

# A square lattice algebraic spin liquid with SO(5) symmetry

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We propose a critical spin liquid ground state for  $S = 1/2$  antiferromagnets on the square lattice. In a renormalization group analysis of the ‘staggered flux’ algebraic spin liquid, we examine perturbations, present in the antiferromagnet, which break its global SU(4) symmetry to SO(5). At physical parameter values, we find an instability towards a fixed point with SO(5) symmetry. We discuss the possibility that this fixed point describes a transition between the Néel and valence bond solid states, and the relationship to the SO(5) non-linear sigma model of Tanaka and Hu.

Critical spin liquids appear in a variety of contexts in studies of correlated electrons in insulators and superconductors. These are states in which spin rotation symmetry is preserved and there is a gapless spectrum of spin excitations which do not have a quasiparticle interpretation. The simplest, and best understood, states [1, 2] are described by the Wilson-Fisher fixed point of the Landau-Ginzburg-Wilson theory of fluctuations of the vector antiferromagnetic (Néel) order parameter  $\mathbf{N}$ . These provide a description of the quantum phase transition of dimerized antiferromagnets between a state with long-range Néel order and a spin-gapped state with  $S = 1$  quasiparticle excitations which are quanta of the field  $\mathbf{N}$ . However, the anomalous dimension of the field  $\mathbf{N}$  is quite small at the critical fixed point, implying that a perturbative description in terms of  $\mathbf{N}$  quasiparticles provides a reasonable description of the zero temperature spectrum.

A separate category of critical (or ‘algebraic’) spin liquids involve a description in terms of neutral  $S = 1/2$  excitations (‘spinons’). The first example of this was the ‘staggered flux’ spin liquid state of Affleck and Marston [3], whose low energy theory involved spinons, represented two-component massless Dirac fermions  $\Psi_\alpha$  ( $\alpha = 1 \dots N = 4$  is a combined SU(2) spin and valley index), coupled to an emergent U(1) gauge field  $a_\mu$ . The spin fluctuations of this theory have been better understood in subsequent work [4, 6, 7]: as we will review below, for  $N$  sufficiently large, the low energy theory is a conformal field theory (CFT) with a global SU(4) symmetry. Another spin liquid state involves bosonic spinons represented by relativistic scalars  $z_p$  ( $p = 1, 2$  is a SU(2) spin index) coupled to a U(1) gauge field [8]. It has been proposed [9] that this is realized as a CFT describing a quantum critical point between states with Néel and valence bond solid (VBS) order.

This paper will propose a critical spin liquid ground state described by a CFT with a global SO(5) symmetry. We begin with the SU(4) CFT of Dirac fermions noted above, and examine the renormalization group (RG) flow of all perturbations which preserve relativistic invariance and at least a global SO(5)  $\subset$  SU(4) symmetry. The global symmetry of the underlying antiferromagnet involves only a continuous SU(2) spin rotation symmetry and various discrete space group symmetries, and all such

SO(5) invariant perturbations will generically be present. Using an expansion defined below, we find that for physical parameter values the SU(4) fixed point is unstable to flow towards a SO(5) invariant fixed point.

Our motivation for examining CFTs with SO(5) symmetry comes from an interesting proposal by Tanaka and Hu [10] (see also the work of Senthil and Fisher [11]). They examined the quantum fluctuations of the 3-component Néel order parameter,  $\mathbf{N}$ , and the complex VBS order parameter  $\Xi$ , and suggested that they be combined into a single five-component real vector  $\Sigma_a$ , with  $a = 1 \dots 5$  which transforms as the fundamental of an enlarged SO(5) group. In a spin liquid with such an SO(5) symmetry, the anomalous dimensions of  $\mathbf{N}$  and  $\Xi$  would be equal, and Sandvik’s numerical results [12] on the quantum critical point between the Néel and VBS states are consistent with such an equality.

