

# Dual vortex theory of doped Mott insulators

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## Abstract

We present a general framework for describing the quantum phases obtained by doping paramagnetic Mott insulators on the square lattice. The undoped insulators are efficiently characterized by the projective transformations of various fields under the square lattice space group (the PSG). We show that the PSG also imposes powerful constraints on the doped system, and on the effective action for the vortex and Bogoliubov quasiparticle excitations of superconducting states. This action can also be extended across transitions to supersolid or insulating states at nonzero doping. For the case of a valence bond solid (VBS) insulator, we show that the doped system has the same PSG as that of elementary bosons with density equal to the density of electron Cooper pairs. We also discuss aspects of the action for a  $d$ -wave superconductor obtained by doping a “staggered-flux” spin liquid state.

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## 1 Introduction

It is now widely accepted that the cuprate superconductors can be understood as doped Mott insulators. Anderson [1] made the early suggestion that appropriate reference Mott insulating ground state should be paramagnetic *i.e.* preserve spin rotation invariance with the electron spin operators obeying  $\langle \mathbf{S}_j \rangle = 0$  on all sites,  $j$ , of the square lattice; these are also loosely referred to as the resonating valence bond (RVB) states of Pauling [2], or as spin liquids. A more subtle and complete understanding of spin liquids on the square lattice has emerged since then. It has been found that some of the physically interesting spin liquids are unstable at low energies to confinement and the emergence of new competing order parameters. The best known examples are  $U(1)$  RVB states with a gap to both spin and charge excitations, which are unstable by general arguments. The instability of the simplest such state leads to the prominent example of competing valence bond solid (VBS) order [3], which will play an important role in the considerations of this paper. Other spin liquids can potentially be stable against symmetry breaking, [4,5] but instead have gapless gauge and fermionic ‘spinon’ excitations which are strongly

coupled together in an ‘algebraic spin liquid’: we will consider the example of the ‘staggered flux’ (sF) state in this paper. The sF state also has strongly enhanced fluctuations of the VBS and other orders.

Many authors have described how doping a paramagnetic Mott insulator with charged carriers leads to  $d$ -wave superconductivity. [6,7,8,9,10] This leads to a number of natural questions on the role of competing orders, such as VBS or charge density wave orders, in the doped system. (i) Can the competing orders have long-range order in the doped system, and how does the order parameter at finite doping relate to the VBS order in the undoped insulator? (ii) Does the long-range order co-exist with superconductivity, or can it appear only in finite doping insulating states? (iii) What are the theories of the quantum critical points between such states, and can they be of the Landau-forbidden[11,12,13] variety? (iv) In the moderate doping  $d$ -wave superconductor, where this is no competing long-range order, what are the correlations of the competing order, and what is their interplay with vortices and quasiparticles? This paper will develop a formalism designed to address such questions. We will do this in as general a setting as possible, relying mainly on symmetry arguments. We will not use any specific microscopic model of the doped antiferromagnet, and so will not make any quantitative predictions of the phase diagram.

Our analysis will focus on two distinct classes of paramagnetic, undoped Mott insulators. In both classes, the Mott insulator is described by the dynamics of a compact U(1) gauge field  $\mathcal{A}_\mu$  ( $\mu = x, y, \tau$  is a spacetime index). The ‘photon’ excitation of this gauge field describes  $S = 0$  excitations above a spin singlet RVB ground state. Over at least a significant intermediate energy range, this gauge field can be described by a conventional, Maxwell action of the form

$$\mathcal{S}_A = -K \sum_{\square} \cos(\epsilon_{\mu\nu\lambda} \Delta_\nu \mathcal{A}_{j\lambda}). \quad (1)$$

