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Holographic metals and the fractionalized Fermi liquid

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We show that there is a close correspondence between the physical properties of holographic metals near charged black holes in anti-de Sitter (AdS) space, and the fractionalized Fermi liquid phase of the lattice Anderson model. The latter phase has a 'small' Fermi surface of conduction electrons, along with a spin liquid of local moments. This correspondence implies that certain mean-field gapless spin liquids are states of matter at non-zero density realizing the near-horizon, $AdS_2\times R^2$ physics of Reissner-Nordström black holes.

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There has been a flurry of recent activity [1–10] on the holographic description of metallic states of non-zero density quantum matter. The strategy is to begin with a strongly interacting conformal field theory (CFT) in the ultraviolet (UV), which has a dual description as the boundary of a theory of gravity in anti-de Sitter (AdS) space. This CFT is then perturbed by a chemical potential (μ) conjugate to a globally conserved charge, and the infrared (IR) physics is given a holographic description by the gravity theory. For large temperatures $T \gg \mu$, such an approach is under good control, and has produced a useful hydrodynamic description of the physics of quantum criticality [11]. Much less is understood about the low temperature limit $T \ll \mu$: a direct solution of the classical gravity theory yields boundary correlation functions describing a non-Fermi liquid metal [4], but the physical interpretation of this state has remained obscure. It has a non-zero entropy density as $T \to 0$, and this raises concerns about its ultimate stability.

This paper will show that there is a close parallel between the above theories of holographic metals, and a class of mean-field theories of the 'fractionalized Fermi liquid' (FFL) phase of the lattice Anderson model.

The Anderson model (specified below) has been a popular description of inter-metallic transition metal or rare-earth compounds: it describes itinerant conduction electrons interacting with localized resonant states representing d (or f) orbitals. The FFL is an exotic phase of the Anderson model, demonstrated to be generically stable in Refs. [12, 13]: it has a 'small' Fermi surface whose volume is determined by the density of conduction electrons alone, while the d electrons form a fractionalized spin liquid state. The FFL was also found in a large spatial dimension mean field theory by Burdin *et al.* [14], and is the ground state needed for a true "orbital-selective Mott transition" [15]. The FFL should be contrasted from the conventional Fermi liquid phase (FL), in which there is a 'large' Fermi surface whose volume counts both the conduction and d electrons: the FL phase is the accepted description of many 'heavy fermion' rare-earth intermetallics. However, recent experiments on $YbRh_2(Si_{0.95}Ge_{0.05})_2$ have observed an unusual phase for which the FFL is an attractive candidate [16].

Here, we will describe the spin liquid of the FFL by the gapless mean-field state of Sachdev and Ye [17] (SY). We will then find that physical properties of the FFL are essentially identical to those of the present theories of holographic metals. Similar comments apply to other gapless quantum liquids [18] which are related to the SY state. This agreement implies that non-zero density matter described by the SY (or a related) state is a realization of the near-horizon, $AdS_2\times R^2$ physics of Reissner-Nordström black holes.

We begin with a review of key features of the present theory of holographic metals. The UV physics is holographically described by a Reissner-Nordström black hole in AdS_4. In the IR, the low energy physics is captured by the near-horizon region of the black hole, which has a $AdS_2\times R^2$ geometry [4]. The UV theory can be written as a $SU(N_c)$ gauge theory, but we will only use gauge-invariant operators to describe the IR physics. We use a suggestive condensed matter notation to represent the IR, anticipating the correspondence we make later. We probe this physics by a 'conduction electron' $c_{\mathbf{k}\alpha}$ (where $\mathbf{k}$ is a momentum and $\alpha =\uparrow,\downarrow$ a spin index), which will turn out to have a Fermi surface at a momentum $k \equiv |\mathbf{k}| = k_F$. The IR physics of this conduction electron is described by the effective Hamiltonian [4, 7]

$$H = H_0 + H_1[\mathfrak{d}, c] + H_{\rm AdS} \quad (1)$$

$$H_0 = \sum_\alpha \int \frac{d^2k}{4\pi^2} (\varepsilon_{\mathbf{k}} - \mu) c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha}, \quad (2)$$

with $c_{\mathbf{k}\alpha}$ canonical fermions and $\varepsilon_{\mathbf{k}}$ their dispersion, and

