Anomalous density fluctuations in a random $t$-$J$ model

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Abstract

A previous work (Joshi et al., Phys. Rev. X 10, 021033 (2020)) found a deconfined critical point at non-zero doping in a $t$-$J$ model with all-to-all and random hopping and spin exchange, and argued for its relevance to the phenomenology of the cuprates. We extend this model to include all-to-all and random density-density interactions of mean-square strength $K$. In a fixed realization of the disorder, and for specific values of the hopping, exchange, and density interactions, the model is supersymmetric; but, we find no supersymmetry after independent averages over the interactions. Using the previously developed renormalization group analysis, we find a new fixed point at non-zero $K$. However, this fixed point is unstable towards the previously found fixed point at $K = 0$ in our perturbative analysis. We compute the exponent characterizing density fluctuations at both fixed points: this exponent determines the spectrum of electron energy-loss spectroscopy.
I. INTRODUCTION

The possibility of a quantum critical point underneath the superconducting dome of high-temperature cuprate materials has been a subject of intense study. Such a critical point, and the corresponding critical theory, possibly holds the key to understanding the enigmatic strange-metal phase at high temperatures. Recently, high resolution experiments have been able to shed more light in this region. Photoemission experiments [1, 2] and thermal Hall measurements [3] have given strong evidence for a transformation in the Fermi surface across a critical value of doping. In this context, we have recently proposed a microscopic model which hosts such a quantum critical point [4]: it was shown to be a deconfined critical point with a SYK-like [5, 6] spin correlations.

The strange-metal phase is also characterized by an absence of quasiparticles and thus one expects a continuum response to many probes. Recently, an anomalous continuum was observed in dynamic charge response measurements [7, 8] on optimally doped Bi$_{2}$Sr$_{1.9}$Ca$_{1.0}$Cu$_{2.0}$O$_{8+x}$ (Bi-2212) using momentum-resolved electron energy-loss spectroscopy (M-EELS). Motivated by these measurements we investigate the density correlation in a model hosting a quantum critical point at finite doping.

The paper is organized as follows. In Sec. II we describe our model and related algebra of the operators. In Sec. III we discuss the mapping of our model to an impurity model, which can be then studied using renormalization group as shown in Sec. IV. We conclude in Sec. V and present an alternative RG calculation in Appendix A. A discussion on possibility of supersymmetry can be found in Appendix B.

II. MODEL

We consider the following Hamiltonian,

$$H_{tJK} = \frac{1}{\sqrt{N}} \sum_{ij} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j} J_{ij} S_i \cdot S_j + \frac{1}{\sqrt{N}} \sum_{i<j} K_{ij} n_in_j - \mu \sum_{i} c_{i\alpha}^\dagger c_{i\alpha}, \quad (2.1)$$

where $N$ is the number of sites, $\mu$ is the chemical potential, $\alpha$ is the spin index ($\uparrow$ or $\downarrow$), $n_i = c_{i\alpha}^\dagger c_{i\alpha}$ and double occupancy on each site is excluded, i.e., $n_i \leq 1$. The hoppings $t_{ij}$, exchange interactions $J_{ij}$, and density-density interactions $K_{ij}$ are random numbers drawn from a Gaussian probability distribution with zero mean value such that $|t_{ij}|^2 = t^2$, $|J_{ij}|^2 = J^2$ and $|K_{ij}|^2 = K^2$. Note that the density-density interactions are present in the familiar derivation of the $t$-$J$ model from the Hubbard model, and are usually ignored. We include them here as independent random couplings, because we are interested in their possible influence on the spectrum of density fluctuations.

To account for the double occupancy constraint, we fractionalize the electron on each site into
a bosonic holon \( b \) and fermionic spinon \( f_a \) degrees of freedom such that,
\[
c_a = f_a b^\dagger, \quad S^a = f^\dagger a \sigma^a_{\alpha\beta} f^\dagger \beta, \quad V = \frac{1}{2} f^\dagger a f_a + b^\dagger b, \quad n = f^\dagger a f_a.
\]
The Hilbert-space constraint of no double occupancy now takes the form: \( f^\dagger a f_a + b^\dagger b = 1 \). Note that \( V_i = 1 - n_i/2 \).

On each site \( i \), the operators \( c, S \) and \( V \) (dropping site indices) define a superalgebra \( SU(1|2) \) as follows:
\[
\{c_\alpha, c^\dagger_\beta\} = 0, \quad \{c_\alpha, c^\dagger_\beta\} = \delta_{\alpha\beta} V + \sigma^a_{\alpha\beta} S^a, \quad [S^a, c_\alpha] = -\frac{1}{2} \sigma^a_{\alpha\beta} c_\beta, \quad [S^a, c^\dagger_\alpha] = \frac{1}{2} \sigma^a_{\alpha\beta} c^\dagger_\beta,
\]
\[
[S^a, S^b] = i\epsilon_{abc} S^c, \quad [S^a, V] = 0, \quad [V, c_\alpha] = \frac{1}{2} c_\alpha, \quad [V, c^\dagger_\alpha] = -\frac{1}{2} c^\dagger_\alpha.
\]

As an aside, note that one can also work with an alternative equivalent representation with a bosonic spinon and fermionic holon, which form a \( SU(2|1) \) superalgebra [4].

The Hamiltonian \( H_{tJK} \) clearly commutes with total spin, \( \sum_i S_i^a \), and total density \( \sum_i V_i \). For the remaining generator of the \( SU(1|2) \) superalgebra, the commutator is simple for for \( t_{ij} = K_{ij}/2 = -J_{ij}/2 \), when we find
\[
[c_\alpha, H_{tJK}] = -\mu c_\alpha.
\]
which implies supersymmetry at fixed particle number. The non-random supersymmetric \( t-J \) model has been studied in the past in one dimension, for instance see Refs. [9–14].

III. LARGE-\( N \) LIMIT AND IMPURITY HAMILTONIAN

In order to make progress, we will now use the same strategy as in Ref. [4]. Using the replica trick and in the limit \( N \to \infty \) we effectively have a single-site problem,
\[
\mathcal{Z} = \int \mathcal{D} c_\alpha(\tau) e^{-S - S_\infty}
\]
\[
S = \int d\tau \left[ c^\dagger_\alpha(\tau) \left( \frac{\partial}{\partial \tau} - \mu \right) c_\alpha(\tau) \right] + t^2 \int d\tau d\tau' R(\tau - \tau') c^\dagger_\alpha(\tau) c_\alpha(\tau')
- \frac{J^2}{2} \int d\tau d\tau' Q(\tau - \tau') S(\tau) \cdot S(\tau') - \frac{K^2}{2} \int d\tau d\tau' P(\tau - \tau') n(\tau)n(\tau'),
\]
where the fields \( R, Q, \) and \( P \) have to be determined self-consistently via,
\[
R(\tau - \tau') = -\langle c_\alpha(\tau)c^\dagger_\alpha(\tau') \rangle_Z, \quad Q(\tau - \tau') = \frac{1}{3} \langle S(\tau) \cdot S(\tau') \rangle_Z, \quad P(\tau - \tau') = \langle n(\tau)n(\tau') \rangle_Z.
\]

To set-up our RG, let us ignore the self-consistency for now. We shall come back to it later. Let us assume that at the criticality the fields have the following power-law decay in imaginary time:
\[
P(\tau) \sim \frac{1}{|\tau|^{d-1}}, \quad Q(\tau) \sim \frac{1}{|\tau|^{d-1}}, \quad R(\tau) \sim \frac{\text{sgn}(\tau)}{|\tau|^{d+1}}.
\]
Now we introduce fermionic and bosonic fields in the same spirit as in Ref. [4] in order to obtain an impurity Hamiltonian. Such an impurity Hamiltonian has been studied in different limits in Refs. [15–22]. In our case we can map the above Hamiltonian to the following impurity and bath Hamiltonians:

\[ H_{\text{imp}} = (s_0 + \lambda) f^\dagger_\alpha f_\alpha + \lambda b^\dagger b + g_0 \left( f^\dagger_\alpha b \psi_\alpha(0) + H.c. \right) + \gamma_0 f^\dagger_\alpha \frac{\sigma^a_\alpha}{2} \beta f_\beta \phi_\alpha(0) + v_0 (f^\dagger_\alpha f_\alpha - n_f) \zeta(0) \]

\[ H_{\text{bath}} = \int |k|^r dk k \psi^\dagger_{\alpha \sigma} \psi_{\alpha \sigma} + \frac{1}{2} \int d^d x \left( \pi^2_\alpha + (\partial_x \phi_\alpha)^2 \right) + \frac{1}{2} \int d^{d'} x \left( \tilde{\pi}^2 + (\partial_x \zeta)^2 \right), \quad (3.4) \]

where \( \lambda \to \infty \) is introduced to handle the constraint \( f^\dagger_\alpha f_\alpha + b^\dagger b = 1 \), and \( n_f = \langle f^\dagger_\alpha f_\alpha \rangle \). We have introduced fermionic bath \( \psi_{\alpha \sigma} \), as well as bosonic baths \( \phi_\alpha \) and \( \zeta \), which upon integrating out gives us the original Hamiltonian. Also, \( \phi_\alpha(0) \equiv \phi_\alpha(x = 0) \), \( \zeta(0) \equiv \zeta(x = 0) \) and \( \psi_\alpha(0) \equiv \int dk |k|^r \psi_{\alpha \sigma} \).