We have discretized spacetime onto the sites of a cubic lattice,  $j$ , with each equal time section mapping onto the sites of the underlying physical square lattice (it should be clear from the context whether  $j$  refers to the square or cubic lattice),  $\square$  represents a sum over elementary plaquettes of the cubic lattice,  $\Delta_\mu$  is a lattice derivative, the indices  $\mu, \nu, \lambda$  extend over  $x, y, \tau$ , and  $K$  is a coupling constant. In addition to the pure gauge fluctuations in Eq. (1), the theory of the undoped Mott insulator must also account for the ‘matter’ fields: the matter associated with the average density of exactly one electron per site. For our purposes, it turns out that the most important effect [15,3] of the matter in the undoped insulator is on its consequences for the *monopoles* in  $\mathcal{A}_\mu$  (the monopoles are discrete tunnelling events in which the net U(1) gauge flux is changed by  $2\pi$ ). The matter causes each monopole to transform non-trivially under the space group symmetries of the square lattice, and we dub these transformations (following Wen, [14] in a slight abuse of mathematical terminology) as the PSG (for projective symmetry group) of

the monopoles.[12,16,17] The monopole PSG specifies the competing order parameters which emerge from the RVB state at low energies.

We turn next to the doped antiferromagnet, where the ground state can be a superconductor. Here, one class of excitations of the superconductor are the vortices in Cooper pair condensate, each carrying electromagnetic flux of  $hc/(2e)$ . It was shown in recent work [16,17] that a general description of competing orders near to a superconducting state is obtained by analyzing the PSG of the vortices. The PSG ties each vortex to fluctuations in competing orders associated with generalized ‘density-wave’ and ‘vorticity-wave’ modulations, and a number of observable consequences of this connection were discussed.

So how do we describe the evolution of the competing orders in the undoped Mott insulator to those in the superconductor ? We will argue in this paper that a very powerful and general method of doing this is provided by the PSG. Quite simply, we combine the PSG of the monopoles of the undoped Mott insulator with the PSG of the vortices in the doped system, and obtain a combined action which is invariant under both PSGs. In principle, this effective action applies in both the undoped insulator and the highly doped superconductors, and at all doping concentrations in between. It is designed to address the questions posed in the second paragraph.

From a different perspective, the approach of this paper is an expansion about a “Mott quantum critical point”, describing a system on the verge of superconductivity due to charge condensation in a non-trivial paramagnetic Mott insulator. The crucial assumption is the validity of this expansion, i.e the closeness in phase space of the physical system to the quantum critical point. The degree of stability of the non-trivial Mott insulator – the “reference state” – is a secondary issue. Indeed, in the first case we consider, the non-trivial Mott insulator is the  $U(1)$  RVB state with fully gapped spin and charge excitations, which is generically unstable to VBS order at low energies. We therefore denote this RVB state as a VBS insulator, though most of our considerations never require the actual occurrence of long-range VBS order of any particular type. Nevertheless, a number of recent works have shown that such a state can be a good starting point for describing the quantum critical regime in which VBS order is weak. The different possible paramagnetic Mott states do, however, generally lead to physically different predictions. A proper choice amongst such states must be made on energetic or phenomenological (empirical) grounds.

The following two subsections briefly introduce the two classes of paramagnetic Mott insulators we consider. For the VBS insulator, introduced in Section 1.1, we will be able to carry our derivation of the action to completion, using the PSG of the monopoles described in Ref. [12]. The sF state will be considered in

Section 1.2: the monopole PSG of this state has not yet been computed and so our analysis of this case will remain incomplete. Nevertheless, we will be able to make substantial progress in describing the structure of the action in this paper, and note significant differences from the VBS case. The final subsection of this introduction will briefly outline the strategy for extending our analysis to finite doping, and discuss the issue of flux quantization.

### 1.1 The Valence Bond Solid Insulator

Our first quantum paramagnet is obtained in a theory of quantum fluctuations about an antiferromagnetically ordered Néel state. This theory focusses on the Néel order parameter as the primary dynamic degree of freedom, and obtains VBS order in the state where quantum fluctuations have “disordered” the Néel order. [3,18]

