

Singularity of the London penetration depth at quantum critical points in superconductors

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We present a general theory of the singularity in the London penetration depth at symmetry-breaking and topological quantum critical points within a superconducting phase. While the critical exponents, and ratios of amplitudes on the two sides of the transition are universal, an overall sign depends upon the interplay between the critical theory and the underlying Fermi surface. We determine these features for critical points to spin density wave and nematic ordering, and for a topological transition between a superconductor with \mathbb{Z}_2 fractionalization and a conventional superconductor. We note implications for recent measurements of the London penetration depth in $\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$ (Hashimoto *et al.*, *Science* **336**, 1554 (2012)).

An important focus of the study of the cuprate high temperature superconductors has been the quantum criticality of the onset of spin density wave (SDW) order within the superconducting phase. A number of neutron scattering experiments have observed such critical spin fluctuations in hole-doped LSCO [1–4] and YBCO [5], and in electron-doped NCCO [6]. The phase diagrams of the iron-based superconductors show a clear overlap between the SDW and superconducting phases [7], and Hashimoto *et al.* [8] have recently provided a careful study of the SDW quantum critical point in $\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$. A novel feature of the latter observations is that the influence of the magnetic critical point appears to have been observed in a property of the superconducting phase, the London penetration depth. Such an observation supports the proposal that the ‘same’ electrons are involved in both the SDW order and superconductivity, and that these two orders are strongly coupled [9].

In this paper, we provide a general theory of the singularity in the superfluid stiffness and the London penetration depth near a wide class of symmetry-breaking or topological transitions within superconductors in two spatial dimensions. Our results are summarized in Fig. 1 for 3 cases labelled A, B, C.

Case A, the SDW transition, is the one best studied in experiments so far [8]. The critical theory is described by the fluctuations of a bosonic SDW order parameter φ with $N = 3$ real components, with an effective Lagrangian which has a relativistic form [10]; the coupling to the fermionic quasiparticle excitations of the superconductor only serves to renormalize the parameters of the Lagrangian. These features allow us to use the powerful critical phenomena technology [11] to make definitive statements on the singularity in the London penetration depth. We find that the London penetration depth λ_L increases as we approach the quantum critical point from the SC phase. This is in agreement with the observations [8], and an independent recent computation [12] which focused on quasiparticle renormalization effects within the SC phase. However, the experiments [8] also observe a peak-like maximum in λ_L , and this is not present in our critical theory for the transition within the superconducting phase. This implies that the observed maximum appears *within* the SDW+SC phase, and not at the quantum critical point, as has been previously suggested [8, 12]. Explaining the maximum will require con-

sideration of other physical properties of the SDW+SC phase, which we will briefly discuss at the end of this paper.

Case B applies to the Ising-nematic transition, that is also observed in $\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$ [13]. This transition has a bosonic order parameter φ with $N = 1$ real component, and the effective theory is otherwise the same as that for the SDW case. However, fermionic quasiparticles which are gapless lead to a distinct critical theory [14], and so our present results apply to the nematic transition only if nodal quasiparticles are absent.

Finally, case C is a more exotic topological transition between a ‘fractionalized’ superconductor [15, 16] (often labeled SC* [17]) and a conventional superconductor (SC). Roughly speaking, in a SC* state some of the electrons have localized into a spin liquid (we consider the case of a \mathbb{Z}_2 spin liquid [18, 19]), while the remaining electrons are in a paired BCS state; there can then be a confinement transition to a SC state which has all the electrons in the BCS state. With the accumulating evidence for a fractionalized metallic state in a number of heavy fermion compounds [20–23], we can expect a SC* phase and SC*-SC transition in the superconducting state at lower temperatures. We will show below that the confinement transition out of the SC* state associated with a \mathbb{Z}_2 spin liquid is described by the theory of a “dual” Ising field φ with a relativistic structure. So ultimately, the critical theory is the same as that considered above for the Ising-nematic case, with the important difference that it is now the SC phase which has $\langle \varphi \rangle \neq 0$.