The Hamiltonian \( H_{\text{imp}} + H_{\text{bath}} \) is our representation of the effective theory after averaging the disorder. We explore the possibility that this Hamiltonian could be supersymmetric in Appendix B, and find no supersymmetry. So supersymmetric is specific to particular realizations of disorder, and does not re-emerge after independent averages over \( t_{ij} \), \( J_{ij} \), and \( K_{ij} \).

**IV. RENORMALIZATION GROUP ANALYSIS**

In this section we present the details of RG analysis of the impurity Hamiltonian introduced in Eq. 3.4. At the tree-level the scaling dimensions are found as follows:

\[ \text{dim}[f] = \text{dim}[b] = 0, \quad \text{dim}[\psi_{\alpha \sigma}] = -\frac{1 + r}{2} \equiv -\text{dim}[\psi_\alpha(0)], \quad \text{dim}[\phi_\alpha] = \frac{d - 1}{2}, \quad \text{dim}[\zeta] = \frac{d' - 1}{2} \]

\[ \text{dim}[g_0] = \frac{1 - r}{2} \equiv \tilde{r}, \quad \text{dim}[\gamma_0] = \frac{3 - d}{2} \equiv \frac{\epsilon}{2}, \quad \text{dim}[v_0] = \frac{3 - d'}{2} \equiv \frac{\epsilon'}{2}. \quad (4.1) \]

This establishes \( r = 1, d = 3, \) and \( d' = 3 \) as upper critical dimensions. Next, the renormalized fields and couplings are defined as follows:

\[ f_\alpha = \sqrt{Z_f} f_{R\alpha}, \quad b = \sqrt{Z_b} b_R, \quad g_0 = \frac{\mu^r \sqrt{Z_g}}{Z_f} g, \quad \gamma_0 = \frac{\mu^{r/2} Z_{\gamma}}{Z_f \sqrt{\tilde{S}_{d+1}}} \gamma, \quad v_0 = \frac{\mu^{r'/2} Z_v}{Z_f \sqrt{\tilde{S}_{d'+1}}} v, \quad (4.2) \]

where \( \tilde{S}_d = \Gamma(d/2 - 1)/(4\pi^{d/2}) \). The bulk-bath fields \( \psi, \phi_\alpha \), and \( \zeta \) do not get renormalized because of the absence of the respective interaction terms. These renormalization factors will be determined in the following sections from the self-energy and vertex corrections. We shall work at zero temperature and tune the system to criticality, i.e., we set \( s_0 = 0 \) and subsequently derive the flow away from it.
FIG. 1: One-loop fermion and boson self-energy diagrams. Fermion self-energy diagrams are shown in (a), (b), and (c), while boson self energy is shown in (d). We use a convention where a solid line denotes $f$ propagator, a dashed line denotes $\psi$ propagator, wavy denotes $b$ propagator, spiral denotes $\phi$ propagator, and red spiral denotes $\zeta$ propagator.

A. Self energy

We begin with the calculation of the fermionic self energy at one-loop level. Note that at this level there are no diagrams involving both the bosonic and the fermionic bath couplings. Here we have three relevant diagrams, shown in Fig. 1 (a), (b) and (c). The diagrams in Fig. 1 (a) and (b) have been evaluated already, and their corresponding expressions can be found in Eqs. (3.3) and (3.4) in Ref. [4], respectively. Below we quote the fermion self-energy corresponding to the diagram in Fig. 1 (c),

$$\Sigma^f_{1(c)} = v_0^2 \frac{1}{\beta} \frac{1}{\sum_{i=0}^{\infty}} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\omega_n^2 + k^2 + i\nu + i\omega - \lambda} = v_0^2 \frac{S_{d'}}{2} \int_0^\infty d\nu \frac{k^d-2}{i\nu - \lambda - k}$$

$$= v_0^2 \frac{S_{d'}}{2} \pi \csc(\pi(d' - 2))(\lambda - i\nu)^{-2+d'}$$

$$= C_\mu v^2 (i\nu - \lambda) \left[ -\frac{1}{\epsilon' + 2(N_0 + 2i\pi)} \right] \quad \text{(with } C_\mu = \mu'^\epsilon(i\nu - \lambda)^{-\epsilon'} \frac{Z_v^2}{Z_b^2}). \quad \text{(4.3)}$$

Here, $N_0 = \gamma_E - 2 \log(2) - \psi^{(0)} (\frac{3}{2})$ with $\gamma_E$ being the Euler’s constant and $\psi^{(0)}$ is the polygamma function.

There is only one diagram contributing to the bosonic self-energy at one-loop level, shown in Fig. 1 (d). It has been evaluated previously and its expression can be found in Eq. (3.8) in Ref. [4].

B. Vertex correction

Firstly, note that there is no one-loop correction to the vertex $g_0$ corresponding to the fermionic bath coupling. So we proceed with calculating the vertex corrections to the bosonic bath couplings $\gamma_0$ and $v_0$. The diagrams corresponding to the vertex correction to $\gamma_0$ are shown in Fig. 2 (a) and (b), while those corresponding to $v_0$ are shown in Fig. 2 (c) and (d). Note that the diagram in
FIG. 2: One-loop diagrams for vertex corrections. Vertex corrections to $\gamma_0$ are shown in (a) and (b), while that for $\nu_0$ are shown in (c) and (d). The convention for different lines is same as introduced in Fig. 1.

Fig. 2 (a) has been evaluated before and its expression can be found in Eq. (3.9) in Ref. [4]. The expressions for the rest of the diagrams in Fig. 2 are as follows:

$$\Gamma_{2(b)}^\gamma = \gamma_0 \nu_0^2 \frac{1}{\beta} \sum_{i_{1n}} \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{\omega_{1n}^2 + k_1^2} \frac{1}{i\Omega_{1n} + i\omega_{1n} - \lambda} \frac{1}{i\Omega_{2n} + i\omega_{1n} - \lambda} = \gamma_0 C_\mu \nu^2 \left[ \frac{1}{\epsilon'} - 1 + \frac{1}{2} (-N_0 - 2i\pi) \right], \quad (4.4)$$

$$\Gamma_{2(c)}^\gamma = \nu_0^3 \frac{1}{\beta} \sum_{i_{1n}} \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{\omega_{1n}^2 + k_1^2} \frac{1}{i\Omega_{1n} + i\omega_{1n} - \lambda} \frac{1}{i\Omega_{2n} + i\omega_{1n} - \lambda} = \nu_0 C_\mu \nu^2 \left[ \frac{1}{\epsilon'} - 1 + \frac{1}{2} (-N_0 - 2i\pi) \right], \quad (4.5)$$

$$\Gamma_{2(d)}^\gamma = \frac{3}{4} \nu_0^2 \gamma_0 \frac{1}{\beta} \sum_{i_{1n}} \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{\omega_{1n}^2 + k_1^2} \frac{1}{i\Omega_{1n} + i\omega_{1n} - \lambda} \frac{1}{i\Omega_{2n} + i\omega_{1n} - \lambda} = \frac{3}{4} \nu_0 B_\mu \gamma^2 \left[ \frac{1}{\epsilon'} - 1 + \frac{1}{2} (-N_0 - 2i\pi) \right]. \quad (4.6)$$