We begin by reviewing the SU(4)-invariant CFT of Dirac fermions, largely following the notation of Ref. 7. The CFT is described by the Euclidean spacetime action  $S_0 = \int d^2r d\tau \mathcal{L}_0$ , where

$$\mathcal{L}_0 = \bar{\Psi}^\alpha \gamma^\mu (\partial_\mu + ia_\mu) \Psi_\alpha \quad (1)$$

where  $\mu = \tau, x, y$  is a spacetime index,  $\gamma^\mu$  are the Dirac matrices,  $\bar{\Psi} = \Psi^\dagger \gamma_0$  and  $a_\mu$  is an emergent U(1) gauge field. As shown in earlier work [4, 7], this action defines an SU( $N$ ) invariant CFT in an expansion in  $1/N$ . The combined Néel-VBS operator,  $\Sigma_a$ , can be written in terms of the  $\Psi_\alpha$  by

$$\Sigma_a = \bar{\Psi} \Gamma_a \Psi \quad (2)$$

where  $\Gamma_a$  are five  $4 \times 4$  matrices from the SU(4) algebra. This algebra can be realized using the tensor product of two independent sets of Pauli matrices,  $\boldsymbol{\mu}$  and  $\boldsymbol{\sigma}$ , and Hermele *et al.* showed that  $\Gamma_a = (\mu^z \sigma^x, \mu^z \sigma^y, \mu^z \sigma^z, \mu^x, \mu^y)$ . A curious, and key, property of the  $\Gamma_a$  is that they anti-commute,  $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$ , and so they are Dirac matrices of five spacetime dimensions. The 10 generators of the SO(5) group, under which  $\Sigma_a$  transforms as a SO(5) fundamental, are obtained from the commutators of the  $\Gamma_a$ :

$$\Gamma_{ab} = \frac{1}{2i} [\Gamma_a, \Gamma_b]. \quad (3)$$

The  $\Gamma_a$  and  $\Gamma_{ab}$  are the complete set of  $SU(4)$  generators.

It will be important for our analysis to be able to generalize these order parameters, and the associated algebraic structure, from  $SU(4)$  to general  $SU(N)$ , so as to allow a systematic  $1/N$  expansion. A similar strategy was used in the context of chiral symmetry breaking of three dimensional QED [18]. However, the above embedding of  $SO(5)$  into  $SU(4)$  relies on the spinor representations of  $SO(5)$ , and this does not have a suitable generalization. However, we note that there is an antisymmetric matrix  $\mathcal{J} = i\sigma^y\mu^x$ , with  $\mathcal{J}^2 = -1$ , under which

$$\mathcal{J}\Gamma_{ab}\mathcal{J} = \Gamma_{ab}^T \quad (4)$$

for all  $ab$ . Eq. (4) is the defining relation for generators of the  $Sp(4)$  subgroup of  $SU(4)$ , and we have just established the well-known congruence  $Sp(4) \cong SO(5)$ . The embedding of  $Sp(N)$  into  $SU(N)$  generalizes easily to all even  $N$ , with an  $N \times N$  antisymmetric  $\mathcal{J}$  matrix obeying  $\mathcal{J}^2 = -1$ . We will therefore study here the  $SU(N)$  invariant CFT in Eq. (1) with  $\alpha = 1 \dots N$ , while allowing perturbations which are invariant under  $Sp(N)$ .

A linear stability analysis of this  $SU(N)$  CFT has been carried out earlier [7] for a limited set of perturbations. For sufficiently large  $N$ , all perturbations are believed to be irrelevant. However, the anomalous dimensions arising at order  $1/N$  can be quite large, and we shall show find below a perturbation which becomes relevant when its scaling dimension is evaluated at  $N = 4$ . We are also interested in finding a systematic approach to determining the fate of such a relevant perturbation, beyond a linear stability analysis. To this end, we will allow the tree-level scaling dimensions to vary as a function of spatial dimensionality,  $d$ , as is common in other critical phenomena contexts. With Dirac fermions, there is the subtle issue of dimensional continuation of the Dirac matrices,  $\gamma^\mu$ ; as is commonly done [13], we will deal with this by applying the Dirac algebra and phase space factors as in  $d = 2$ . Our stability analysis of spin liquids and their perturbations is formally justified by taking  $(d-1) \propto 1/N$ , and then expanding in  $1/N$ .

It is also interesting to consider application of this method to antiferromagnets in  $d = 1$ . Although we will not describe the computation here, it is necessary to adapt our results to Dirac matrices in  $d = 1$ . From such a computation, we reproduced the results of Affleck [14] on the spectrum of scaling dimensions of operators with  $SU(N)$  and  $Sp(N)$  symmetry at the fixed points described by WNZW models.

