Universal thermal and electrical transport near the superconductor-metal quantum phase transition in nanowires

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We describe the thermal (κ) and electrical (σ) conductivities of quasi-one dimensional wires, across a quantum phase transition from a superconductor to a metal induced by pairbreaking perturbations. Fluctuation corrections to BCS theory motivate a field theory for quantum criticality. We describe deviations in the Wiedemann-Franz ratio $\kappa/\sigma T$ (where T is the temperature) from the Lorenz number $(\pi^2/3)(k_B/e)^2$, which can act as sensitive tests of the theory. We also describe the crossovers out of the quantum critical region into the metallic and superconducting phases.

The Wiedemann-Franz law relates the low temperature (T) limit of the ratio $W \equiv \kappa/(\sigma T)$ of the thermal (κ) and electrical (σ) conductivities of metals to the universal Lorenz number $L_0 = (\pi^2/3)(k_B/e)^2$. This remarkable relationship is independent of the strength of the interactions between the electrons, relates macroscopic transport properties to fundamental constants of nature, and depends only upon the Fermi statistics and charge of the elementary quasiparticle excitations of the metal. It has been experimentally verified to high precision in a wide range of metals [1], and realizes a sensitive macroscopic test of the quantum statistics of the charge carriers.

It is interesting to note the value of the Wiedemann-Franz ratio in some other important strongly interacting quantum systems. In superconductors, which have low energy bosonic quasiparticle excitations, σ is infinite for a range of $T > 0$, while κ is finite in the presence of impurities [2], and so $W = 0$. At quantum phase transitions described by relativistic field theories, such as the superfluid-insulator transition in the Bose Hubbard model, the low energy excitations are strongly coupled and quasiparticles are not well defined; in such theories the conservation of the relativistic stress-energy tensor implies that κ is infinite, and so $W = \infty$ [3]. Li and Orignac [4] computed W in disordered Luttinger liquids, and found deviations from L_0, and found a non-zero universal value for W at the metal-insulator transition for spinless fermions.

The present paper will focus on the quantum phase transition between a superconductor and a metal (a SMT). We will consider quasi-one dimensional nanowires with a large number of transverse channels (so that the electronic localization length is much larger than the mean free path (ℓ)) which can model numerous recent experiments [5, 6, 7, 8, 9, 10, 11, 12, 13]. We will describe universal deviations in the value of W from L_0, which can serve as sensitive tests of the theory in future experiments.

The mean-field theory for the SMT goes back to the early work [14] of Abrikosov and Gorkov (AG): in one of the earliest discussions of a quantum phase transition, they showed that a large enough concentration of magnetic impurities could induce a SMT at $T = 0$. It has since been shown that such a theory applies in a large variety of situations with 'pair-breaking' perturbations: anisotropic superconductors with non-magnetic impurities [15], lower-dimensional superconductors with magnetic fields oriented in a direction parallel to the Cooper pair motion [16], and s-wave superconductors with inhomogeneity in the strength of the attractive BCS interaction [17]. Indeed, it is expected that pair-breaking is present in any experimentally realizable SMT at $T = 0$: in the nanowire experiments, explicit evidence for pair-breaking magnetic moments on the wire surface was presented recently by Rogachev *et al.* [13].

Fluctuations about the AG theory have been considered [16, 18, 19] in the metallic state, and lead to the well-known Aslamazov-Larkin (AL), Maki-Thomson (MT) and Density of States (DoS) corrections to the conductivity. At the SMT, field-theoretic analyses [20, 21] show that the AG theory, along with the AL, MT and DoS corrections, is inadequate in spatial dimension $d \leq 2$, and additional self-interactions among Cooper pairs have to be included. Here, d defines the dimensionality of the Cooper pair motion, while the metallic fermionic quasiparticles retain a three-dimensional character; therefore, the confining dimension, R, is larger than the inverse Fermi wavevector, but smaller than a superconducting coherence length or Cooper pair size, ξ. The behavior of W has been considered in this field-theoretic framework [21], and it was found that there were logarithmic corrections to the Lorenz number in $d = 2$. Here we will examine the $d = 1$ case in some detail: the transition is described by a strongly-coupled field theory of bosonic Cooper pairs, overdamped by their coupling to the fermionic quasiparticles. Remarkably, all important couplings between the bosons and the fermions scale to universal values, and consequently the Wiedemann-Franz ratio of this theory also approaches a universal constant which we compute in a $1/N$ expansion (the physical case