C. Beta functions

In the expressions for the renormalized vertices and the $f/b$ Green's functions, we look at the cancellation of poles at the external frequency $i\nu - \lambda = \mu$. We thus obtain the following expressions of the renormalizing factors,

$$Z_f = 1 - \frac{g^2}{2\bar{\rho}} - \frac{3\gamma^2}{4\epsilon} - \frac{\nu^2}{\epsilon'}, \quad (4.7)$$

$$Z_b = 1 - \frac{g^2}{\bar{\rho}}, \quad (4.8)$$

$$Z_\gamma = 1 + \frac{\gamma^2}{4\epsilon} - \frac{\nu^2}{\epsilon'}, \quad (4.9)$$

$$Z_\nu = 1 - \frac{\nu^2}{\epsilon'} - \frac{3\gamma^2}{4\epsilon}. \quad (4.10)$$
Note that \( Z_g = 1 \) at this level due to no one-loop vertex correction to \( g_0 \). It is now straightforward to obtain the beta functions using Eqs. (4.7-4.10),

\[
\beta(g) = -\bar{r}g + \frac{3}{2}g^3 + \frac{3}{8}g\gamma^2 + \frac{1}{2}v^2 g, \tag{4.11}
\]

\[
\beta(\gamma) = -\frac{\epsilon}{2} \gamma + \gamma^3 + g^2 \gamma, \tag{4.12}
\]

\[
\beta(v) = -\frac{\epsilon'}{2} v + g^2 v. \tag{4.13}
\]

D. Fixed points and stability

By analyzing where the beta functions vanish, we obtain the following fixed points, \((FP \equiv (g^*, \gamma^*, v^*))\):

\[
FP_1 = (0, 0, 0), \tag{4.14}
\]

\[
FP_2 = \left(0, \frac{\epsilon}{2}, 0\right), \tag{4.15}
\]

\[
FP_3 = \left(\frac{2\bar{r}}{3}, 0, 0\right), \tag{4.16}
\]

\[
FP_4 = \left(\frac{\epsilon'}{2}, 0, 2\bar{r} - \frac{3}{2} \frac{\epsilon'}{2}\right), \tag{4.17}
\]

\[
FP_5 = \left(-\frac{\epsilon}{6} + \frac{8\bar{r}}{9}, \frac{2\epsilon}{3} - \frac{8\bar{r}}{9}, 0\right), \tag{4.18}
\]

\[
FP_6 = \left(\frac{\epsilon'}{2}, -\frac{\epsilon'}{2}, 2\bar{r} - \frac{3}{8} \frac{\epsilon'}{2} - \frac{9}{8} \frac{\epsilon'}{2}\right). \tag{4.19}
\]

For \( FP_5 \) to be real, we need \( 3\epsilon/8 < 2\bar{r} < 3\epsilon/2 \). While for \( FP_6 \) to be real we need \( \epsilon > \epsilon' > 0 \) and \( 2\bar{r} > (3\epsilon + 9\epsilon')/8 \).

We will now do the stability analysis of the fixed points by looking at the eigenvalues of the following stability matrix:

\[
J \equiv \begin{bmatrix}
J_1 & J_2 & J_3 \\
J_4 & J_5 & J_6 \\
J_7 & J_8 & J_9
\end{bmatrix}, \tag{4.20}
\]

where,

\[
J_1 \equiv \frac{\partial \beta(g)}{\partial g} = -\bar{r} + \frac{9}{2} g^2 + \frac{3}{8} \gamma^2 + \frac{v^2}{2}, \quad J_2 \equiv \frac{\partial \beta(\gamma)}{\partial \gamma} = \frac{3}{4} g\gamma, \quad J_3 \equiv \frac{\partial \beta(g)}{\partial v} = vg, \tag{4.21}
\]

\[
J_4 \equiv \frac{\partial \beta(g)}{\partial \gamma} = 2g\gamma, \quad J_5 \equiv \frac{\partial \beta(\gamma)}{\partial \gamma} = -\frac{\epsilon}{2} + 3\gamma^2 + g^2, \quad J_6 \equiv \frac{\partial \beta(\gamma)}{\partial v} = 0, \tag{4.21}
\]

\[
J_7 \equiv \frac{\partial \beta(v)}{\partial g} = 2gv, \quad J_8 \equiv \frac{\partial \beta(v)}{\partial \gamma} = 0, \quad J_9 \equiv \frac{\partial \beta(v)}{\partial v} = -\frac{\epsilon'}{2} + g^2. \tag{4.21}
\]
From the eigenvalues of the above matrix, it is immediately clear that for \( \bar{r} > 0, \epsilon > 0 \) and \( \epsilon' > 0 \), the Gaussian fixed point \( FP_1 \) is always unstable.

For \( FP_5 \) to be a stable fixed point, we require \( \epsilon > 0, 3\epsilon/8 < 2\bar{r} < 3\epsilon/2, \) and \( 2\bar{r} > (3\epsilon + 9\epsilon')/8 \). The second inequality is trivially satisfied as soon as \( FP_5 \) is real. If we use in addition the self-consistency condition \( \epsilon = 2\bar{r} = 1 \) (to be discussed in Section IV F), this implies that \( FP_5 \) is stable if \( \epsilon' < 5/9 \) (although we cannot trust the present expansion at values of \( \epsilon' \) of order unity).

For \( FP_6 \) the eigenvalues of the stability matrix are given by the following characteristic polynomial:

\[ 3 + A^2 + B^2 + C. \]

The corresponding coefficients are as follows:

\[ A = -\epsilon - \frac{\epsilon'}{2}, \quad B = \epsilon'\left(\frac{3\epsilon}{2} - 2\bar{r}\right), \quad C = \frac{\epsilon'}{8}(\epsilon - \epsilon')(16\bar{r} - 3\epsilon - 9\epsilon'). \]  

(4.22)

From the condition for \( FP_6 \) to be real it is clear that \( C > 0 \) which implies that at least one eigenvalue is negative if \( FP_6 \) is real. Therefore the non-trivial fixed point \( FP_6 \) is unstable. If this fixed point is real it always has one relevant direction.

**E. Anomalous dimension of \( f \) and \( b \) operators**

We now calculate the anomalous dimension of the \( f \) and \( b \) propagators, defined as follows:

\[ \eta_f = \mu \frac{d \ln Z_f}{d \mu}|_{FP}, \quad \eta_b = \mu \frac{d \ln Z_b}{d \mu}|_{FP} \]  

(4.23)

In our case,

\[ \mu \frac{d \ln Z_f}{d \mu} = g^2 + \frac{3}{4} \gamma^2 + \nu^2, \quad \mu \frac{d \ln Z_b}{d \mu} = 2g^2. \]  

(4.24)

Thus we find the following anomalous dimension at the fixed points,

\[ FP_1: \eta_f = 0 \quad \eta_b = 0; \]  

(4.25)

\[ FP_2: \eta_f = \frac{3}{8} \epsilon, \quad \eta_b = 0; \]  

(4.26)

\[ FP_3: \eta_f = \frac{2}{3} \bar{r}, \quad \eta_b = \frac{4}{3} \bar{r}; \]  

(4.27)

\[ FP_4: \eta_f = 2\bar{r} - \epsilon', \quad \eta_b = \epsilon'; \]  

(4.28)

\[ FP_5: \eta_f = \frac{1}{3} \epsilon + \frac{2}{9} \bar{r}, \quad \eta_b = -\frac{1}{3} \epsilon + \frac{16}{9} \bar{r}; \]  

(4.29)

\[ FP_6: \eta_f = 2\bar{r} - \epsilon', \quad \eta_b = \epsilon'. \]  

(4.30)

**F. Anomalous dimension of spin, electron and density operators**

We are interested in the anomalous dimensions of the gauge-invariant operators, \( S, c, \) and \( n. \) For this purpose we can look at the correlators \( \langle S(\tau) \cdot S(0) \rangle, \langle c_\alpha(\tau)c_\alpha(0) \rangle, \) and \( \langle n(\tau)n(0) \rangle \) made
from the composite operators $f^\dagger _\alpha \sigma^\alpha _\beta f^\beta _\beta /2$, $f^\dagger _\alpha b$, and $f^\dagger _\alpha f_\alpha$ respectively. In order to proceed, we first introduce these composite operator terms in the action, such that,

$$ S(D) = \frac{1}{\beta} \sum_{\omega_n} \left( \Lambda_S f^\dagger _\alpha \sigma^\alpha _\beta f^\beta _\beta + \Lambda_c [f^\dagger _\alpha b + H.c.] + \Lambda_n f^\dagger _\alpha f_\alpha \right) + S_{\text{rest}}(D), $$