We now present our RG results for perturbations of the CFT in Eq. (1). We begin by considering perturbations which are invariant under  $SU(N)$ . To the order we are working, there are only two independent perturbations, which we write as

$$\mathcal{L}_1 = \frac{\lambda_1}{N} (\bar{\Psi}^\alpha \Psi_\alpha) (\bar{\Psi}^\beta \Psi_\beta) + \frac{\lambda_2}{N} (\bar{\Psi}^\alpha \gamma^\mu \Psi_\alpha) (\bar{\Psi}^\beta \gamma_\mu \Psi_\beta) \quad (5)$$

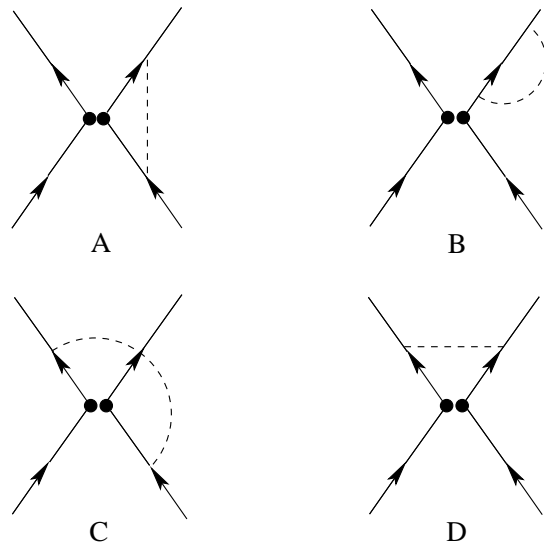


FIG. 1: Feynman diagrams contribute to the linear orders in both Eqs. 6 and 9. The dashed lines are dressed photon propagators, and the full circles denote the trace in Dirac space.

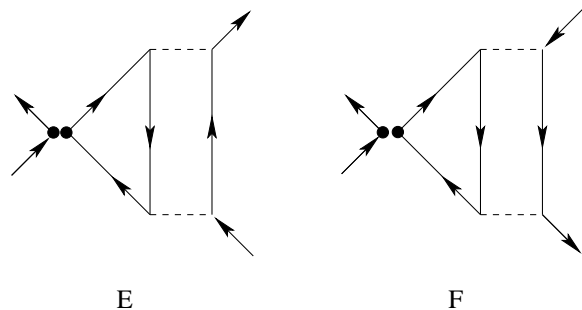


FIG. 2: Feynman diagrams which only contribute to the linear orders in Eqs. 6, but not in Eqs. 9.

where the circular brackets indicate a trace over indices in Dirac space. Other possible terms, such as  $(\bar{\Psi}^\alpha \Psi_\beta) (\bar{\Psi}^\beta \Psi_\alpha)$  and  $(\bar{\Psi}^\alpha \gamma^\mu \Psi_\beta) (\bar{\Psi}^\beta \gamma_\mu \Psi_\alpha)$ , can be shown to be linearly related to the terms in Eq. (5).

From the diagrams shown in Fig. 1, Fig. 2 and Fig. 3, we obtained the following RG equations for a rescaling by a factor  $e^\ell$ :

$$\begin{aligned} \frac{d\lambda_1}{d\ell} &= \left(1 - d - \frac{256}{3N\pi^2}\right) \lambda_1 + \frac{64}{N\pi^2} \lambda_2 - \frac{2}{\pi^2} \lambda_1^2, \\ \frac{d\lambda_2}{d\ell} &= (1 - d) \lambda_2 + \frac{64}{3N\pi^2} \lambda_1 + \frac{2}{3\pi^2} \lambda_2^2. \end{aligned} \quad (6)$$

In the terms linear in the  $\lambda$  on the right-hand-side, we have computed co-efficients to order  $1/N$ , and the  $1/N$  corrections come from the dressed photon propagator [5]

$$G_{\mu\nu}(p) = \frac{16}{Np} (\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}). \quad (7)$$

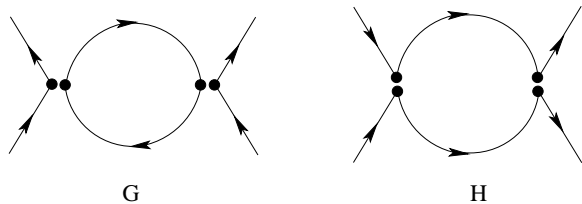


FIG. 3: Feynman diagrams which contribute to the quadratic order of the RG equations (6) and (9). Notice that since we only calculate to the order of unity in the quadratic terms, diagram G only contributes to equation (6) but not (9), and diagram H only contributes to equation (9) but not (6).

