

# Infinite randomness fixed point of the superconductor-metal quantum phase transition

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We examine the influence of quenched disorder on the superconductor-metal transition, as described by a theory of overdamped Cooper pairs which repel each other. The self-consistent pairing eigenmodes of a quasi-one dimensional wire are determined numerically. Our results support the recent proposal by Hoyos *et al.* (Phys. Rev. Lett. **99**, 230601 (2007)) that the transition is characterized by the same strong disorder fixed point describing the onset of ferromagnetism in the random quantum Ising chain in a transverse field.

Numerous recent experiments [1, 2, 3, 4, 5, 6] have measured the electrical transport properties of quasi-one dimensional nanowires. While thicker wires have vanishing resistance in the low temperature ( $T$ ) limit, thinner wires do not display superconductivity even at the lowest  $T$ . The superconducting wires display clear signatures of thermal phase fluctuations of the Cooper pair order parameter,  $\Psi$ , at low  $T$ . Quantum fluctuations of the phase and amplitude of  $\Psi$  increase with decreasing wire thickness, leading to a transition to a non-superconducting state.

Recent work [7, 8] has proposed that these experiments should be described by a quantum superconductor-metal transition (SMT) in the pair-breaking universality class. Arguments based upon microscopic BCS theory were used to propose a model of  $\Psi$  fluctuations damped by decay into single electron excitations of the metal [9, 10, 11, 12, 13]. In this paper, we will present the results of a numerical study of the influence of quenched disorder on this model. The role played by disorder near the quantum SMT is of considerable interest, as disorder correlations are of infinite range in the imaginary time direction and can lead to unusual critical phenomena [14].

In a renormalization group (RG) analysis of this overdamped Cooper pair model with disorder, Hoyos *et al.* [15] have recently argued that the SMT is described by a strong disorder fixed point. The latter exhibits activated dynamic scaling where the logarithm of characteristic frequencies of  $\Psi$  fluctuations grows as a power of their characteristic length scale. They argued further that the strong disorder fixed point is in the same universality class as the one describing the onset of ferromagnetism in the quantum random transverse field Ising model (RT-FIM) in one spatial dimension. Many exact results were obtained by Fisher [16] for this fixed point, and carry through to a large extent to the case of the SMT. Note that this is a non-trivial result, since the RTFIM contains *no dissipation*, and possesses a *discrete symmetry*. All our numerical results below confirm the remarkable predictions of Hoyos *et al.*, providing strong evidence for the applicability of their strong randomness RG.

Our analysis was carried out on a lattice discretization of the disordered overdamped Cooper pair model of Ref. 15 at  $T = 0$ . The degrees of freedom are  $\Psi_j(\tau)$ ,

which are complex functions of imaginary time,  $\tau$ , on the sites,  $j$ , of a one-dimensional chain with action

$$\mathcal{S} = \sum_j \int d\tau \left[ D_j |\Psi_j - \Psi_{j+1}|^2 + \alpha_j |\Psi_j|^2 + \frac{u_j}{2} |\Psi_j|^4 \right] + \int \frac{d\omega}{2\pi} \sum_j \gamma_j |\omega| |\Psi_j(\omega)|^2, \quad (1)$$

where  $\Psi_j(\omega)$  is the Fourier transform of  $\Psi_j(\tau)$ , and the couplings in  $\mathcal{S}$  are all random functions of  $j$ . The quartic coefficients  $u_j$  are all positive to ensure stability and repulsion between Cooper pairs. The dissipation into the metallic bath is represented by  $\gamma_j$ , which is also required to be positive by causality. Finally, we can choose a gauge such that  $D_j > 0$ . A more careful analysis and suitable rescalings [17] allow us to reduce the randomness to the spatial dependence of  $D_j$  (uniformly distributed on  $(0, 1]$ ) and  $\alpha_j$  (taken to be Gaussian), while setting  $u_j = u$  and  $\gamma_j = 1$ . At zero temperature, the SMT can be tuned by reducing the mean of the  $\alpha_j$  distribution,  $\bar{\alpha}$ , while keeping its variance constant at 0.25 in units of  $\gamma^2$ .

Equivalently, we can also work in a lattice model of fluctuating phases with  $\Psi_j(\tau) = e^{i\theta_j(\tau)}$  of unit magnitude [7, 12]; this should have the same properties as  $\mathcal{S}$ , but our analysis proceeds more conveniently by also allowing for magnitude fluctuations.