$$H_1[\mathfrak{d}, c] = \sum_\alpha \int \frac{d^2k}{4\pi^2} \left[V_{\mathbf{k}} \mathfrak{d}^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} + V^*_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} \mathfrak{d}_{\mathbf{k}\alpha} \right], \quad (3)$$

with $V_{\mathbf{k}}$ a 'hybridization' matrix element. The $\mathfrak{d}_{\mathbf{k}\alpha}$ are non-trivial operators controlled by the strongly-coupled IR CFT associated with the AdS_2 geometry, and described by $H_{\rm AdS}$; their long imaginary-time (τ) correlation under $H_{\rm AdS}$ is given by [4, 7, 19] (for $0 < \tau < 1/T$)

$$\left\langle \mathfrak{d}_{\mathbf{k}\alpha}(\tau) \mathfrak{d}^\dagger_{\mathbf{k}\beta}(0) \right\rangle_{H_{\rm AdS}} \sim \left[\frac{\pi T}{\sin(\pi T \tau)} \right]^{2\Delta_k}, \quad (4)$$

where Δ_k is the scaling dimension of $\mathfrak{d}_{\mathbf{k}\alpha}$ in the IR CFT. The $T > 0$ functional form in Eq. (4) is dictated by conformal invariance. This $\mathfrak{d}_{\mathbf{k}\alpha}$ correlator implies a singular self-energy for the conduction electrons; after accounting for it, many aspects of 'strange metal' phenomenology can be obtained [8]. The marginal Fermi liquid phenomenology [20] is obtained for $\Delta_k = 1$.

The important characteristics of the above holographic description of metals, which we will need below, are: (i) a conduction electron self-energy which has no singular dependence on $k - k_F$, (ii) a dependence of the self energy on frequency (ω) and T which has a conformal form (obtained by a Fourier transform of Eq. (4)), and (iii) a non-zero ground state entropy associated with the $\mathrm{AdS}_2 \times \mathrm{R}^2$ geometry.

Let us now turn to the lattice Anderson model. To emphasize the correspondence to the holographic theory, we continue to use $c_{\mathbf{k}\alpha}$ for the conduction electrons, while $d_{\mathbf{k}\alpha}$ are canonical fermions representing the d orbitals (these will be connected to the $\mathfrak{d}_{\mathbf{k}\alpha}$ below). Then the Hamiltonian is $H_A = H_0 + H_1[d, c] + H_U$, where the first two terms are still specified by Eqs. (2) and (3), and

$$
\begin{aligned}
H_U &= \sum_i \Big[U n_{di\uparrow} n_{di\downarrow} + (\varepsilon_d - U/2 - \mu)\, d_{i\alpha}^\dagger d_{i\alpha} \Big] \\
&\quad - \sum_{i \neq j} t_{ij} d_{i\alpha}^\dagger d_{j\alpha}, \qquad (5)
\end{aligned}
$$

where $d_{i\alpha}$ is the Fourier transform of $d_{\mathbf{k}\alpha}$ on the lattice sites i at spatial positions $\mathbf{r}_i$ with $d_{i\alpha} = \int_k d_{\mathbf{k}\alpha} e^{i\mathbf{k}\cdot\mathbf{r}_i}$, $n_{di\alpha} = d_{i\alpha}^\dagger d_{i\alpha}$ is the d number operator, and t_{ij} are hopping matrix elements for the d electrons. We consider H_A as the UV theory of the lattice Anderson model; it clearly differs greatly from the UV AdS_4 SU(N_c) CFT considered above. We will now show that, under suitable conditions, both theories have the same IR limit.