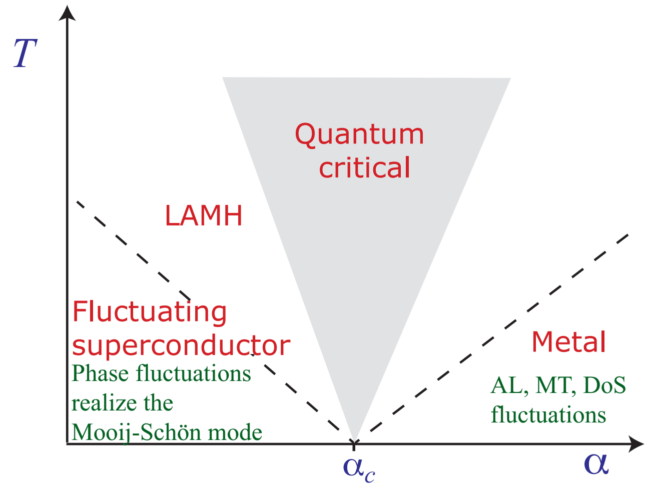

FIG. 1: (Color online) Crossover phase diagram of the superconductor-metal transition in a quasi-one dimensional superconductor. The "Metal" is described by the perturbative theory of Ref. 16. The "Quantum critical" region is described by $\mathcal{S}$ and realizes our result for W in Eq. (1). The Mooij-Schön mode is present everywhere, but couples strongly to superconducting fluctuations only in the "Fluctuating superconductor" regime, where it is described by Eq. (7); note that $\mathcal{S}$ does *not* apply here. The dashed lines are crossover boundaries which, by Eq. (4) occur at $T \sim |\alpha - \alpha_c|^{z\nu}$.

is $N = 1$)

$$W = \left(\frac{k_B}{e}\right)^2 \left(0.2820 + \frac{0.0376}{N}\right). \tag{1}$$

Our present computation of W ignores the influence of disorder on quantum criticality, and this may require the clean limit $\xi \ll \ell$ [22].

We also discuss the nature of the crossovers from this universal quantum critical physics to previously studied regimes at low T about the superconducting and metallic phases: these are summarized in Fig. 1. On the metallic side, there is a crossover to a low T regime described by the theory [16] of AL+MT+DoS corrections in $d = 1$. On the superconducting side, there is a regime of intermediate temperatures where the classical phase slip theory of Langer, Ambegaokar, McCumber, and Halperin (LAMH) applies [23, 24], and eventually another crossover at still lower temperatures to a phase fluctuating regime whose description requires a non-linear σ-model of fermion pair fluctuations coupled to the superconducting order [25]. Here the phase fluctuations are essentially equivalent to a plasma charge oscillation, which in $d = 1$ is the Mooij-Schön mode [26]. A number of works [27, 28, 29, 30] have examined the destruction of superconductivity due to quantum phase slips in such a phase fluctuating regime: we maintain that the phase with phase slip proliferation in such models is an *insulator* at $T = 0$, and so such theories describe a superconductor-insulator quantum transition, and may be appropriate in inhomogenous systems. Our theory includes amplitude and phase fluctuations on an equal footing, along with strong damping from the fermionic modes, and describes the transition into a metallic phase at $T = 0$: this is the case even though physics of the Mooij-Schön mode is present in the "fluctuating superconductor" regime in Fig. 1.