(4.31)

where $S_{\text{rest}}$ has all the other terms in the action analyzed before. We define the renormalized couplings and the renormalized composite operators

$$ \Lambda_S = \frac{Z_{ff} \Lambda_{SR, S}}{Z_f}, \quad \Lambda_c = \frac{Z_{f b} \Lambda_{cR, c}}{\sqrt{Z_f Z_b}}, \quad \Lambda_n = \frac{Z_{f f 1} \Lambda_{n, R}}{Z_f}, $$

(4.32)

$$ \hat{S} = \sqrt{Z_S S_R}, \quad c = \sqrt{Z_{c c R}}, \quad n = \sqrt{Z_n n_R}. $$

(4.33)

We find that the diagrams required to evaluate the vertex corrections to $S$, $c$, and $n$ are exactly those that we used in the calculation of $Z_f$, $Z_g$, and $Z_v$ respectively. Therefore,

$$ Z_S = \left( \frac{Z_f}{Z_{\gamma}} \right)^2, \quad Z_c = \frac{Z_f Z_b}{Z_g^2}, \quad Z_n = \left( \frac{Z_f}{Z_v} \right)^2. $$

(4.34)

This readily gives us,

$$ Z_S = 1 - \frac{g^2}{r} - \frac{2\gamma^2}{\epsilon}, $$

(4.35)

$$ Z_c = 1 - \frac{3g^2}{2r} - \frac{3\gamma^2}{4\epsilon} - \frac{v^2}{\epsilon'}, $$

(4.36)

$$ Z_n = 1 - \frac{g^2}{r}. $$

(4.37)

We can now evaluate the anomalous dimensions as,

$$ \eta_S \equiv \frac{d \ln Z_S}{d \ln \mu} = \frac{1}{Z_S} \left[ \frac{\partial Z_S}{\partial g} \beta(g) + \frac{\partial Z_S}{\partial \gamma} \beta(\gamma) + \frac{\partial Z_S}{\partial v} \beta(v) \right] = 2(g^2 + \gamma^2), $$

(4.38)

$$ \eta_c \equiv \frac{d \ln Z_c}{d \ln \mu} = \frac{1}{Z_c} \left[ \frac{\partial Z_c}{\partial g} \beta(g) + \frac{\partial Z_c}{\partial \gamma} \beta(\gamma) + \frac{\partial Z_c}{\partial v} \beta(v) \right] = 3g^2 + \frac{3}{4} \gamma^2 + v^2, $$

(4.39)

$$ \eta_n \equiv \frac{d \ln Z_n}{d \ln \mu} = \frac{1}{Z_n} \left[ \frac{\partial Z_n}{\partial g} \beta(g) + \frac{\partial Z_n}{\partial \gamma} \beta(\gamma) + \frac{\partial Z_n}{\partial v} \beta(v) \right] = 2g^2. $$

(4.40)

The anomalous dimensions at the fixed points are listed in Table I. Just as shown in Ref. [4], we can also make an exact statement here. To all orders in $\epsilon$, $\epsilon'$, and $\bar{r}$: If $g \neq 0$ then $\eta_c = 2\bar{r}$, if $\gamma \neq 0$ then $\eta_S = \epsilon$, and if $v \neq 0$ then $\eta_n = \epsilon'$. Thus at the non-trivial fixed point, $FP_6$, $\eta_S = \epsilon$, $\eta_c = 2\bar{r}$, and $\eta_n = \epsilon'$ to all orders in $\epsilon$, $\epsilon'$ and $\bar{r}$. While at the non-trivial fixed point $FP_5$, $\eta_S = \epsilon$ and $\eta_c = 2\bar{r}$ to all orders, but $\eta_n$ cannot be evaluated exactly to all orders.

We now recall the self-consistency condition, Eq. (3.2). We now match the exponents found here to those in Eq. (3.2) for the respective spin, electron and density correlator. At the fixed
point $FP_5$ (i.e., the DQCP FP from Ref. [4]) we find that $\epsilon = 1$ and $\bar{\epsilon} = 1/2$ by matching the exponents of $Q$ and $R$ respectively in Eq. (3.2) to those of $\eta_S$ and $\eta_c$ found above (see Table I). However, at $FP_5$ since $K = 0$ there is no self-consistency condition on $\eta_n$ and so the value of $\epsilon'$ is not fixed. Since the exponents $\eta_c$ and $\eta_S$ are obtained exactly, their values of $\eta_c = 2\bar{\epsilon} = 1$ and $\eta_S = \epsilon = 1$ can be trusted. But the exponent $\eta_n$ is not exact and will have corrections from higher order expansion in $\bar{\epsilon}$ and $\epsilon$ (it does not depend upon $\epsilon'$ at $FP_5$). We can choose any $\epsilon' < 5/9$ so that $FP_5$ is stable. We then obtain our main result that $\eta_n = 5/9$, using Eq. (4.40) or Table I.

Note that at the other non-trivial fixed point, $FP_6$, the exponents $\eta_c = 2\bar{\epsilon}$, $\eta_S = \epsilon$ and $\eta_n = \epsilon'$ are obtained exactly. Imposing the self-consistency condition, Eq. (3.2), at this fixed point leads to $2\bar{\epsilon} = \epsilon = \epsilon' = 1$. Hence, at this fixed point $\eta_c = \eta_S = \eta_n = 1$. For these large values of $\bar{\epsilon}$, $\epsilon$ and $\epsilon'$ the fixed point $FP_6$ becomes complex and is not stable at one loop order, but there is no justification for using the one loop results at these large values.

### G. Flow of $s$

At one-loop level, we can derive the flow of $s$. The corresponding beta function is as follows:

$$
\beta(s) = -s + 3sg^2 - g^2 + \frac{3}{4}\gamma^2 + v^2.
$$

(4.41)

This governs the flow away from the critical point, discussed above for $s_0 = 0$. It turns out that $s$ is always a relevant parameter, which in fact tunes the phase transition from a metallic spin glass phase to a disordered Fermi liquid [4].

<table>
<thead>
<tr>
<th>Fixed point</th>
<th>$\eta_S$</th>
<th>$\eta_c$</th>
<th>$\eta_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$FP_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$FP_2$</td>
<td>$\epsilon$</td>
<td>$\frac{3}{5}\epsilon$</td>
<td>0</td>
</tr>
<tr>
<td>$FP_3$</td>
<td>$\frac{1}{3}\bar{\epsilon}$</td>
<td>$2\bar{\epsilon}$</td>
<td>$\frac{4}{3}\bar{\epsilon}$</td>
</tr>
<tr>
<td>$FP_4$</td>
<td>$\epsilon'$</td>
<td>$2\bar{\epsilon}$</td>
<td>$\epsilon'$</td>
</tr>
<tr>
<td>$FP_5$</td>
<td>$\epsilon$</td>
<td>$2\bar{\epsilon}$</td>
<td>$\frac{10}{3}\bar{\epsilon} - \frac{\epsilon}{3}$</td>
</tr>
<tr>
<td>$FP_6$</td>
<td>$\epsilon$</td>
<td>$2\bar{\epsilon}$</td>
<td>$\epsilon'$</td>
</tr>
</tbody>
</table>

**TABLE I:** Anomalous dimensions at fixed points.
V. CONCLUSION

This paper has presented a renormalization group analysis of the $t$-$J$-$K$ model in (2.1), a model for the cuprates with random and infinite-range interactions. This model was previously studied without the density-density interaction, $K$, in Ref. 4: they found a deconfined critical point at a non-zero doping $p = p_c$, separating a metallic spin glass for $p < p_c$, from a disordered Fermi liquid for $p > p_c$. In the present paper, examined the fate of this fixed point for non-zero $K$, and also computed the exponent characterizing density correlations.