We need to study the IR limit of H_A to establish this claim. We work in the limit of U larger than all other parameters, when the occupation number of each d site is unity. As is well-known [21, 22], to leading order in the t_{ij} and $V_{\mathbf{k}}$, we can eliminate the coupling to the doubly-occupied and empty d sites by a canonical transformation $\mathcal{U}$, and derive an effective low-energy description in terms of a Kondo-Heisenberg Hamiltonian. Thus $H_A \to \mathcal{U} H_A \mathcal{U}^{-1}$, where the $d_{i\alpha}$ are now mapped as $\mathfrak{d}_{i\alpha} = \mathcal{U} d_{i\alpha} \mathcal{U}^{-1}$ which yields [22, 23]

$$
\begin{aligned}
H_A &= H_0 + H_1[\mathfrak{d}, c] + H_J \qquad (6) \\
\mathfrak{d}_{i\alpha} &= \frac{\sigma^a_{\alpha\beta}}{2} \int \frac{d^2k}{4\pi^2} \left[\frac{U V_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}_i}}{U^2/4 - (\varepsilon_d - \varepsilon_{\mathbf{k}})^2} \right] c_{\mathbf{k}\beta}\, \hat{S}^a_i. \qquad (7)
\end{aligned}
$$

Here σ^a ($a = x, y, z$) are the Pauli matrices and the $\hat{S}^a_i$ are operators measuring the spin of the d local moment on site i. The $\hat{S}^a_i$ operators should be considered as abstract operators acting on the local moments: they are fully defined by the commutation relations $[\hat{S}^a_i, \hat{S}^b_j] = i\epsilon^{abc}\delta_{ij}\hat{S}^c_i$ and the length constraint $\sum_a \hat{S}^{a2}_i = 3/4$. The IR physics directly involves only the metallic $c_{\mathbf{k}\alpha}$ fermions (which remain canonical), and the spin operators $\hat{S}^a_i$. The Schrieffer-Wolff transformation [22] implies that $\mathfrak{d}_{\mathbf{k}\alpha}$ is a composite of these two low-energy (and gauge-invariant) operators, and is not a canonical fermion. The canonical transformation $\mathcal{U}$ also generates a direct coupling between the $\hat{S}^a_i$ which is

$$
H_J = \sum_{i<j} J_{ij} \hat{S}^a_i \hat{S}^a_j \qquad (8)
$$

where $J_{ij} = 4|t_{ij}|^2/U$. Also note that after substituting Eq. (7) into H_1 we obtain the Kondo exchange between the conduction electron and the localized spins: here, we have reinterpreted this Kondo interaction as the projection of the d electron to the IR via Eq. (7).

More generally, we can view the correspondence $\mathfrak{d} \sim c\hat{S}$ in Eq. (7) as the simplest operator representation consistent with global conservation laws. We need an operator in the IR theory which carries both the electron charge and spin $S = 1/2$. The only simpler correspondence is $\mathfrak{d} \sim c$, but this can be reabsorbed into a renormalization of the c dispersion.

We now focus on the FFL phase of H_A in Eq. (6). In this phase the influence of H_1 can be treated perturbatively [12] in $V_{\mathbf{k}}$, and so we can initially neglect H_1. Then the $c_{\mathbf{k}\alpha}$ form a 'small' Fermi surface defined by H_0, and the spins of H_J are required [12] to form a spin liquid. As discussed earlier, we assume that H_J realizes the SY gapless spin liquid state. Such a state was formally justified [17] in the quantum analog of the Sherrington-Kirkpatrick model, in which all the J_{ij} are infinite-range, independent Gaussian random variables with variance J^2/N_s (N_s is the number of sites, i). However, it has also been shown [14, 24] that closely related mean-field equations apply to frustrated antiferromagnets with non-random exchange interactions in the limit of large spatial dimension [25, 26]. We will work here with the SY equations as the simplest representative of a class that realize gapless spin liquids. The SY state of H_J is described by a single-site action $\mathcal{S}$, describing the self-consistent quantum fluctuation of the spin $\hat{S}^a(\tau)$ in imaginary time. We express the spin in terms of a unit-length vector $n^a(\tau) = 2\hat{S}^a(\tau)$ and then we obtain the coherent state path integral

$$
\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}n^a(\tau)\, \delta(n^{a2}(\tau) - 1) \exp(-\mathcal{S}) \qquad (9) \\
\mathcal{S} &= \frac{i}{2} \int d\tau \mathcal{A}^a \frac{dn^a}{d\tau} - \frac{J^2}{2} \int d\tau d\tau' Q(\tau - \tau') n^a(\tau) n^a(\tau').
\end{aligned}
$$