It is useful to place our results in the context of recent microscopic computations in BCS theory [16] on the metallic side, with the pairbreaking parameter α larger than critical α_c of the SMT. These results were obtain in the dirty limit ($\ell \ll \xi$), but we expect the same theory (the action $\mathcal{S}$ below) to apply in the quantum critical regime in the both the clean and dirty limits (although there are distinctions in the "fluctuating superconductor" regime of Fig. 1). For the conductivity, these results are [16]

$$\begin{aligned}\sigma &= \sigma_0 + \frac{e^2}{\hbar}\left(\frac{k_B T}{\hbar D}\right)^{-1/2}\left[\frac{\pi}{12\sqrt{2}}\left(\frac{k_B T}{\hbar(\alpha-\alpha_c)}\right)^{5/2}\right] \\ &\quad + \frac{e^2}{\hbar}\left(\frac{k_B T}{\hbar D}\right)\left[c\frac{\hbar(\alpha-\alpha_c)}{k_B T}\right]\end{aligned} \tag{2}$$

where σ_0 is a background metallic conductivity, c is a non-universal constant, D is the diffusion constant in the metal, and the remaining corrections from pairing fluctuations have been written in the form of a power of T times a factor within the square brackets which depends only upon the ratio $\hbar(\alpha - \alpha_c)/k_B T$. This way of writing the results allows us to deduce the importance of the fluctuations corrections, in the renormalization group sense, to the SMT. The first square bracket represents the usual AL correction, and has a prefactor of a negative power of T, and so is a relevant perturbation; this is so even though this correction vanishes as $T \to 0$. The second square bracket arises from the additional AL, MT and DoS corrections: the prefactor has no divergence as a power of T, and so this correction is formally irrelevant at the SMT. Note, however, the complete second term has a finite limit as $T \to 0$, and so becomes larger than the formally relevant AL term at sufficiently low T in the metal. We therefore identify the second term as *dangerously irrelevant* in critical phenomena parlance: *i.e.* important for the properties of the low T metallic region, but can be safely neglected in the shaded quantum critical region of Fig. 1.

Armed with the above insights, we focus on the fluctuations associated with the usual AL correction. These have [16] a Cooper pair propagator $(\widetilde{D}q^2 + |\omega| + \alpha)^{-1}$ at wavevector q and imaginary frequency ω in the metal in both the clean and dirty limits. This motivates the quantum critical theory of Ref. 20 for a field $\Psi(x, \tau)$ representing the local Cooper pair creation operator:

$$\mathcal{S} = \int dx \Bigg[d\tau \left(\widetilde{D}|\partial_x \Psi|^2 + \alpha|\Psi|^2 + \frac{u}{2}|\Psi|^4 \right) + \int \frac{d\omega}{2\pi} |\omega| |\Psi(x,\omega)|^2 \Bigg]. \quad (3)$$

This theory will apply to quasi-one dimensional wires for $R < (\hbar\widetilde{D}/k_B T)^{1/2}$. In the dirty limit, $\ell \ll \xi$, we have $\widetilde{D} = D$, but the value of $\widetilde{D}$ is different in the clean case. All couplings in $\mathcal{S}$ are random functions of position; in particular, randomness in α is expected to be relevant at the quantum critical point. We neglect this randomness in our quantitative results in the quantum critical region, and so they only apply above a $T = T_{\rm dis}$ which can be made arbitrarily small in the clean limit.

From $\mathcal{S}$, we obtain[20] the singular contribution to the conductivity in the vicinity of the quantum critical region

$$\sigma_{\rm sing} = \frac{e^2}{\hbar} \left(\frac{k_B T}{\hbar \widetilde{D}} \right)^{-1/z} \Phi_\sigma \left(\frac{\hbar(\alpha - \alpha_c)}{(k_B T)^{1/(z\nu)}} \right) \quad (4)$$