Recent momentum-resolved electron energy-loss spectroscopy (M-EELS) experiments [7, 8] have observed anomalous density fluctuations near optimal doping in the cuprates. In our theory, the critical density fluctuations are characterized by the $T = 0$ spectral density

$$\chi''_n(\omega) \sim \text{sgn}(\omega) |\omega|^{\eta_n - 1},$$

and similarly for the spin fluctuations with exponent $\eta_S$. At non-zero $T$, the spectrum is characterized by a ‘Planckian’ frequency scale, and (5.1) is multiplied by a universal function of $\hbar \omega/(k_B T)$ so that (5.1) holds for $\hbar \omega \gg k_B T$, while $\chi''_n \sim \omega/T^{2-\eta_n}$ for $\hbar \omega \ll k_B T$. We also note that in a Fermi liquid $\chi''_n(\omega) \sim \omega$.

In this paper, we found a new fixed point, $FP_6$, with $K \neq 0$, at which the exponents can be determined to all loop order: we obtained the ‘marginal’ value $\eta_n = \eta_S = 1$. However, at least the one loop order at which our computations were carried out, this fixed point was unstable to the previously found [4] fixed point at $K = 0$, labeled $FP_5$ here. But it cannot be ruled out that at strong coupling $FP_6$ is the appropriate fixed point, and we expect $\eta_n = \eta_S = 1$ to continue to hold exactly at any such fixed point with $K \neq 0$.

At the $K = 0$ fixed point $FP_5$, we previously showed that $\eta_S = 1$ to all loop order [4]. In the present paper, we are only able to determine $\eta_n$ at $FP_5$ to one loop (there is no corresponding argument to extend the computation of $\eta_n$ to all orders): the result is shown in Table I. At the self-consistent values of the expansion parameters, $\epsilon = 2\bar{r} = 1$, the exponent evaluates to $\eta_n = 5/9$. However, our computation is first order in $\epsilon$, $\bar{r}$ (both of the same order), and so the accuracy of this result is not clear.

We hope that numerical studies of Hamiltonians like (2.1) will shed further light on the existence and nature of the finite doping deconfined critical point.

ACKNOWLEDGEMENT

We thank Chenyuan Li, Grigory Tarnopolsky, and Antoine Georges for collaboration on previous work [4]. We also thank P. Abbamonte and M. Mitrano for discussions of the M-EELS experiments.
Appendix A: RG in terms of gauge-invariant operators

In this appendix we present an alternative RG analysis directly in terms of the gauge-invariant operators. This also has the advantage that we can present our results for a general $M$ and $M'$, which generalizes SU(1|2) to SU($M'|M$). We have the following impurity Hamiltonian as before,

$$H_{\text{imp}} = g_0 \left( c_{\ell \alpha}^\dagger \psi_{\alpha \ell} (0) + H.c. \right) + \gamma_0 S^a \phi_a (0) + v_0 \tilde{n} \zeta (0) + \int |k|^r d k k \psi_{\kappa \alpha}^\dagger \psi_{\kappa \alpha} + \frac{1}{2} \int d^d x \left[ \pi_a^2 + (\partial_x \phi_a)^2 \right] + \frac{1}{2} \int d^d x \left[ \tilde{\pi}^2 + (\partial_x \zeta)^2 \right], \quad (A1)$$

where $\alpha = 1, ..., M$, $\ell = 1, ..., M'$ and $a = 1, ..., M^2 - 1$. This Hamiltonian is a large $M$, $M'$ generalization of Eq. 3.4. In the above Hamiltonian, $\tilde{n} \equiv n - n_f$ with $n \equiv f_\alpha^\dagger f_\alpha$ and $n_f \equiv \langle f_\alpha^\dagger f_\alpha \rangle_0 = 2/3$. To proceed with RG, we first introduce the following renormalization factors,

$$S^a = \sqrt{Z_s S^a_R}, \quad c_{\rho a} = \sqrt{Z_c c_{R, \rho a}}, \quad \tilde{n} = \sqrt{Z_n \tilde{n}_R}, \quad n = \sqrt{Z_n n_R},$$

$$\gamma_0 = \frac{\mu^{r/2} \tilde{Z}_\gamma}{\sqrt{Z_s \tilde{S}_d + 1}}, \quad g_0 = \frac{\mu^{r/2} \tilde{Z}_g}{\sqrt{Z_c \Gamma (r + 1)}}, \quad v_0 = \frac{\mu^{r/2} \tilde{Z}_v}{\sqrt{Z_n \tilde{S}_d + 1}}. \quad (A2)$$

In what follows we will also make use of the following expression for expectation values:

$$\mathcal{I}_{m, m'} = \left\langle \left( f_\alpha^\dagger f_\alpha \right)^m \left( b_\ell^\dagger b_\ell \right)^{m'} \right\rangle = \frac{1}{\mathcal{D}(M, M', P)} \int_{|z| = |c| < 1} d z \frac{1}{2 \pi i} \frac{d}{d z} \left[ \left( z \frac{d}{d z} \right)^m \left( 1 + z \right)^M \right] \left[ \left( z \frac{d}{d z} \right)^{m'} \left( 1 - z \right)^{M'} \right]. \quad (A3)$$

For more details we refer to Ref. [4]. We just recall that $\mathcal{I}_{0,0} = 1$ and the values for $M = 2$, $P = 1$, and $M' = 1$, which is the case of interest to us are as follows:

$$\mathcal{I}_{m,0} = 2 \frac{2}{3}, \quad m \geq 1; \quad \mathcal{I}_{0,m'} = 1 \frac{1}{3}, \quad m' \geq 1; \quad \mathcal{I}_{m,m'} = 0, \quad m \geq 1 \text{ and } m' \geq 1. \quad (A4)$$

1. Spin correlator

Here we calculate the spin correlator, $\langle O_{1} \rangle \equiv \langle S^a (\tau) S^a (0) \rangle$, which will give us $Z_s$. We will follow the strategy from Ref. [4, 16], which relies on explicit evaluation of operator traces rather
than the Wick’s theorem, such that $\langle O_1 \rangle = N_1 / D$. We evaluate the denominator and numerator in $\langle O_1 \rangle$ using the diagrams shown in Figs. 3 and 4 respectively to obtain,

$$D = 1 + \gamma_0^2 L_0 (D_{1\phi} + D_{2\phi} + D_{3\phi}) + g_0^2 L_1' (D_{1\psi} + D_{2\psi} + D_{3\psi})$$

$$+ g_0^2 L_2'' (D_{1\psi}'' + D_{2\psi}'' + D_{3\psi}'') + v_0^2 L''_0 (D_{1\zeta} + D_{2\zeta} + D_{3\zeta}) , \quad (A5)$$

$$N_1 = L_0 + \gamma_0^2 (L_1 D_{1\phi} + L_2 D_{2\phi} + L_3 D_{3\phi}) + g_0^2 (L_1' D_{1\psi} + L_2' D_{2\psi} + L_3' D_{3\psi})$$

$$+ g_0^2 (L_2'' D_{1\psi}' + L_2'' D_{2\psi}' + L_3'' D_{3\psi}') + v_0^2 (L''_0 D_{1\zeta} + L''_0 D_{2\zeta} + L''_0 D_{3\zeta}) . \quad (A6)$$

The diagrams in Figs. 3 (a)-(d) and Figs. 4 (a)-(j) have been evaluated before in Ref. [4]. The expressions for $L_i$, $L_i'$ and $L_i''$ can be found in Eqs. (B5)-(B16) in Ref. [4], while those for $D_i$, $D_i'$ and $D_i''$ can be found in Eqs. (B17)-(B25) in Ref. [4]. We quote here the previously not evaluated expressions,

$$L''_0 = \langle \tilde{n}\tilde{n} \rangle = \mathcal{I}_{2,0} - 2 n_f \mathcal{I}_{1,0} + n_f^2 , \quad (A7)$$

$$L''_1 = \langle S^a \tilde{n}\tilde{n} S^a \rangle = \frac{M + 1}{2 M} (M \mathcal{I}_{3,0} - \mathcal{I}_{1,0} - 2 n_f (M \mathcal{I}_{2,0} - \mathcal{I}_{3,0}) + n_f^2 (M \mathcal{I}_{1,0} - \mathcal{I}_{2,0})) , \quad (A8)$$

$$L''_2 = \langle S^a \tilde{n}\tilde{n} S^a \rangle = \frac{M + 1}{2 M} (M \mathcal{I}_{3,0} - \mathcal{I}_{1,0} - 2 n_f (M \mathcal{I}_{2,0} - \mathcal{I}_{3,0}) + n_f^2 (M \mathcal{I}_{1,0} - \mathcal{I}_{2,0})) , \quad (A9)$$

$$L''_3 = \langle S^a \tilde{n}\tilde{n} S^a \rangle = \frac{M + 1}{2 M} (M \mathcal{I}_{3,0} - \mathcal{I}_{1,0} - 2 n_f (M \mathcal{I}_{2,0} - \mathcal{I}_{3,0}) + n_f^2 (M \mathcal{I}_{1,0} - \mathcal{I}_{2,0})) . \quad (A10)$$