While  $\mathcal{S}$  is a suitable model for describing the influence of disorder on the fluctuating Cooper pair states, we also have to consider the effect of randomness on the single electron states. We have estimated such effects in the framework of weak-coupling BCS theory: at criticality, we find that on a scale parametrically smaller than the single electron localization length, the gain in condensation energy can offset the cost in elastic energy when order parameter fluctuations take advantage of randomness in the  $\alpha_j$ . This justifies our focus on the influence of disorder in a purely bosonic overdamped Cooper pair theory. Details will be provided later (see also Ref. 11).

The RG analysis [15] was carried out in a model with an  $N$ -component order parameter and it was found that flows had only an irrelevant dependence on the value of  $N$  [18]. Thus the exact critical properties can be obtained by studying the model in the large  $N$  limit. This

is equivalent to approximating  $\mathcal{S}$  by the Gaussian action

$$\mathcal{S}_0 = \sum_j \int \frac{d\omega}{2\pi} [D_j |\Psi_j - \Psi_{j+1}|^2 + (r_j + |\omega|) |\Psi_j|^2], \quad (2)$$

where the  $r_j$  are determined self-consistently by solving

$$r_j = \alpha_j + u \langle |\Psi_j(\tau)|^2 \rangle_{\mathcal{S}_0}. \quad (3)$$

We set  $u = 1$  to reach a strong coupling regime and use fixed but random boundary conditions, similar to those employed in Ref. 7. Solving the innocuous looking Eq. (3) for a large number of disorder realizations and large system sizes was the primary time-consuming numerical step in obtaining the results of this study. Similar numerical large- $N$  methods have been used previously for disordered systems with conventional (power law) dynamic scaling [19, 20] but the presence of activated scaling leads to sluggish dynamics and the necessity to properly include spurious disorder configurations that, although exponentially rare, can make large contributions to thermodynamic properties. The numerical solution is facilitated via the implementation of a method which we have dubbed the *solve-join-patch* (SJP) procedure. We begin by generating a realization of disorder for  $L$  sites with  $L$  large. Near the critical point, characterized by the condition that the correlation length  $\xi \sim L$ , the direct iterative solution of Eq. (3) is computationally quite costly. This is a result of the fact that the eigenmodes of  $\mathcal{S}_0$  begin to delocalize and have a characteristic energy scale that is exponentially small in the distance from criticality, requiring that the solutions  $r_j$  must be computed with exponentially increasing precision.

To cope with this difficulty, the system consisting of  $L$  sites is broken up into a group of smaller sub-systems with boundary conditions adjusted to reflect their location in the larger chain. The sub-systems are *solved*, then *joined* together in groups of two. The grouped sub-systems are now close to satisfying Eq. (3) and they can be quickly brought into accord by *patching*, which involves re-solving a mini-system around the joint consisting of a small number of sites. The joined and patched sub-system can now be easily solved and the SJP procedure is iterated until a full solution to Eq. (3) is obtained for the complete chain of  $L$  sites. We have considered up to 3000 realizations of disorder for system sizes  $L = 16, 32, 64, \text{ and } 128$ .

Fisher's remarkable solution of the RTFIM [16] includes asymptotically exact results for the exponents and correlation functions at the infinite randomness fixed point, and many directly translate to the RG calculations by Hoyos *et al.* [15] for the dissipative model considered here. In particular, one expects activated dynamic scaling with  $\ln(1/\Omega) \sim L^\psi$  where  $\Omega$  is a characteristic energy scale and  $\psi = 1/2$  is a tunneling exponent. This reflects the fact that at an infinite randomness fixed point, the dynamical critical exponent  $z$  is formally infinite. The

RG approach defines a real space decimation procedure that either creates or destroys *clusters* or *bonds* as the energy scale is reduced. The typical moment of a surviving cluster scales like  $\mu \sim \ln^\phi(1/\Omega)$  at criticality, where  $\phi = (1 + \sqrt{5})/2 \simeq 1.62$  is the golden mean. Average correlations are described by a correlation length which diverges as  $\xi \sim |\delta|^{-\nu}$  with  $\nu = 2$  and  $\delta$  a measure of the distance from criticality. From Ref. 15,  $\delta$  is expected to be proportional to  $\overline{\ln(r_i/r_c)}$  where  $r_c$  is some critical value. Our numerical study reveals that close to criticality this quantity is linearly related to the detuning of the average  $\bar{\alpha}$  from its quantum critical value,  $\bar{\alpha}_c$  (which has yet to be determined) and it further demonstrates that correlations among the  $r_i$  due to their self-consistency does not affect the strong randomness RG flow.

The remainder of this paper will present a numerical confirmation of the results of Ref. 15 by providing arguments for the presence of dynamically activated scaling at the quantum SMT, characterized by exponents  $\nu$ ,  $\psi$  and  $\phi$  taking on their RTFIM values. The evidence comes from an analysis of equal time correlations, energy gap statistics and dynamic susceptibilities in the weakly disordered quantum Griffiths phase [21].