For the terms quadratic in  $\lambda$  to be of the same order as the linear terms, we need only compute the co-efficients to order unity, as is the case above.

The RG equations in Eq. (6) have several fixed points, but we begin by considering the fixed point at  $\lambda_1 = \lambda_2 = 0$ . The eigenvalues at this fixed point are  $1 - d - (128 \pm 64\sqrt{7})/(3N\pi^2)$ . At the physical values of  $d = 2$  and  $N = 4$ , these eigenvalues evaluate to  $-0.651$  and  $-3.510$ . So both are negative and the  $\lambda_1 = \lambda_2 = 0$  fixed point is stable. None of the other fixed points of Eq. (6) were found to be stable at these values of  $d$  and  $N$ . By examining the  $N$  dependence of the eigenvalues at  $\lambda_1 = \lambda_2 = 0$  we conclude that the  $SU(N)$  CFT defined by Eq. (1) is stable to  $SU(N)$ -invariant perturbations for  $N > 1.40/(d-1)$ .

Next we consider the additional perturbations of  $\mathcal{L}_0$  when the global symmetry is reduced from  $SU(N)$  to  $Sp(N)$ . A simple analysis shows there is only one allowed term

$$\mathcal{L}_2 = \frac{\lambda_3}{N} \mathcal{J}_{\alpha\gamma} \mathcal{J}^{\beta\delta} (\bar{\Psi}^\alpha \Psi_\beta) (\bar{\Psi}^\gamma \Psi_\delta). \quad (8)$$

A second possible term  $\mathcal{J}_{\alpha\gamma} \mathcal{J}^{\beta\delta} (\bar{\Psi}^\alpha \gamma^\mu \Psi_\beta) (\bar{\Psi}^\gamma \gamma_\mu \Psi_\delta)$  reduces to the above term after application of Fierz identities.

From the diagrams in Fig. 1 and diagram H in Fig. 3, the RG equations for  $\mathcal{L}_2$  reads (Notice that diagrams in Fig. 2 and diagram G in Fig. 3 do not contribute to the leading order of  $1/N$  expansion)

$$\frac{d\lambda_3}{dl} = \left(1 - d + \frac{64}{N\pi^2}\right) \lambda_3 - \frac{1}{3\pi^2} \lambda_3^2. \quad (9)$$

This has fixed points at  $\lambda_3 = 0$  and  $\lambda_3 = \lambda_3^* = 3\pi^2(1 - d + 64/(N\pi^2))$ . At  $d = 2$  and  $N = 4$  we now find a result which is very different from the  $SU(N)$  perturbations above. The  $\lambda_3 = 0$  fixed point is unstable with RG eigenvalue  $0.621$ , while the fixed point at  $\lambda_3 = \lambda_3^* > 0$  is stable; for general  $N$ , we find that the stability of the  $\lambda_3 = \lambda_3^*$  fixed point holds for  $N < 6.48/(d-1)$ . So for  $1.40/(d-1) < N < 6.48/(d-1)$ , the theory  $\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$  flows to a fixed point with  $\lambda_1 = \lambda_2 = 0$  and  $\lambda_3 = \lambda_3^*$  which describes our advertised  $Sp(N)$ -invariant critical spin liquid.

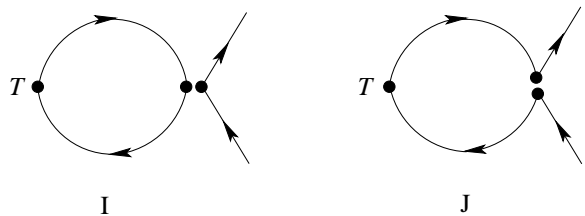


FIG. 4: Feynman diagrams which contribute to the difference of scaling dimensions of fermion bilinears  $\bar{\Psi}\Gamma_a\Psi$  and  $\bar{\Psi}\Gamma_{ab}\Psi$ .