Also,

$$D_{1\zeta} = \int_0^\tau d\tau_1 \int_{\tau_1}^\tau d\tau_2 G_\zeta (\tau_1 - \tau_2) = -\frac{\tilde{S}_{d+1} \tau^\epsilon}{\epsilon' (1 - \epsilon')} , \quad (A11)$$

$$D_{2\zeta} = \int_\tau^\beta d\tau_1 \int_{\tau_1}^\beta d\tau_2 G_\zeta (\tau_1 - \tau_2) = -\frac{\tilde{S}_{d+1} \tau^\epsilon}{\epsilon' (1 - \epsilon')} , \quad (A12)$$
FIG. 4: Diagrams used in the evaluation of the numerator, \( N_1 \) (Eq. A6), of \( \langle O_1 \rangle = \langle S^a(\tau)S^a(0) \rangle \). Here, the external \( S^a \) operator is represented by an open circle. Apart from this the rest of the conventions are same as in Fig. 3.

\[
D_{3\zeta} = \int_{0}^{\tau} d\tau_1 \int_{\tau}^{\beta} d\tau_2 G_\zeta(\tau_1 - \tau_2) = \frac{2\tilde{S}_{d+1} \tau^\prime}{\epsilon(1 - \epsilon^\prime)}, \tag{A13}
\]

\[
G_\zeta(\tau) = \int \frac{d^d k \ d\omega \ e^{-i\omega\tau}}{(2\pi)^d \ 2\pi k^2 + \omega^2} = \frac{\tilde{S}_{d+1}}{|\tau|^{d-1}}. \tag{A14}
\]

Using Eqs. A5 and A6 we get,

\[
\langle O_1 \rangle = \frac{N_1}{D} = L_0 \left\{ 1 + \gamma_0^2 \left[ \left( \frac{L_1}{L_0} - L_0 \right) D_{1\phi} + \left( \frac{L_2}{L_0} - L_0 \right) D_{2\phi} + \left( \frac{L_3}{L_0} - L_0 \right) D_{3\phi} \right] + g_0^2 \left[ \left( \frac{L_1}{L_0} - L_0 \right) D_{1\psi} + \left( \frac{L_2}{L_0} - L_0 \right) D_{2\psi} + \left( \frac{L_3}{L_0} - L_0 \right) D_{3\psi} \right] + g_0^2 \left[ \left( \frac{L_1}{L_0} - L_0 \right) D_{1\chi} + \left( \frac{L_2}{L_0} - L_0 \right) D_{2\chi} + \left( \frac{L_3}{L_0} - L_0 \right) D_{3\chi} \right] \right\}. \tag{A15}
\]

We thus obtain,

\[
Z_S = 1 - \frac{\gamma^2}{\epsilon} L_\gamma - \frac{g^2}{2\tilde{r}} L_g - \frac{v^2}{\epsilon^\prime} L_v, \tag{A16}
\]

where

\[
L_\gamma = \frac{L_1 + L_2 - 2L_3}{L_0}, \tag{A17}
\]

\[
L_g = \frac{L_1' + L_2' + L_3' - 2L_3}{L_0}, \tag{A18}
\]

\[
L_v = \frac{L_1'' + L_2'' - 2L_3''}{L_0}. \tag{A19}
\]
We find that $L_\gamma = L_g = 2$ and $L_v = 0$ for $M = 2, M' = 1$. Thus, for $M = 2, M' = 1$,

$$Z_S = 1 - \frac{2\gamma^2}{\epsilon} - \frac{g^2}{\bar{r}}.$$ (A20)

### 2. Electron correlator

In this subsection we will calculate the electron correlation, $\langle O_2 \rangle \equiv \langle c(\tau)c^\dagger(0) \rangle = N_2/D$. The denominator, $D$, has been already evaluated in Eq. A5. The numerator, $N_2$, is evaluated using the diagrams shown in Fig. 5. Thus we obtain,

$$N_2 = P_0 + \gamma_0^2 (P_1D_1\phi + P_2D_2\phi + P_3D_3\phi) + g_0^2 \left( P_1'D_1'\psi + P_2'D_2'\psi + P_3'D_3'\psi \right)$$
$$+ g_0^2 \left( P_1''D_1''\psi + P_2''D_2''\psi + P_3''D_3''\psi \right) + \nu_0^2 \left( P_1'''D_1\zeta + P_2'''D_2\zeta + P_3'''D_3\zeta \right).$$ (A21)

The diagrams in Fig. 5 (a)-(j) have been previously evaluated. The expressions for $P_1, P_1'$ and $P_1''$ can be found in Eqs. (B33)-(B42) in Ref. [4]. For the rest we have,

$$P_1'' = \langle c_{\alpha\alpha_0}^{\dagger} \bar{n} \bar{n} c_{\alpha_0} \rangle = M'(\mathcal{I}_{3,0} - 2\mathcal{I}_{2,0} + \mathcal{I}_{1,0} - 2n_f(\mathcal{I}_{2,0} - \mathcal{I}_{1,0}) + n_f^2\mathcal{I}_{1,0})$$
$$+ \mathcal{I}_{3,1} - 2\mathcal{I}_{2,1} + \mathcal{I}_{1,1} - 2n_f(\mathcal{I}_{2,1} - \mathcal{I}_{1,1}) + n_f^2\mathcal{I}_{1,1},$$ (A22)

$$P_2'' = \langle c_{\alpha\alpha_0}^{\dagger} \bar{n} c_{\alpha_0} \bar{n} \rangle = M'(\mathcal{I}_{3,0} - 2n_f\mathcal{I}_{2,0} + n_f^2\mathcal{I}_{1,0}) + \mathcal{I}_{3,1} - 2n_f\mathcal{I}_{2,1} + n_f^2\mathcal{I}_{1,1},$$ (A23)

$$P_3'' = \langle c_{\alpha\alpha_0}^{\dagger} \bar{n} c_{\alpha_0} \bar{n} \rangle = M'(\mathcal{I}_{3,0} - \mathcal{I}_{2,0} - n_f(2\mathcal{I}_{2,0} - \mathcal{I}_{1,0}) + n_f^2\mathcal{I}_{1,0})$$
$$+ \mathcal{I}_{3,1} - \mathcal{I}_{2,1} - n_f(2\mathcal{I}_{2,1} - \mathcal{I}_{1,1}) + n_f^2\mathcal{I}_{1,1}.$$ (A24)
From Eqs. A5 and A21 we have,

\[
\langle O_2 \rangle = \frac{N_2}{D} = P_0 \left\{ 1 + \gamma_0^2 \left[ \left( \frac{P_1}{P_0} - L_0 \right) D_{1\phi} + \left( \frac{P_2}{P_0} - L_0 \right) D_{2\phi} + \left( \frac{P_3}{P_0} - L_0 \right) D_{3\phi} \right] + g_0^2 \left[ \left( \frac{P_1'}{P_0} - L_0' \right) D_{1\psi} + \left( \frac{P_2'}{P_0} - L_0' \right) D_{2\psi} + \left( \frac{P_3'}{P_0} - L_0' \right) D_{3\psi} \right] + g_0^2 \left[ \left( \frac{P_1''}{P_0} - L_0'' \right) D_{1\psi} + \left( \frac{P_2''}{P_0} - L_0'' \right) D_{2\psi} + \left( \frac{P_3''}{P_0} - L_0'' \right) D_{3\psi} \right] + v_0^2 \left[ \left( \frac{P_1'''}{P_0} - L_0''' \right) D_{1\xi} + \left( \frac{P_2'''}{P_0} - L_0''' \right) D_{2\xi} + \left( \frac{P_3'''}{P_0} - L_0''' \right) D_{3\xi} \right] \right\}.
\]