We now turn to a derivation of the results in Fig. 1. We have already argued that all three cases are described by the familiar φ^4 field theory of a N -component field φ with imaginary time (τ) action

$$\mathcal{S}_\varphi = \int d^2x d\tau \left[(\partial_\tau \varphi)^2 + c^2 (\nabla_x \varphi)^2 + (g - g_c^0) \varphi^2 + u (\varphi^2)^2 \right]$$

where c is a velocity of the collective mode excitations of the ordered phase, g_c^0 is the bare critical point, and u is a strongly-relevant self-interaction between φ fluctuations. The transition is taking place within the superconducting phase, but we can ignore the fluctuations of the phase of the superconducting order because these are suppressed by the long-range Coulomb interactions.

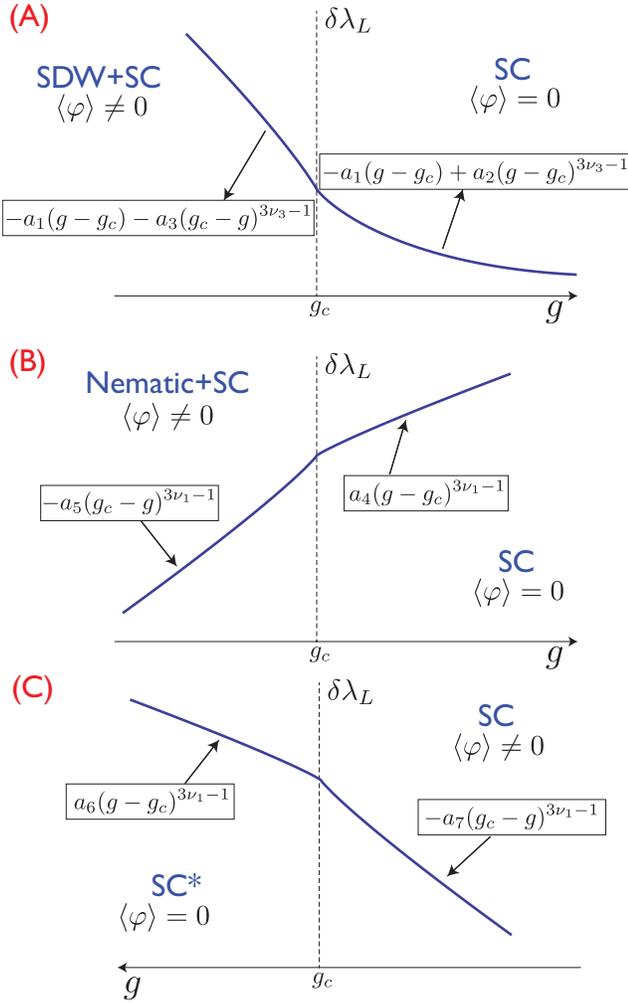


FIG. 1: Main results for the singular behavior of the London penetration depth, λ_L in (A) onset of spin density wave order leading to transition from SC to SDW+SC, (B) onset of nematic order, and (C) a deconfinement transition to a fractionalized SC* phase. These transitions are tuned by a generic zero temperature parameter g , and the quantum critical point is at $g = g_c$. The coupling C_2 in Eq. (1) has values which are (A) negative, (B) positive, (C) positive. All the prefactors a_{1-7} are positive, and some of their ratios are universal: $a_2/a_3 = 1.52$ and $a_4/a_5 = a_6/a_7 = 0.52$. The correlation length exponents are $\nu_1 = 0.631$ and $\nu_3 = 0.710$. For case A, we have included a non-singular term $-a_1(g - g_c)$ because it is larger than the singular terms; cases B,C can also have such non-singular terms, but here they are subdominant to the singular terms.