where z is the dynamic critical exponent, ν is a correlation length exponent, and Φ_σ is a universal scaling function. In a Gaussian approximation, the Kubo formula yields $z = 2$, $\nu = 1/2$, and $\Phi_\sigma(y) = (\pi/(12\sqrt{2}))y^{-5/2}$, and so this result is in precise correspondence with the first term in Eq. (2). In the limit $T \ll (\alpha - \alpha_c)^{z\nu}$, we have already seen that this term is subdominant to the dangerously irrelevant second term in Eq. (2). However, moving into the quantum critical region where $T \gg (\alpha - \alpha_c)^{z\nu}$, the contribution from Eq. (4) dominates all other terms, and we have $\sigma \sim T^{-1/z}\Phi_\sigma(0)$. The microscopic analysis of Ref. 16 obtained $\sigma \sim T^{-2}$, which is valid only at T large enough where u can be neglected. Going beyond the Gaussian theory, the values of z and ν have been determined in a $d = 2 - \epsilon$ expansion [20, 31], and also in quantum Monte Carlo simulations [32] with excellent agreement. Here, we have obtained these exponents in a theory with N complex fields Ψ directly in $d = 1$, and obtained to order $1/N$

$$z = 2 - \frac{0.131}{N} \quad : \quad \nu = 1 - \frac{0.389}{N}, \quad (5)$$

to be compared with the Monte Carlo estimates of $z = 1.97$ and $\nu = 0.689$ [32]. The value of $\Phi_\sigma(0)$ has a nonuniversal cutoff dependence associated with anomalous dimension $2 - z$. Note that the results above for σ in the metallic and quantum-critical regimes imply a nonmonotonic T dependence for $\alpha > \alpha_c$, possibly consistent with the observations of Ref. 11.

Similar reasoning can be applied to the thermal conductivity κ, which can be computed from $\mathcal{S}$ using a separate Kubo formula [33]. The scaling form analogous to Eq. (4) is

$$\kappa_{\rm sing} = \frac{k_B^2 T}{\hbar} \left(\frac{k_B T}{\hbar \widetilde{D}} \right)^{-1/z} \Phi_\kappa \left(\frac{\hbar(\alpha - \alpha_c)}{(k_B T)^{1/(z\nu)}} \right) \quad (6)$$

with Φ_κ another universal function. We have verified that the Gaussian prediction from $\mathcal{S}$ again agrees with the perturbative AL contribution of the microscopic theory at low T. Our main result for the Wiedemann-Franz ratio in Eq. (1) follows from $W = (k_B/e)^2 \Phi_\kappa(0)/\Phi_\sigma(0)$, as the nonuniversal prefactor cancels out in ratio of these scaling functions; the $1/N$ expansion was carried out by generalizing the methods of Ref. 34.

We now turn to an important conceptual issue: the role of charge conservation and associated normal modes. From hydrodynamic arguments we know that a one-dimensional metal or superconductor should support a gapless plasmon, or a Mooij-Schön normal mode [26]. In $d = 1$, this mode is gapless and disperses as $\omega \sim q \ln^{1/2}(1/(qR))$. In our theory for the quantum critical region, the Cooper pair field Ψ carries charge $2e$ but only exhibits diffusive dynamics with $\omega \sim \tilde{D}q^2$, and there is no Mooij-Schön mode in the dynamics of the action $\mathcal{S}$. The answer to this puzzle is contained in arguments made in Refs. 35 and 36 on the role of conservation laws in the critical fluctuations of quantum transitions in metallic systems for which the order parameter is overdamped (as is the case here). These early works considered the onset of spin-density wave order in a metal; in the quantum critical region, the spin excitations consisted of diffusive paramagnons whose dynamics did not conserve total spin. However, Ioffe and Millis [35] argued that the Ward identities associated with spin conservation only imposed significant constraints on the effective action at $\omega \gtrsim q$, and played little role in the $\omega \sim q^2$ regime important for the critical fluctuations. Essentially the same argument can be applied here: the Mooij-Schön mode is present only at relatively high frequencies $\omega \sim q$, and the critical theory $\mathcal{S}$ describes the overdamped Cooper pair modes in the distinct region of phase space with $\omega \sim q^2$. The Mooij-Schön fluctuations lead to oscillations in the local electrochemical potential, but these remain essentially decoupled from the critical modes described by $\mathcal{S}$ [35, 36] (see however, Eq. (8) below). It must be noted that the action $\mathcal{S}$ is *not* valid for $\omega \sim q$ and a complete description in terms of the underlying fermions is necessary to obtain the proper dynamics, which will contain the Mooij-Schön mode, as required.