(A25)

Thus we obtain,

\[
Z_c = 1 - \frac{\gamma^2}{\epsilon} P_\gamma - \frac{g^2}{2 \epsilon} P_g - \frac{v^2}{\epsilon} P_v ,
\]

(A26)

where

\[
P_\gamma = \frac{P_1 + P_2 - 2P_3}{P_0}, \quad (A27)
\]

\[
P_g = \frac{P_1' + P_2' - 2P_3' + P_1'' + P_2'' - 2P_3''}{P_0}, \quad (A28)
\]

\[
P_v = \frac{P_1''' + P_2''' - 2P_3'''}{P_0}. \quad (A29)
\]

We obtain \(P_g = 3\), \(P_\gamma = 3/4\) and \(P_v = 1\) for \(M = 2\), \(M' = 1\). Thus, for \(M = 2\), \(M' = 1\),

\[
Z_c = 1 - \frac{3 \gamma^2}{4 \epsilon} - \frac{3 g^2}{2 \epsilon} - \frac{v^2}{\epsilon^2} .
\]

(A30)

### 3. Density correlator

In this subsection we will evaluate the density correlation, \(\langle O_4 \rangle \equiv \langle n(\tau) n(0) \rangle = N_4/D\). Apart from a constant \(\langle \tilde{n}(\tau) \tilde{n}(0) \rangle\) has the same form as \(\langle n(\tau) n(0) \rangle\). The numerator, \(N_4\), is evaluated using the diagrams shown in Fig. 6. We thus have,

\[
N_4 = T_0 + \gamma_0^2 \left( T_1 D_{1\phi} + T_2 D_{2\phi} + T_3 D_{3\phi} \right) + g_0^2 \left( T_1' D_{1\psi} + T_2' D_{2\psi} + T_3' D_{3\psi} \right) + g_0^2 \left( T_1'' D_{1\psi} + T_2'' D_{2\psi} + T_3'' D_{3\psi} \right) + v_0^2 \left( T_1''' D_{1\xi} + T_2''' D_{2\xi} + T_3''' D_{3\xi} \right) ,
\]

(A31)

where,

\[
T_0 = \langle nn \rangle = T_{2,0} ,
\]

(A32)

\[
T_1 = \langle nS^a S^a n \rangle = \frac{M + 1}{2M} \left( M T_{3,0} - T_{4,0} \right) ,
\]

(A33)
Using Eqs. A5 and A31 we have, \( n_4 \) (Eq. A31), of \( \langle O_4 \rangle = \langle n(\tau)n(0) \rangle \).

Here, the external \( n \) operator is represented by an open hexagon, while the rest of the conventions are same as in Fig. 3.

\[
T_2 = \langle nnS^nS^n \rangle = \frac{M + 1}{2M} (M\mathcal{I}_{3,0} - \mathcal{I}_{4,0}), \quad (A34)
\]

\[
T_3 = \langle nS^n nS^n \rangle = \frac{M + 1}{2M} (M\mathcal{I}_{3,0} - \mathcal{I}_{4,0}), \quad (A35)
\]

\[
T'_1 = \langle nc_{\beta}c_{\beta}n \rangle = M\mathcal{I}_{2,1} - \mathcal{I}_{3,1}, \quad (A36)
\]

\[
T'_2 = \langle nncc_{\beta}c_{\beta} \rangle = M\mathcal{I}_{2,1} - \mathcal{I}_{3,1}, \quad (A37)
\]

\[
T'_3 = \langle nc_{\beta}nc_{\beta} \rangle = M\mathcal{I}_{1,1} + (M - 1)\mathcal{I}_{2,1} - \mathcal{I}_{3,1}, \quad (A38)
\]

\[
T''_1 = \langle nc_{\beta}c_{\beta}n \rangle = M'\mathcal{I}_{3,0} + \mathcal{I}_{3,1}, \quad (A39)
\]

\[
T''_2 = \langle nncc_{\beta}c_{\beta} \rangle = M'\mathcal{I}_{3,0} + \mathcal{I}_{3,1}, \quad (A40)
\]

\[
T''_3 = \langle nc_{\beta}nc_{\beta} \rangle = M'(\mathcal{I}_{3,0} - \mathcal{I}_{1,0}) + \mathcal{I}_{3,1} - \mathcal{I}_{1,1}, \quad (A41)
\]

\[
T''_4 = \langle nncc_{\beta}c_{\beta} \rangle = M'(\mathcal{I}_{3,0} - \mathcal{I}_{1,0}) + \mathcal{I}_{3,1} - \mathcal{I}_{1,1}, \quad (A42)
\]

\[
T''_5 = \langle n\tilde{n}\tilde{n}n \rangle = \mathcal{I}_{4,0} - 2n_f\mathcal{I}_{3,0} + n_f^2\mathcal{I}_{2,0}, \quad (A43)
\]

\[
T''_6 = \langle n\tilde{n}\tilde{n}n \rangle = \mathcal{I}_{4,0} - 2n_f\mathcal{I}_{3,0} + n_f^2\mathcal{I}_{2,0}, \quad (A44)
\]

Using Eqs. A5 and A31 we have,

\[
\langle O_4 \rangle = \frac{N_4}{D} = T_0 \left\{ 1 + \gamma_0^2 \left[ \left( \frac{T_1}{T_0} - L_0 \right) D_{1\phi} + \left( \frac{T_2}{T_0} - L_0 \right) D_{2\phi} + \left( \frac{T_3}{T_0} - L_0 \right) D_{3\phi} \right] 
+ g_0^2 \left[ \left( \frac{T'_1}{T_0} - L'_0 \right) D'_{1\psi} + \left( \frac{T'_2}{T_0} - L'_0 \right) D'_{2\psi} + \left( \frac{T'_3}{T_0} - L'_0 \right) D'_{3\psi} \right] 
+ g_0^2 \left[ \left( \frac{T''_1}{T_0} - L''_0 \right) D''_{1\psi} + \left( \frac{T''_2}{T_0} - L''_0 \right) D''_{2\psi} + \left( \frac{T''_3}{T_0} - L''_0 \right) D''_{3\psi} \right] 
+ \nu_0^2 \left[ \left( \frac{T''_1}{T_0} - L''_0 \right) D''_{1\zeta} + \left( \frac{T''_2}{T_0} - L''_0 \right) D''_{2\zeta} + \left( \frac{T''_3}{T_0} - L''_0 \right) D''_{3\zeta} \right] \right\}. \quad (A45)
\]
Therefore, we obtain,

$$Z_n = Z_{\tilde{n}} = 1 - \frac{\gamma^2}{\epsilon} T_\gamma - \frac{g^2}{2r} T_g - \frac{v^2}{\epsilon} T_v,$$

(A46)

where

$$T_\gamma = \frac{T_1 + T_2 - 2T_3}{T_0},$$

(A47)

$$T_g = \frac{T'_1 + T'_2 - 2T'_3 + T''_1 + T''_2 - 2T''_3}{T_0},$$

(A48)

$$T_v = \frac{T'''_1 + T'''_2 - 2T'''_3}{T_0}.$$

(A49)

We find that $T_g = 2$, $T_\gamma = 0$ and $T_v = 0$ for $M = 2$, $M' = 1$. Thus, for $M = 2$, $M' = 1$,

$$Z_n = Z_{\tilde{n}} = 1 - \frac{g^2}{r}.$$

(A50)

4. Beta functions

With the renormalization factors for the gauge-invariant operators at hand, we can obtain the beta functions in a straightforward manner. Note that due to the absence of interaction terms the renormalization factors for the coupling constants are all unity, i.e., $\tilde{Z}_g = \tilde{Z}_\gamma = \tilde{Z}_v = 1$. Now using Eq. A2 we find,

$$\frac{\epsilon}{2} \gamma Z_S + \left[ Z_S - \frac{\gamma}{2} \frac{\partial Z_S}{\partial \gamma} \right] \beta(\gamma) - \frac{\gamma}{2} \frac{\partial Z_S}{\partial g} \beta(g) - \frac{\gamma}{2} \frac{\partial Z_S}{\partial v} \beta(v) = 0,$$