As written, the theory \mathcal{S}_φ has no direct coupling to the external magnetic field for cases A and B, and so cannot influence the superfluid stiffness and the London penetration depth. In these two cases, the vector potential \mathbf{A} of the external field couples to the underlying electrons, as does the order parameter φ . We will describe below the theory of the electrons coupled to both φ and \mathbf{A} , and obtain an effective Lagrangian by integrating out the electronic degrees of freedom. While performing this integration we assume that wavevectors, q , of

φ and \mathbf{A} , obey $q\xi_{sc} \ll 1$ where ξ_{sc} is the coherence length of the superconductor. At the same time, q is of order the correlation length, ξ^{-1} , of fluctuations of the bosonic mode φ ; consequently, the validity of our theory is limited to the critical region where $\xi \gg \xi_{sc}$. With $q\xi_{sc} \ll 1$, we can safely evaluate all gapped fermion loops at $q = 0$, and this leads to a simple and local effective Lagrangian after the fermions have been accounted for. Moreover, in case C the external magnetic field couples to the “dual” Ising field φ directly so that in all the three cases, one ends up with,

$$\mathcal{L}_{\text{eff}}[\mathbf{A}, \varphi] = C_1 \mathbf{A}^2 + C_2 \mathbf{A}^2 \varphi^2, \quad (1)$$

where $C_{1,2}$ are constants to be evaluated below. From this, we obtain the superfluid stiffness as

$$\frac{\hbar^2 c^2}{4e^2} \rho_s = C_1 + C_2 \langle \varphi^2 \rangle_{\mathcal{S}_\varphi} \quad (2)$$

where our notation indicates that the expectation value of φ^2 is to be computed in the field theory \mathcal{S}_φ . So our final results depend upon two distinct computations. The first is the computation of C_2 : we will turn to this below and show how its magnitude and sign depend upon the structure of the underlying fermionic excitations. The second is the computation of $\langle \varphi^2 \rangle$ for which numerous precise results are readily available [11]. Specifically we have $\langle \varphi^2 \rangle \sim A_\pm |g - g_c|^{3\nu-1} + \dots$ as $g - g_c \rightarrow \pm 0$, where ν is the exponent of the correlation length $\xi \sim |g - g_c|^{-\nu}$, the ratio A_+/A_- is universal, and the ellipses indicate non-universal terms analytic in $g - g_c$. Crucial constraints on the signs of the various coefficients arise from the fact that $-\partial \langle \varphi^2 \rangle / \partial g$ is proportional to the “specific heat”, C_V , of the classical statistical system described by the Euclidean field theory \mathcal{S}_φ , and so must be positive (note that C_V is unrelated to the specific heat of the quantum model we are studying). If $\nu > 2/3$, C_V has only a cusp-like singularity at $g = g_c$, and we assume for this case that C_V is a local maximum at $g = g_c$. After assembling these constraints with the values of C_2 computed below, it is a simple matter to obtain the results in Fig. 1.

We now describe the computation of C_2 for case A with SDW order. We consider models appropriate for the cuprates and pnictides, and also the model in Ref. [24], of electrons $c_{i,a\alpha}$ on sites i with orbital index a and spin index α in a SC state described by

$$H = - \sum_{\substack{ij \\ ab}} t_{ab,ij} c_{i,a\alpha}^\dagger c_{j,b\alpha} + \sum_{\mathbf{k},a} \Delta_{\mathbf{k}}^a c_{\mathbf{k},a\uparrow} c_{-\mathbf{k},a\downarrow} + \text{H.c.} \quad (3)$$

where $t_{ab,ij}$ are the hopping matrix elements, and $\Delta_{\mathbf{k}}^a$ is the pairing amplitude. These electrons are coupled to the 3-component SDW order parameter φ_m via

$$H_{sdw} = g \sum_i \varphi_m(i) c_{i,a\alpha}^\dagger \sigma_{\alpha\beta}^m c_{i,b\beta} e^{i\mathbf{K}\cdot\mathbf{r}_i} \quad (4)$$

where σ^m are the Pauli matrices and \mathbf{K} is the SDW ordering wavevector. We choose the superconducting pairing function

