma(\omega \gg T)]}{N^2 v_F^2 g^2 |\omega|} = \left[ \frac{6|\omega|}{\left( 2 + \frac{N g^2}{6\pi} \ln \left( \frac{e\Lambda^3}{c_b|\omega|} \right) \right)^2 + \frac{N^2 g^4 |\omega|^4}{36}} \right]^{-1}. \]

The transport scattering rate is therefore still linear in \( |\omega| \) (and hence \( T \)), up to logarithms, and there is no residual resistivity when \( v = 0 \) despite the presence of disorder in \( g' \).

Crossovers. For \( v \neq 0 \), and energy \( (E) \) scales larger than \( E_{c,1} \sim \Gamma^2/(v_F c_d) \), but smaller than \( E_{c,2} \sim g^4/(g^6 v_F^2 N^2) \) (assuming \( E_{c,1} < E_{c,2} \)), the leading frequency dependence of the inelastic part of the fermion self energy changes from \( i\omega \ln(1/|\omega|) \) to \( i\text{sgn}(\omega)|\omega|^{2/3} \), as in the theory with \( v = 0 \) described above. However, as shown above, the \( |\omega| \) or \( T \) dependence of the transport scattering continues to be linear (up to logarithms), but with a different slope that is a factor of \( \sim 2/3 \) smaller.

For energy scales larger than \( E_{c,2} \sim g^4/(g^6 v_F^2 N^2) \), there is an additional crossover to the theory with \( g = 0 \) considered in Refs. \[29, 31\], which also has a linear \( |\omega| \) or
The dimensionless number \( \alpha \) appearing in a Drude formula for the resistivity. In our theory. However, on the other hand, if \( E > E_{c,1} \), with no change of slope of the transport scattering rate.

Planckian behavior. Experimental analyses \([6, 7]\) have compared the slope of the linear-\( T \) resistivity to the renormalization of the effective mass in a proximate Fermi liquid, and so deduced a ‘scattering time’ \( \tau_{\text{st}}^* \) appearing in a Drude formula for the resistivity. In our theory, we obtain

\[
\frac{1}{\tau_{\text{st}}} = \alpha \frac{k_B T}{\hbar}.
\]

The dimensionless number \( \alpha \) has been computed previously \([29, 31]\) in the limit \( g' \gg g \) to be \( \alpha \approx (\pi/2) \times \text{(ratio of logarithms of } T \text{)}. For smaller \( g' \) we find (at \( v \neq 0 \))

\[
\alpha \approx \frac{\pi}{2} \frac{g'^2}{g^2 L_1(T)} + \frac{2g^2}{\pi TN} L_2(T), \quad L_{1,2}(T) \sim -\ln T.
\]

Therefore, ‘Planckian behavior’ (\( \alpha \) depending only slowly on \( T \) and non-universal parameters) only occurs in the regime of large \( g' \) considered in Refs. \([29, 31]\). Otherwise, \( \alpha \ll 1 \) when \( g \) is the largest interaction coupling. For \( v = 0 \), \( \alpha \ll 1 \) and has a power-law dependence on \( T \), and therefore there is manifestly no Planckian behavior. It is worth noting that quantum critical \( T \)-linear resistivity with \( \alpha \ll 1 \) has been recently observed in experiments on heavy fermion materials \([7]\).

Boson mass disorder. Finally, we propose a route to accounting for disorder in the boson ‘mass’ \( m_b \). Such a term is not allowed for emergent gauge fields, but it can appear as a fluctuation in the position of the quantum critical point for the cases where \( \phi \) is a symmetry breaking order parameter.

\[
S_w = \int d\tau \frac{1}{2\sqrt{N}} \int d^2 r \sum_{ij=1}^N w_{ij}(r)\phi_i(r, \tau)\phi_j(r, \tau) \tag{16}
\]

with

\[
\frac{w_{ij}(r)w_{lm}(r')}{w_{ij}^c(r)} = \frac{w^2}{2} \delta(r - r') (\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) \tag{17}
\]

The large \( N \) analysis shows that \( S_w \) is strongly relevant. Consequently it is appropriate to account for \( S_w \) first, by transforming to the bases of eigenmodes of \( \phi \) which are eigenstates of the harmonic terms for \( \phi \) in a given disorder realization. In this new basis, we will obtain a theory which has the same form as \( S_g + S_v + S_{g'} \) with additional spatial disorder in the couplings, including in \( K \). However, it is not difficult to show that spatial disorder in \( K \) is unimportant. So we conclude that \( S_w \) can be effectively absorbed in a renormalization of the values of \( v \) and \( g' \), and we can continue to use our results for \( S_g + S_v + S_{g'} \).

Discussion.

\( i \) A phenomenologically attractive feature of our theory is that the residual resistivity and the slope of the linear-\( T \) resistivity are determined by different types of disorder: respectively, the potential disorder \( v \) (which determines the elastic scattering rate \( \Gamma \)) and the interaction disorder \( g' \) (which determines the inelastic self energy in the last term of (9)).

\( ii \) All the computed diagrams, and their associated cancellations, apply also for \( N = 1 \); the large \( N \) method is mainly a very useful guide towards systematically selecting a consistent set of diagrams to resum.

\( iii \) Our theory of the influence of spatial disorder is non-perturbative in the disorder strength, unlike the perturbative disorder analysis of earlier memory function treatments \([14, 16]\).

\( iv \) When the values of the interaction couplings and \( T \) are large enough to make the fermion self energy \( \Sigma \) comparable to the Fermi energy, we expect the theories described here to cross over into a locally critical ‘bad
metal’ regime \[37\]; it would be interesting to examine the relationship to the transport properties of such a regime.

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