The first term in $\mathcal{S}$ is the spin Berry phase, with $\mathcal{A}^a$ any function of n^a obeying $\epsilon^{abc}(\partial\mathcal{A}^b/\partial n^c) = n^a$. The function Q is to determined self-consistently by the solution of

$$
Q(\tau - \tau') = \langle n^a(\tau) n^a(\tau') \rangle_{\mathcal{Z}}\,. \qquad (10)
$$

The equations (9) and (10) define a strong-coupling problem for which no complete solution is known. However, these equations have been extensively studied

[17, 27–29] in the framework of a $1/N$ expansion in which the SU(2) spins are generalized to SU(N) spins, and some scaling dimensions are believed to be known to all orders in $1/N$ [29]. Note that the SU(N) is a global 'flavor' symmetry. For the SU(N) case, we can consider general spin representations described by rectangular Young tableaux with m columns and qN rows. For such spins, the generators of SU(N), $\hat{S}_\alpha^\beta$, (now $\alpha, \beta = 1 \ldots N$) can be written in terms of 'slave' fermions $f^{s\alpha}$ (with $s = 1 \ldots m$) by [31] $\hat{S}_\alpha^\beta = \sum_{s=1}^m f_{s\alpha}^\dagger f^{s\beta}$ along with the constraint $\sum_{\alpha=1}^N f_{s\alpha}^\dagger f^{s'\alpha} = \delta_s^{s'} qN$. When expressed in terms of such fermions, the original lattice model H_J defines a U(m) gauge theory [31]. It is worth emphasizing that the $f_{s\alpha}$ are the *only* gauge-dependent operators considered in this paper, and the U(m) gauge transformation acts on the s index. For $\mathcal{Z}$ in Eq. (9), the slave fermion representation enables a solution in the limit of large N, at fixed q and m. Remarkably, the IR limit of this solution has the structure of a conformally-invariant (0+1)-dimensional boundary of a 1+1 dimensional CFT [27, 28]. In particular, for the fermion Green's function $G_f(\tau) = \left\langle f^{s\alpha}(\tau) f_{s\alpha}^\dagger(0) \right\rangle$ we find the conformal form [17, 28, 29]

$$G_f(\tau) \sim \left[\frac{\pi T}{\sin(\pi T \tau)} \right]^{1/2}. \tag{11}$$

In the large N limit, $Q(\tau) \propto G_f(\tau) G_f(-\tau)$, and therefore

$$Q(\tau) \sim \frac{\pi T}{\sin(\pi T \tau)}. \tag{12}$$

This implies the non-trivial result that the scaling dimension of the spin operator $\hat{S}_\beta^\alpha$ is 1/2. It has been argued that this scaling dimension holds to all orders in $1/N$ [29, 30], and so for SU(2) we also expect $\dim[\hat{S}^a] = 1/2$. Other mean-field theories of H_A have been studied [18, 24, 25, 27, 30], and yield related gapless quantum liquids with other scaling dimensions, although in most cases the solution obeys the self-consistency condition in Eq. (10) only with the exponent in Eq. (12).

With the knowledge of Eq. (12), we can now compute the physical properies of the FFL phase of H_A associated with the SY state. These can be computed perturbatively in $V_{\mathbf{k}}$, as was discussed by Burdin *et al.* [14]. They reproduced much of the 'marginal Fermi liquid' phenomenology of Ref. [20], including the linear-T resistivity. Note that no exponent was adjusted to achieve this (unlike Ref. [8]); the linear resistivity is a direct consequence of the scaling dimension $\text{sdim}[\hat{S}^a] = 1/2$.

We are now in a position to compare the IR limit of the theory of holographic metals to the FFL phase of H_A associated with Eq. (12):
(i) For H_A, we can easily compute the two-point $d_{\mathbf{k}\alpha}$ correlator from Eq. (7) as a product of the $c_{\mathbf{k}\alpha}$ and $\hat{S}^a$ correlators. For the latter, we use (12) for the on-site correlation, and drop the off-site correlations which average to zero in the SY state (and in large dimension limits); it is this limit which leads to the absence of a singular $\mathbf{k}$-dependence in the $d_{\mathbf{k}\alpha}$ correlator. For the electron, we use the Fermi liquid result

$$\left\langle c_{i\alpha}(\tau) c_{i\alpha}^\dagger(0) \right\rangle_{H_0} \sim \frac{\pi T}{\sin(\pi T \tau)}, \tag{13}$$