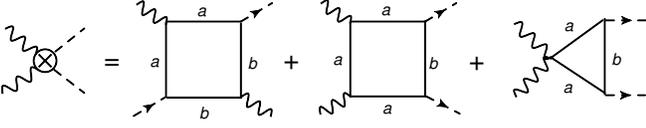


FIG. 2: The four-point coupling (crossed circle) between \mathbf{A} and φ after integrating out the Fermions. The wavy and the dashed lines represent the gauge field and the φ fields respectively. The solid lines represent Fermions having a Nambu structure, with $a \neq b$ for case (A), while $a = b$ for case (B).

$\Delta_{\mathbf{k}}^a$ so that points on the Fermi surface connected by \mathbf{K} always have pairing amplitudes with opposite sign: this is the type of spin-singlet superconductivity found in both the cuprates and the pnictides. Finally, we introduce an external magnetic field by a vector potential \mathbf{A} in the gauge $\nabla \cdot \mathbf{A} = 0$ via a Peierls substitution in the hopping matrix elements t_{ij} . Then conventional many-body perturbation theory in the coupling g leads to the Feynman diagrams in Fig 2. It is worth noting that a three-point coupling, $\sim C_3 \mathbf{A}_\perp \varphi^2$, where \mathbf{A}_\perp is the transverse component of \mathbf{A} , is also generated upon integrating out the fermions, but $C_3 = 0$ within the critical region $\xi \gg \xi_{sc}$. We have evaluated C_2 numerically for different band-structures that resemble the pnictide and cuprate Fermi surfaces. Remarkably, we find that $C_2 < 0$ for all the cases that we have considered here; a specific example of a Fermi-surface with the corresponding $\delta\lambda_L$ are shown in Figs. 3 and 4. In the numerical computations, $\langle \varphi^2 \rangle$ has been computed at the Gaussian level on the disordered side of the critical point. Since the qualitative features of the results do not seem to depend on the specific details of the underlying band-structure, it is likely that the observed behavior in $\delta\lambda_L$ is present in the vicinity of all $(2+1)$ -dimensional spin-density wave quantum critical points in a superconductor. We also note that our present methods, which focus only on the longest wavelength fluctuations of φ , cannot accurately account for the *analytic* dependence of λ_L on $g - g_c$ contained in the a_1 term in Fig. 1A; Ref [12] accounts for the short wavelength fluctuations of φ more completely, and so their methods give a better estimate of a_1 , and of the enhancement of λ_L upon approaching the critical point from the SC.

A similar analysis applies to the nematic ordering transition in case B. The order parameter φ has only one component, and its coupling to the electrons in the pairing channel is

$$H_{\text{nematic}} = g\varphi \sum_{\mathbf{k}, a} c_{\mathbf{k}, a}^\dagger c_{-\mathbf{k}, a} + \text{H.c.} \quad (5)$$

In this case, $C_2 > 0$ for all band-structures that were considered and the behavior of $\delta\lambda_L$ for the Fermi-surface chosen in Fig.3(A) is shown in the inset of Fig. 3(B). Once again, it is likely that the observed increase in $\delta\lambda_L$ away from the critical point is present at all $(2+1)$ -dimensional nematic quantum critical points.