The scaling dimensions of all 16 fermion bilinears  $\bar{\Psi}T_a\Psi$  ( $T_a$  are  $SU(4)$  generators with  $a = 1, \dots, 15$ ) equal at the fixed point with  $\lambda_i = 0$  which respects the  $SU(4)$  symmetry. At the order of  $1/N$ , the scaling dimensions read:

$$\begin{aligned} \Delta(\bar{\Psi}T_a\Psi) &= 2 - \frac{64}{3N\pi^2}, \\ \Delta(\bar{\Psi}\Psi) &= 2 + \frac{128}{3N\pi^2}. \end{aligned} \quad (10)$$

with  $N = 4$ , the difference between the two scaling dimensions above is from the diagrams similar to the ones in Fig. 3 [7] with two photon propagators and a trace in the fermion flavor space, which only contributes to fermion bilinear  $\bar{\Psi}\Psi$ . At the  $Sp(4)$  symmetric fixed point, the scaling dimensions of fermion bilinears are classified as the representation of  $Sp(4) \simeq SO(5)$  group:  $\bar{\Psi}\Psi$ ,  $\bar{\Psi}\Gamma_a\Psi$  and  $\bar{\Psi}\Gamma_{ab}\Psi$  form scalar, vector and adjoint representations of  $SO(5)$  group respectively, and the scaling dimensions of fermion bilinears within the same representation are equal to each other.

For larger  $N$ , the scaling dimensions of the fermion bilinears at the  $Sp(N)$  fixed point deviate from their value at the  $SU(N)$  fixed point at the order of  $1/N^2$ , and requires a lot more calculations. But their differences at  $1/N^2$  order can be calculated readily from diagrams in Fig. 4:

$$\Delta(\bar{\Psi}\Gamma_a\Psi) - \Delta(\bar{\Psi}\Gamma_{ab}\Psi) = \frac{6\lambda_3^*}{\pi^2 N}. \quad (11)$$

Here  $\Gamma_a$  and  $\Gamma_{ab}$  together form a fundamental representation of  $SU(N)$  algebra, and  $\Gamma_{ab}$  form the spinor representation of  $Sp(N)$  subalgebra.

To fully analyze the physical implications of this fixed point, we have to examine the fate of all perturbations which further reduce the global symmetry from  $Sp(4)$  down to those required by the  $SU(2)$  spin rotation symmetry and the square lattice space group. There are a large number of such additional perturbations, and analyzing them all would require an analysis of daunting complexity. We also need a procedure for generalizing such perturbations to general  $Sp(N)$  operators to enable a  $1/N$  expansion, and there is no unique and natural choice like the one we have used so far; the results will

depend upon the particular choices made for the invariant subgroups of  $\text{Sp}(N)$ . We will therefore not present such an analysis here. Additional perturbations which break Lorentz invariance are also possible; there were examined by Hermele *et al.* [7], and found to be irrelevant.

Should no relevant perturbations emerge at the  $\text{Sp}(4)$  fixed point, it would describe a stable critical spin liquid phase. Otherwise it would be a (multi-) critical point between ordered phases, with the dimensionality of the phase diagram determined by the number of relevant operators. An intriguing possibility is that there is only one relevant perturbation, which drives the system to a Néel or a VBS state on opposite sides of the  $\text{Sp}(4)$ -invariant critical point.

Such a  $\text{Sp}(4) \cong \text{SO}(5)$  fixed point separating Néel and VBS states was suggested by Tanaka and Hu [10]. They further proposed a  $\text{SO}(5)$  non-linear sigma model, with a Wess-Zumino term which could realize that a critical state. However, our  $\text{Sp}(4)$  critical point also has a  $\text{U}(1)$  gauge field, and an associated conserved topological current, and there is no analog of this conserved current in the Tanaka-Hu sigma model. So it is likely that our  $\text{Sp}(4)$  critical spin liquid is distinct from their proposal [15].

A large number of possible spin liquid ground states have been proposed for the square lattice antiferromagnet. All previous proposals have been associated with a mean-field saddle point of a theory of electrically neutral spinons which are either fermions or bosons. This paper has proposed a novel type of a spin liquid, which does not have a direct mean-field realization, but is induced by the gauge fluctuations about a mean-field saddle point. The only numerical evidence so far of a spin liquid state on the square lattice for  $\text{SU}(2)$  antiferromagnets is in the studies of the transition point between Néel and VBS states [12, 16, 17]. Our  $\text{SO}(5)$  spin liquid is a candidate for this state, as it can explain the possible equality of the scaling dimensions of the Néel and VBS operators. A further testable property of our spin liquid is that the 10

observable operators [7] defined by  $\bar{\Psi}\Gamma_{ab}\Psi$  all have equal scaling dimensions, which are distinct from those of the Néel and VBS orders.

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