and then we find that the $d_{\mathbf{k}\alpha}$ correlator has the form of the holographic result in Eq. (4) with $\Delta_k = 1$. As expected from the results for H_A, this is the value of Δ_k corresponding to the marginal Fermi liquid [8].
(ii) The SY state has a finite entropy density at $T = 0$. This entropy has been computed in the large N limit [29], and the results agree well with considerations based upon the boundary entropy of 1+1 dimensional CFTs [27]. The holographic metal has also a finite entropy density, associated with the horizon of the extremal black hole. However, a quantitative comparison of the entropies of these two states is not yet possible. The entropy of the SY state is quantitatively computed [29] in the limits of large N (where SU(N) is a flavor group) and large spatial dimension, but at fixed m and q. In contrast, the holographic metal computation is in the limit of large N_c (where SU(N_c) is the gauge group).

The above correspondences in the IR limit of the electron correlations and the thermodynamics support our main claim that the SY-like spin liquids realize the physics of $\text{AdS}_2 \times \text{R}^2$.

It is interesting to compare our arguments with the recent results of Kachru *et al.* [32]. They used an intersecting D-brane construction to introduce point-like impurities with spin degrees of freedom which were coupled to a background CFT. For each such impurity there was an asymptotic AdS_2 and an associated degeneracy of the ground state; a lattice of impurities led to a non-zero entropy density at $T = 0$. Thus working from their picture, it is very natural to associate $\text{AdS}_2 \times \text{R}^2$ with a lattice of interacting spins; with supersymmetry [32], or in a mean-field theory [17], such a model can have a non-zero entropy density. The similarity between these theories leads us to conjecture that a possible true ground state of the quantum gravity theory of the holographic metal is a spontaneously formed crystal of spins coupled to the Fermi surface of conduction electrons. This would then be an example of quantum "order-from-disorder" [33], with the quantum ground state having a lower translational symmetry than that of the classical gravity theory.

Below, we accept our main claim connecting the holographic metal to the FFL phase with a SY-like spin liquid, and discuss further implications.

From the perspective of H_A, it is not likely that the SY state is stable beyond its large spatial dimension limit [29]; the $d_{\mathbf{k}\alpha}$ propagator should acquire a singular $\mathbf{k}$-dependence in finite dimensions. However, the remarkable emergence of the large dimension SY state in the very different holographic context suggests a certain robustness, and so perhaps it should be taken seriously as a description over a wide range of intermediate energy scales. Ultimately, it is believed that at sufficiently low energies we must cross over to a gauge-theoretic description of a stable spin liquid with zero ground state entropy density [12, 13]. Associated with this stable spin liquid would be a stable FFL phase in finite dimension, whose ultimate IR structure was described earlier [12, 13]. It is clearly of interest to find the parallel instabilities of the holographic metal on $AdS_2 \times R^2$. The geometry should acquire corrections which are compatible with a $\mathbf{k}$-dependent self energy, and this should ultimately lift the ground state entropy. Some of the considerations of Refs. [6, 10] may already represent progress in this discussion.

Refs. [12, 13] also discussed the nature of the quantum phase transition between the FFL and FL phases. It was argued that this was a Higgs transition which quenched gauge excitations of the FFL spin liquid. Consequently, we conclude that a holographic Fermi liquid can be obtained by a Higgs transition in the boundary theory. In string theory, the Higgs transition involves separation of co-incident D-branes, and it would be useful to investigate such a scenario here. The transition from FFL to FL involves an expansion in the size of the Fermi surface from 'small' to 'large', so that the Fermi surface volume accounts for all the fermionic matter. It is no longer permissible to work perturbatively in $V_{\mathbf{k}}$ in the FL phase: instead we have to renormalize the band structure to obtain quasiparticles that have both a $c_{\mathbf{k}\alpha}$ and a $d_{\mathbf{k}\alpha}$ character. Present theories of holographic metals have an extra 'bath' of matter outside the Fermi surface which can be accounted for perturbatively in $V_{\mathbf{k}}$—indeed, these were key reasons for identifying them with the FFL phase. It would be interesting to obtain the Fermi surface expansion in the holographic context.

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