We begin by studying the disorder averaged equal-time correlation function  $\overline{C}(x) = \langle \overline{\Psi_x^*(\tau) \Psi_0(\tau)} \rangle_{\mathcal{S}_0}$ , which can be computed from the quadratic effective action  $\mathcal{S}_0$  once the full set of solutions  $\{r_j\}$  has been obtained. In the disordered phase, where  $\delta \equiv \bar{\alpha} - \bar{\alpha}_c > 0$  the asymptotic form of  $\overline{C}(x)$  for the RTFIM has been predicted to describe both exponential as well as stretched exponential decay in addition to power law behavior [16]

$$\overline{C}(x) \sim \frac{\exp[-(x/\xi) - (27\pi^2/4)^{1/3}(x/\xi)^{1/3}]}{(x/\xi)^{5/6}}. \quad (4)$$

If we use this expression to define the correlation length  $\xi$ , we can perform fits for each value of  $L$  and various  $\bar{\alpha}$  to extract  $\xi(L, \bar{\alpha})$  as is seen in Fig. 1 for  $L = 64$ . We find remarkable agreement (solid lines) with Eq. (4) over six orders of magnitude for all system sizes considered.

As mentioned above, the length scale which describes average correlations is expected to diverge like  $\xi \sim |\delta|^{-\nu}$  as the critical point is approached. We have employed this result to perform a log-log fit to the finite size scaled correlation length (data extrapolated to  $L \rightarrow \infty$ ) as a function of  $\delta$ , as is shown in the inset of Fig. 1. The value of  $\bar{\alpha}_c$  was found from the mean of the critical  $\alpha_j$  distribution which minimized the least square error of power law fits involving  $\delta = \bar{\alpha} - \bar{\alpha}_c$ . This leads to a value of  $\bar{\alpha}_c = -0.93(3)$  for the critical point and  $\nu = 1.9(2)$  for the correlation length exponent with the number in brackets indicating the uncertainty in the last digit computed from the fitting procedure. The obtained value of  $\nu$  is in accord with the value of 2 predicted for the RTFIM. The correlation length could also have been defined via the exponential tail of  $\overline{C}(x)$  at large separations which yields compatible values for both  $\bar{\alpha}_c$  and  $\nu$ .

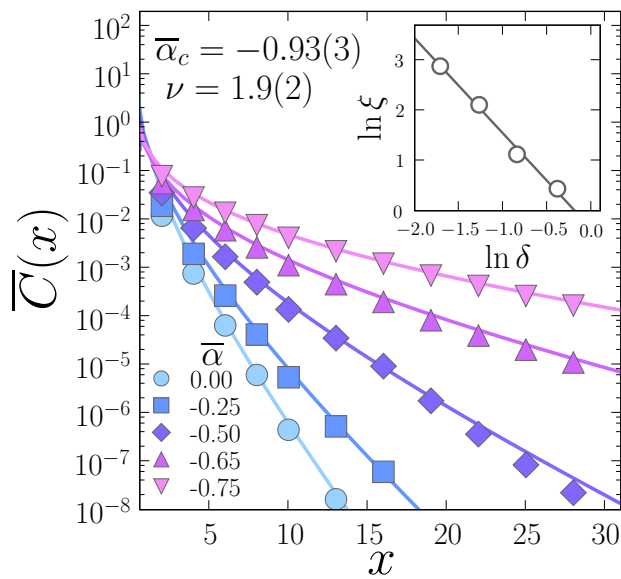


FIG. 1: (Color online) The equal-time disorder averaged correlation functions for  $L = 64$  and five values of the mean of the  $\alpha_j$  distribution,  $\bar{\alpha}$ . The solid lines are fits to the asymptotic form described in Eq. (4) via  $\xi$  and an overall scale parameter. The inset shows the result of a fit to the power law form of the finite size scaled correlation length leading to an estimate for the location of the critical point  $\bar{\alpha}_c = -0.93(3)$  and the correlation length exponent  $\nu = 1.9(2)$ .