Further insight into this issue is gained by lowering the temperature from the quantum critical regime into the "fluctuating superconductor" regime of Fig. 1 for $\alpha < \alpha_c$. When $k_B T < (\alpha_c - \alpha)$ the action $\mathcal{S}$ does *not* apply for the smallest wavevectors and frequencies. The reasons for this are again analogous to arguments made for the spin-density-wave ordering transition in metals, as discussed in Ref. 37. For the latter case, it was argued that with the emergence of long-range spin density wave order, the low energy fermionic particle-hole excitations at the ordering wavevector were gapped out, and so the diffusive paramagnon action applied only for energies larger than this gap. At energies smaller than the gap, spin-waves with dispersion $\omega \sim q$ emerge, as a consequence of Ward identities associated with spin conservation. In the superconducting case of interest here, there is no true long-range superconducting order at any $T > 0$, but the order is disrupted primarily by 'renormalized classical' thermal fluctuations of the phase, ϕ of the complex Ψ field. We can now assume that there is a local pairing amplitude in the fermion spectrum, analogous to the spin-density wave order. Charge conservation plays an important role in the effective action of ϕ fluctuations, and the 'spin-waves' in this case are, of course, just the Mooij-Schön excitations. The low energy effective action for ϕ cannot be obtained from $\mathcal{S}$; rather, we have to integrate the fermions out in the presence of a local pairing, as outlined above [37], and we expect an action

$$\begin{aligned}\mathcal{S}_\phi &= \int \frac{dq}{2\pi} \int \frac{d\omega}{2\pi} \frac{|A_\tau(q,\omega)|^2}{4\ln(1/(qR))} \\ &+ \int dx d\tau \Big[K_1 (\partial_\tau \phi - 2eA_\tau)^2 + K_2 (\partial_x \phi)^2 \Big] \end{aligned} \quad (7)$$

For the spin density wave case [36] it was found that $K_1 \sim (\alpha_c - \alpha)^{1/2}$ and $K_2 \sim (\alpha_c - \alpha)$, and we expect similar behavior here. We have explicitly included the action of the scalar potential iA_τ which mediates the Coulomb interaction in one dimension. The normal modes of $\mathcal{S}_\phi$ are the Mooij-Schön oscillations which are now identified with the fluctuations of the superconducting order. This should be contrasted from the situation in the shaded quantum critical region of Fig. 1, where the Mooij-Schön oscillations were decoupled from the critical modes described by $\mathcal{S}$.

Actually, there is a weak coupling between the critical modes of Ψ in $\mathcal{S}$ and the Mooij-Schön mode due to particle-hole asymmetry (there is no analog of this phenomenon for the spin density wave case). In general, the critical Ψ action contains the perturbation [20]

$$\mathcal{S}_1 = \int dx d\tau \left[\gamma \Psi^* \left(\frac{\partial}{\partial \tau} - 2eiA_\tau \right) \Psi \right] \quad (8)$$

where γ is proportional to the energy derivative of the density of states, and hence small. Indeed, the same ratio of the pairing to Fermi energy, which justified the present quasi-one dimensional treatment of Cooper pairs but not electrons, also causes γ to be small. We examined the renormalization group fate of the perturbation $\mathcal{S}_1$ at the fixed point of $\mathcal{S}$: we computed the two-loop flow of γ in the $d = (2 - \epsilon)$ expansion and found

$$\frac{d\gamma}{d\ell} = \frac{\epsilon^2}{100} \left(\pi^2 - 8 \right) \gamma. \quad (9)$$

So γ is relevant, but the scaling dimension is extremely small. Along with small bare value of γ, such particle-hole asymmetric effects can justifiably be ignored in experimental applications.

The main experimentally testable results of this paper are: the crossover phase diagram in Fig. 1, the Wiedemann-Franz ratio in Eq. (1) which applies in the shaded region, and the non-monotonic T dependence of σ for $\alpha > \alpha_c$.

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