Turning to case C, we consider the computation of C_2 for the case of the SC*-SC topological transition, where a rather

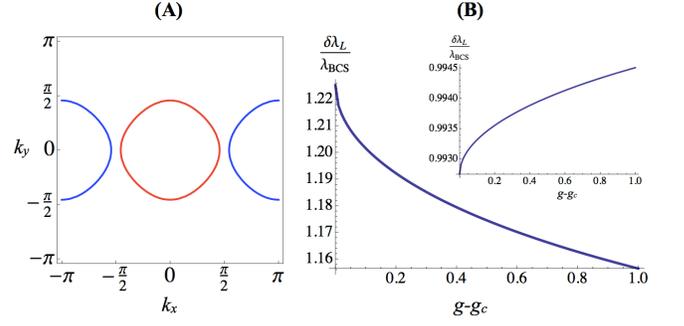


FIG. 3: (A) A two-orbital band structure of the form $\varepsilon_1(\mathbf{k}) = -2t_1 \cos(k_x) - 2t_2 \cos(k_y) - \mu$, $\varepsilon_2(\mathbf{k}) = 2t_2 \cos(k_x) - 2t_1 \cos(k_y) - \mu$, with $t_1 = t_2 = 0.22$, $\mu = -0.5$, $g = 0.5$, $\Delta_1 = -\Delta_2 = 1.0$ (B) The singular correction to $\delta\lambda_L/\lambda_{\text{BCS}}$ as a function of $g > g_c$ for the SDW quantum critical point. Inset: $\delta\lambda_L/\lambda_{\text{BCS}}$ for $g > g_c$ for the nematic quantum critical point.

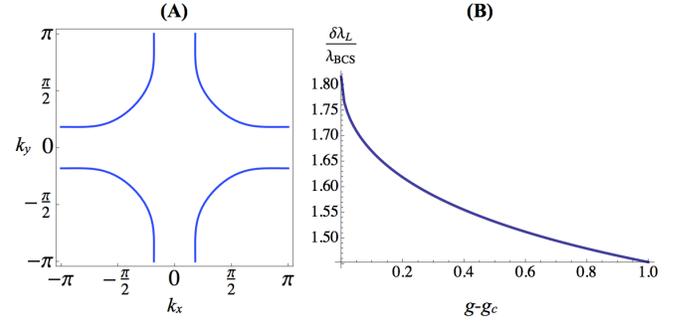


FIG. 4: (a) Fermi surface for the one-band model, $\varepsilon(\mathbf{k}) = -2t_1(\cos(k_x) + \cos(k_y)) - 4t_2 \cos(k_x) \cos(k_y) - 2t_3(\cos(2k_x) + \cos(2k_y)) - \mu$, with parameters $t_1 = 1.0$, $t_2 = -0.32$, $t_3 = 0.128$, $\mu = -1.11856$. The gap function is taken to be $\Delta(\mathbf{k}) = \Delta_0(\cos(k_x) - \cos(k_y))$ with $\Delta_0 = 1.0$. Other parameters are $g = 2.0$, $c = 1.0$. (b) $\delta\lambda_L/\lambda_{\text{BCS}}$ is shown for $g > g_c$ for the $\mathbf{Q} = (\pi, \pi)$ SDW QCP.

different treatment is required. We consider models appropriate for heavy-fermion materials consisting of itinerant conduction electrons and localized spins. Writing the spin operators in terms of fermionic spinons, $\vec{S}_i = f_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} f_{i\beta}/2$, leads to the emergence of an internal vector potential, \mathbf{a} , in addition to the electromagnetic gauge field, \mathbf{A} . The conduction electrons and the spinons can be described by Hamiltonians similar to that in Eq. (3): for the conduction electrons the pairing amplitude Δ represents a superconducting pairing, while the pairing amplitude for the spinons is necessary to obtain a \mathbb{Z}_2 spin liquid [18, 19]. The crucial new ingredient is the Kondo coupling between the conduction electrons and spinons

$$H_{\text{Kondo}} = \sum_i (\Phi_i^* c_{i\alpha}^\dagger f_{i\alpha} + \Phi_i f_{i\alpha}^\dagger c_{i\alpha}), \quad (6)$$

where Φ_i is a complex field, which carries gauge charge $(-1_{\mathbf{A}}, 1_{\mathbf{a}})$ and whose condensation leads to confinement. Because the fermions remain gapped on both sides of the transition, we can integrate them out completely and the action for