For each realization of disorder and each value of  $\bar{\alpha}$  we define the gap  $\Omega(L)$  to be the smallest excitation energy in the system, which in general corresponds to the most delocalized mode of  $\mathcal{S}_0$ . Rare disorder configurations cause clusters to behave as if they were much more critical than the global value of  $\delta$  would suggest. These clusters dominate the critical modes and exhibit abnormally small gaps that make large contributions to disorder averages of  $\ln \Omega$ , leading to the highly anisotropic scaling relationship between space and time that is the hallmark of strong disorder fixed points. An analysis of the probability distribution for the logarithm of the energy gap in the RTFIM was carried out by Young and Rieger [22] where they found cogent evidence for  $z = \infty$ . We have performed a similar analysis here, with the same result. In addition we find that  $\ln \Omega$ , as the minimal excitation energy, is naturally Gumbel distributed. If activated dynamic scaling is indeed present, the disorder averaged value of the logarithm of the gap should scale like  $|\overline{\ln \Omega}| \sim \xi^\psi \sim \delta^{-\nu\psi}$  where we have used the scaling form of the correlation length. Such divergent behavior for the finite size scaled value of  $|\overline{\ln \Omega}|$  is demonstrated in Fig. 2. The possibility of conventional scaling was considered but ultimately excluded through the examination of the maximum likelihood estimator for a wide range of power law fits. Using the previously determined values of  $\bar{\alpha}_c$  and  $\nu$ , the tunneling exponent can be extracted from

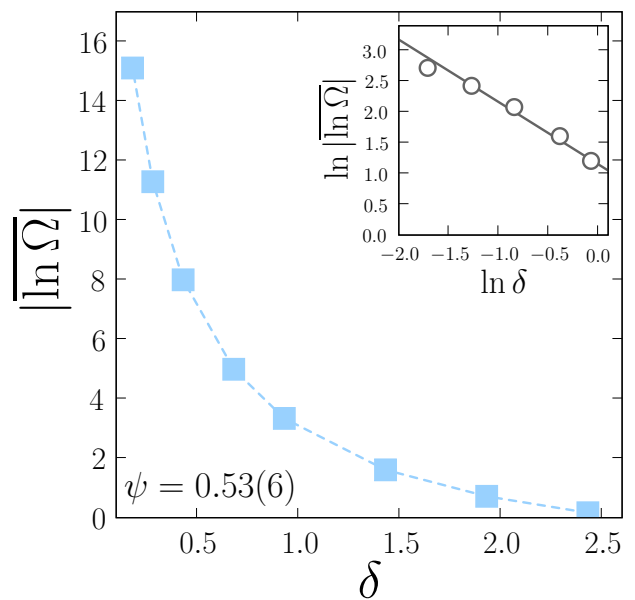


FIG. 2: (Color online) The finite size scaled value of the disorder averaged logarithm of the minimum excitation energy plotted against the distance from the critical point  $\delta$ . We observe divergence consistent with the scaling form  $|\overline{\ln \Omega}| \sim \delta^{-\nu\psi}$  and using the value of  $\bar{\alpha}_c$  and  $\nu$  found above we determine  $\psi = 0.53(6)$  from a log-log linear fit (inset).

a log-log linear fit of the average logarithmic spectrum as shown in the inset of Fig. 2, producing  $\psi = 0.53(6)$  which is consistent with the RTFIM prediction of  $1/2$ .

To confirm full agreement with the universality class of the RTFIM, we must finally determine the value of the exponent  $\phi$  which controls the average moment,  $\mu \sim |\ln \omega|^\phi$ , of a cluster fluctuating with frequency  $\omega$ . This can be accomplished by investigating the imaginary part of the disorder averaged dynamical order parameter susceptibilities after they have been analytically continued to real frequencies. We are interested in the average ( $k = 0$ ) and local susceptibilities defined by

$$\text{Im } \bar{\chi}(\omega) = \text{Im } \frac{1}{L} \sum_j \overline{\langle \Psi_j^*(i\omega) \Psi_0(i\omega) \rangle}_{\mathcal{S}_0} \Big|_{i\omega \rightarrow \omega + i\epsilon} \quad (5)$$

$$\text{Im } \bar{\chi}_{\text{loc}}(\omega) = \text{Im } \overline{\langle \Psi_j^*(i\omega) \Psi_j(i\omega) \rangle}_{\mathcal{S}_0} \Big|_{i\omega \rightarrow \omega + i\epsilon} \quad (6)$$

where  $\langle \dots \rangle_{\mathcal{S}_0}$  indicates an average with respect to the large- $N$  action in Eq. (2) as well as a site average. Note that  $\omega$  is now a real frequency, and we point out that our facile access to such dynamical quantities is one of the prerequisites of the numerical approach we have taken. All frequencies are measured with respect to an ultra-violet cutoff  $\Lambda_\omega$  which is required for convergence when computing the set of solutions to Eq. (3). Physically, one can argue that at criticality, the average cluster moment will be given by the ratio of the average to local susceptibility