(A51)

$$\tilde{\epsilon} g Z_c + \left[ Z_c - \frac{\tilde{\epsilon} g}{2} \frac{\partial Z_c}{\partial \gamma} \right] \beta(g) - \frac{\tilde{\epsilon} g}{2} \frac{\partial Z_c}{\partial g} \beta(g) - \frac{\tilde{\epsilon} g}{2} \frac{\partial Z_c}{\partial v} \beta(v) = 0,$$

(A52)

$$\frac{\epsilon'}{2} v Z_{\tilde{n}} + \left[ Z_{\tilde{n}} - \frac{\epsilon'}{2} \frac{\partial Z_{\tilde{n}}}{\partial v} \right] \beta(v) - \frac{\epsilon'}{2} \frac{\partial Z_{\tilde{n}}}{\partial g} \beta(g) - \frac{\epsilon'}{2} \frac{\partial Z_{\tilde{n}}}{\partial \gamma} \beta(\gamma) = 0.$$  

(A53)

We now solve the above three equations using Eqs. A20, A30 and A50, and obtain the one-loop beta functions,

$$\beta(g) = -\tilde{\epsilon} g + \frac{3}{2} g^3 + \frac{3}{8} g^2 \gamma^2 + \frac{1}{2} g v^2,$$

(A54)

$$\beta(\gamma) = -\frac{\epsilon}{2} \gamma + \gamma^3 + g^2 \gamma,$$

(A55)

$$\beta(v) = -\frac{\epsilon'}{2} v + g^2 v.$$

(A56)

These are exactly the same as obtained earlier via a different RG procedure in Sec. IV C. The calculation of the rest of the details such as the fixed points and anomalous dimensions follow exactly as discussed in the main text.
Appendix B: Supersymmetry

In this appendix, we explore the possibility that averaged Hamiltonians $H_{\text{imp}} + H_{\text{bath}}$ in (3.4) exhibit $SU(1|2)$ supersymmetry. We were unable to define a suitable supersymmetry operation, as we discuss below. The difficult lies in making the bath supersymmetric. One approach is try to implement a spacetime supersymmetry on the bath fermions $\psi_\alpha$ and the bosons $\phi$ and $\zeta$; however that does not work because the scaling dimensions of fermions and bosons are not equal in this supersymmetry, whereas equality of the power-laws in (3.3) requires them to have the same scaling dimensions.

More progress is possible in an approach which fractionalizes the bath operators, in a manner which parallels the impurity site. So we write

\begin{align*}
\psi_\alpha(0) &= \frac{1}{\Omega} \sum_k \tilde{f}_{k\alpha} \tilde{b}_k^\dagger \\
\phi_\alpha(0) &= \frac{1}{\Omega} \sum_k \tilde{f}_{k\alpha} \frac{\sigma^{\alpha\beta}}{2} \tilde{f}_{k\beta} \\
\zeta(0) &= \frac{1}{\Omega} \sum_k \tilde{f}_{k\alpha} \tilde{f}_{k\alpha},
\end{align*}

(B1)

where $\Omega$ is a suitable normalization of the sum over $k$. The Green’s functions of the partons

\begin{align*}
\tilde{G}_f(k, \tau) \delta_{\alpha\beta} &= -\left\langle \tilde{f}_{k\alpha}(\tau) \tilde{f}_{k\beta}^\dagger(0) \right\rangle \\
\tilde{G}_b(k, \tau) &= -\left\langle \tilde{b}_k(\tau) \tilde{b}_k^\dagger(0) \right\rangle,
\end{align*}

(B2)

can then be used to obtain the fields in (3.2)

\begin{align*}
R(\tau) &= -\frac{1}{\Omega} \sum_k \tilde{G}_f(k, \tau) \tilde{G}_b(k, -\tau) \\
Q(\tau) &= -\frac{1}{2\Omega} \sum_k \tilde{G}_f(k, \tau) \tilde{G}_f(k, -\tau) \\
P(\tau) &= -\frac{2}{\Omega} \sum_k \tilde{G}_f(k, \tau) \tilde{G}_f(k, -\tau).
\end{align*}

(B3)

Finally, we replace the bath Hamiltonian in (3.4) by

\begin{equation}
\tilde{H}_{\text{bath}} = \frac{1}{\Omega} \sum_k \epsilon_f(k) \tilde{f}_{k\alpha} \tilde{f}_{k\alpha} + \frac{1}{\Omega} \sum_k \epsilon_b(k) \tilde{b}_k^\dagger \tilde{b}_k.
\end{equation}

(B4)

Now we consider generators of the $SU(1|2)$ superalgebra as the sum of impurity and bath terms,
replacing (2.2,2.3) by

\[ C_\alpha = f_\alpha b^\dagger + \frac{1}{\Omega} \sum_k \tilde{f}_{\alpha k} \tilde{b}^\dagger_k \]

\[ S^a = f^\dagger_\alpha \frac{\sigma^a_{\alpha\beta}}{2} f_\beta + \frac{1}{\Omega} \sum_k \tilde{f}^\dagger_{\alpha k} \frac{\sigma^a_{\alpha\beta}}{2} \tilde{f}_{\beta k} \]

\[ \mathcal{V} = \frac{1}{2} f_\alpha^\dagger f_\alpha + b^\dagger b + \frac{1}{2\Omega} \sum_k \tilde{f}^\dagger_{\alpha k} \tilde{f}_{\alpha k} + \frac{1}{\Omega} \sum_k \tilde{b}^\dagger_k \tilde{b}_k \]

(B5)

It is now easy to see that \( H_{\text{imp}} \) and \( \tilde{H}_{\text{bath}} \) both commute with \( S^a \) and \( \mathcal{V} \). We can also find by explicit evaluation that

\[ [C_\alpha, H_{\text{bath}}] = 0, \quad \text{for } \epsilon_f(k) = \epsilon_b(k). \]  

(B6)

Further,

\[ [C_\alpha, H_{\text{imp}}] = (s_0 + \lambda) c_\alpha - \lambda c_\alpha + g_0 (\delta_{\alpha\beta} V + \sigma_{\alpha\beta}^a) S^a \psi_\beta(0) + g_0 (\delta_{\alpha\beta} \tilde{V} + \sigma_{\alpha\beta}^a \phi_\alpha(0)) c_\beta \]

\[ + \gamma_0 \left( \frac{\sigma^a_{\alpha\beta}}{2} c_\beta \phi_\alpha(0) + \frac{\sigma^a_{\alpha\beta}}{2} \sigma^a_{\alpha\beta} \psi_\beta(0) \right) + v_0 (c_\alpha \zeta(0) + f_\beta^\dagger f_\beta \psi_\alpha(0)) - n_f v_0 \psi_\alpha(0), \]

(B7)

where \( \tilde{V} = (1/\Omega) \sum_k (\tilde{f}^\dagger_{\alpha k} \tilde{f}_{\alpha k} / 2 + \tilde{b}^\dagger_k \tilde{b}_k) \). Now, recall that \( f_\beta^\dagger f_\beta = 2 - 2V \), using Eq. 2.2 and the constraint \( f_\beta^\dagger f_\beta + b^\dagger b = 1 \). For the bath operators we include a chemical potential such that \( (1/\Omega) \sum_k (f^\dagger_{\beta k} f_{\beta k} + b^\dagger b) = 1 \); then one can write \( \zeta(0) = 2 - 2\tilde{V} \). In this case, for \( s_0 = -n_f v_0 \), \( \gamma_0 = -2g_0 \), and \( g_0 = 2v_0 \) we obtain,

\[ [C_\alpha, H_{\text{imp}}] = s_0 C_\alpha, \]  

(B8)

imply supersymmetry for \( H_{\text{imp}} \).

However, the condition in (B6) leads to an issue with supersymmetry in the class of models studied in the body of the paper. To obtain the ansatz in (3.3), with \( R(\tau) \) an odd function of \( \tau \) and \( P(\tau), Q(\tau) \) even functions of \( \tau \), we need \( \epsilon_f(k) \) to be an odd function of \( k \), while \( \epsilon_b(k) \) needs to be positive for stability. This is incompatible with the requirements of supersymmetry.