the gauge fields is given by,

$$\mathcal{S}_0[\mathbf{A}, \mathbf{a}] = \int d^2x d\tau \left(\frac{\mathbf{A}_\perp \Pi^c \mathbf{A}_\perp}{2} + \frac{\mathbf{a}_\perp \Pi^f \mathbf{a}_\perp}{2} \right), \quad (7)$$

where $\Pi^{c,f} > 0$ denote the bare ‘‘superfluid’’ stiffnesses. The condensation of $\langle cc \rangle$ and $\langle ff \rangle$ break the $U(1)_\mathbf{A}$ and $U(1)_\mathbf{a}$ to $Z_{2\mathbf{A}}$ and $Z_{2\mathbf{a}}$ respectively. The operator Φ^2 is neutral under both Z_2 gauge fields. However, once Φ condenses the gauge invariance $Z_{2\mathbf{A}} \times Z_{2\mathbf{a}}$ is broken to its diagonal $Z_{2(\mathbf{A}-\mathbf{a})}$.

We can express Φ in terms of the real fields (φ_0, φ) . The component φ_0 remains gapped even across the critical point, and we can safely integrate it out. Therefore, the critical theory is governed by \mathcal{S}_φ with $N = 1$. The coupling between φ and the gauge-fields is given by,

$$\mathcal{S}[\varphi, \mathbf{A}, \mathbf{a}] = \int d^2x d\tau \frac{1}{2} (\mathbf{A}_\perp - \mathbf{a}_\perp)^2 \varphi^2. \quad (8)$$

It is now a simple matter to integrate \mathbf{a} out from $\mathcal{S}_0[\mathbf{A}, \mathbf{a}] + \mathcal{S}[\varphi, \mathbf{A}, \mathbf{a}]$, which yields the action in Eq. (1) with

$$C_1 = \frac{\Pi^c}{2}, \text{ and, } C_2 = \frac{\Pi^f}{2(\Pi^f + \varphi^2)} > 0. \quad (9)$$

In the limit of large Π^f and sufficiently close to the critical point, $C_2 \approx 1/2$.

This paper has provided signatures of various quantum critical points in superconductors in Fig. 1. For the important case A of the SDW+SC transition, our result agrees with observations in $\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$ [8] in that λ_L increases upon approaching the critical point from the SC phase. However, one of our key results is that there is no maximum in λ_L at the quantum critical point, only a change in sign of its second derivative. Therefore the observed maximum should be within the SDW+SC phase, and involves physics at scales $\xi \sim \xi_c$ or smaller. In this regime, there are renormalizations of the spectrum and lifetime of the fermionic excitations, such as those that are crucial for the onset of the SDW order in a metal [25, 26]. Therefore, it is necessary to evaluate diagrams like those in Fig. 2 while including Fermi surface reconstruction and fermion self energy corrections.

For case B, there isn’t enough evidence yet from the experiments on the pnictides whether the penetration depth has any non-analytic features in the vicinity of the nematic critical point. However, nematic phases are ubiquitous in these systems and hence more careful experiments in the near future are likely to reveal interesting features close to such quantum critical points.

Finally for case C, the heavy fermion compounds with indications of fractionalized metallic states [20–23] are good candidates for realizing a SC* superconductor. Moreover, in $\text{CeCu}_2(\text{Si}_{1-x}\text{Ge}_x)_2$, experiments have found two distinct superconducting domes as a function of pressure along with similar results in $\text{Ce}(\text{Rh}, \text{Ir}, \text{Co})\text{In}_5$ as a function of doping [27]. The nature of one of these SC states remains mysterious and is often attributed to valence fluctuations. It would be interesting to carry out penetration depth measurements in these materials and compare with the present study.

We hope that the signatures reported here will be useful in experimental identifications of various quantum critical points in the pnictides, cuprates and heavy-fermion superconductors in the near future.

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