due to the extra sum over sites in Eq. (5). We thus define

$$R(\omega) = \frac{\text{Im}\bar{\chi}(\omega)}{\text{Im}\bar{\chi}_{\text{loc}}(\omega)} \sim |\ln\omega|^\phi \mathcal{F}(\delta^{\nu\psi}|\ln\omega|), \quad (7)$$

and expect that the scaling function  $\mathcal{F}$  approaches a constant when the dimensionless variable  $\delta^{\nu\psi}|\ln\omega| \ll 1$ . In the quantum disordered phase with  $\delta^{\nu\psi}|\ln\omega| \gg 1$ , a scaling analysis predicts  $\mathcal{F}(x) \sim x^{1-\phi}$  [23] and hence  $R \sim \delta^{\nu\psi(1-\phi)}|\ln\omega|$  [16, 24]. In order to determine the value of  $\phi$ , it is useful to consider a rescaled value of the susceptibility ratio  $\tilde{R}(\delta) = R(\omega)/(\delta^{\nu\psi}|\ln\omega|)$  which should be frequency independent according to the predicted scaling form for  $R(\omega)$  as  $\omega \rightarrow 0$ . We plot the finite size scaled susceptibility ratio in Fig. 3 for the three smallest values of  $\delta$ , and find confirmation of its  $|\ln\omega|$  dependence. The inset of Fig. 3 confirms the frequency

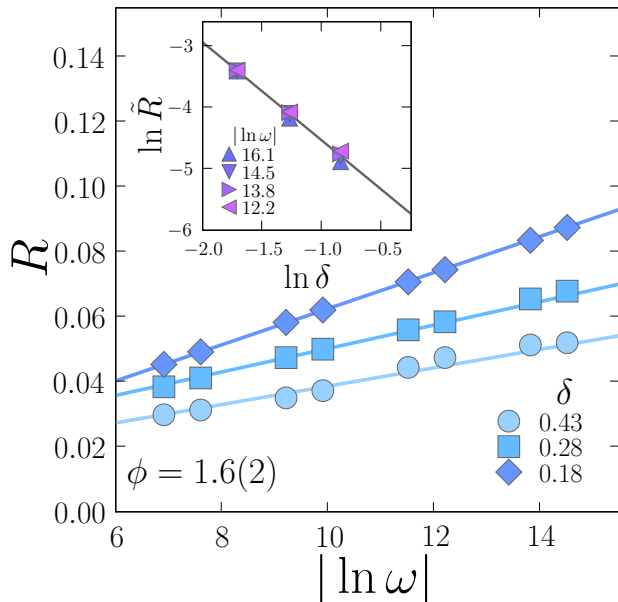


FIG. 3: (Color online) The real frequency dependence of the finite size scaled value of the disorder averaged susceptibility ratio defined in Eq. (7) for three values of the  $\delta = \bar{\alpha} - \bar{\alpha}_c$ . We observe the predicted  $|\ln\omega|$  behavior. After a suitable rescaling described in the text we find that  $\tilde{R}$  does not depend on frequency as  $\omega \rightarrow 0$  (inset), and a log-log linear fit gives the value of the cluster exponent to be  $\phi = 1.6(2)$ .

independence of  $\tilde{R}$  and by determining the best linear fit of  $\ln\tilde{R}$  to  $\ln\delta$  for  $\omega \leq 10^{-3}$  with  $\nu\psi = 1.0(1)$ , we find a cluster exponent  $\phi = 1.6(2)$  which is very close to the predicted RTFIM value of  $(1 + \sqrt{5})/2$ .

The results of the above analysis, as highlighted in Figs. 1–3, provide compelling evidence for the applicability of the real space RG analysis of Hoyos *et al.*, and further reproduces a number of results of [16] to unexpected accuracy. This confirms that the considered model for overdamped repulsive Cooperon fluctuations

in the presence of quenched disorder near a SMT exhibits dynamically activated scaling and is controlled by an infinite randomness fixed point in the same universality class as the RTFIM. The transition is characterized by the numerically computed critical exponents  $(\nu, \psi, \phi) \simeq (1.9, 0.53, 1.6)$  which are entirely consistent with those of the one dimensional random quantum Ising model in a transverse field  $(2, 1/2, (1 + \sqrt{5})/2)$ .

In closing, we note that while our discussion has been framed in the context of the SMT, models similar to  $\mathcal{S}$  describe the onset of a wide variety of orders in metallic systems [14]. Furthermore, the flow to the strong-disorder RTFIM fixed point is expected to also hold in higher dimensions [15]. We thus propose that our results provide strong support for the applicability of the RTFIM physics to many experiments involving the onset of spin- and charge-density wave orders in metals.

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