The Néel state is described by the Néel order parameter,  $\mathbf{n}_j$ , which is a unit vector in spin space, related to the electron spin operator by

$$\mathbf{S}_j = S\eta_j\mathbf{n}_j, \quad (2)$$

where  $S = 1/2$  is the spin, and  $\eta_j = \pm 1$  takes opposite signs on the two sublattices. The action for quantum spin fluctuations of the  $\mathbf{n}$  has a familiar description in terms of the O(3) non-linear  $\sigma$  model with Berry phases:

$$\mathcal{S}_{\mathbf{n}} = \frac{1}{2g} \sum_j (\Delta_\mu \mathbf{n})^2 + iS \sum_j \eta_j \mathbf{A}_D(\mathbf{n}_j) \cdot \Delta_\tau \mathbf{n}_j, \quad (3)$$

where  $\mathbf{A}_D(\mathbf{n}) = \mathbf{A}_D(-\mathbf{n})$  is the Dirac monopole function which is used to yield a Berry phase proportional to the area enclosed by the worldline of each spin. The coupling constant  $g$  can be tune to ‘disorder’ the Néel state: for small  $g$ , the ground state has Néel order with  $\langle \mathbf{n} \rangle \neq 0$ , while for large  $g$ , there are strong fluctuations of  $\mathbf{n}$  leading to a RVB state with  $\langle \mathbf{n} \rangle = 0$ . However, this “disordered” state nevertheless has non-trivial quantum dynamics which is expressed most easily in terms of the gauge field  $\mathcal{A}_\mu$  which is defined here by

$$\mathcal{A}_\mu \equiv \frac{1}{2} \mathbf{A}_D(\mathbf{n}) \cdot \Delta_\mu \mathbf{n}. \quad (4)$$

As has been argued elsewhere, [18] for large  $g$ , the effective action for  $\mathcal{A}_\mu$  in this RVB state is  $\mathcal{S}_{\mathcal{A}} + \mathcal{S}_B$ , where  $\mathcal{S}_{\mathcal{A}}$  was in Eq. (1) and the Berry phase term

$$\mathcal{S}_B = i2S \sum_j \mathcal{A}_{j\tau} \quad (5)$$

descends directly from the second term in Eq. (3). The properties of  $\mathcal{S}_{\mathcal{A}} + \mathcal{S}_B$  have been described in detail elsewhere,[3,19] and will be reviewed here in

Section 2.1. Briefly, the Berry phase endow the monopoles in  $\mathcal{A}_\mu$  with non-trivial transformations under the PSG, so that they transform like a VBS order parameter.[3,12] Further, at low energies, the monopoles “condense” leading to VBS order in the ground state. The full details of the PSG will be presented in Section 2, but here we note the transformation of  $\mathcal{A}_\mu$  under the time-reversal operation,  $\mathcal{T}$ . This can be deduced from Eqs. (3) and (4): we have  $\mathbf{n} \rightarrow -\mathbf{n}$ , and hence

$$\mathcal{T} : \mathcal{A}_\tau \rightarrow \mathcal{A}_\tau ; \mathcal{A}_x \rightarrow -\mathcal{A}_x ; \mathcal{A}_y \rightarrow -\mathcal{A}_y. \quad (6)$$

We will observe shortly that  $\mathcal{A}_\mu$  has a distinct transformation under  $\mathcal{T}$  for our second quantum paramagnet: this distinction plays an important role in the two theories of vortex dynamics in the  $d$ -wave superconductor.

### 1.2 The Staggered Flux spin liquid

This RVB state, which we denote by the sF spin liquid, has seen much recent discussion in the literature,[4,5,9,20] and its basic properties will be reviewed in Section 3. Here, we note that the low energy (fixed point) properties of the sF spin liquid are described by  $\mathcal{S}_\mathcal{A} + \mathcal{S}_\Psi$ , where  $\mathcal{S}_\Psi$  describes the coupling of  $\mathcal{A}_\mu$  to 4 two-component, fermionic, massless Dirac fields,  $\Psi$ ; schematically, the action is  $\mathcal{S}_\Psi = \int d^2r d\tau \mathcal{L}_\Psi[\mathcal{A}_\mu]$ , where

$$\mathcal{L}_\Psi[\mathcal{A}_\mu] = -i\bar{\Psi}\gamma_\mu (\partial_\mu + i\mathcal{A}_\mu) \Psi, \quad (7)$$

where  $\gamma_\mu$  are the Dirac matrices. The PSG properties of the sF spin liquid are described in Section 3.1. Under  $\mathcal{T}$ , the gauge field  $\mathcal{A}_\mu$  transforms, as expected from the relativistic structure of  $\mathcal{S}_\Psi$ , as a conventional Lorentz vector

$$\mathcal{T} : \mathcal{A}_\tau \rightarrow -\mathcal{A}_\tau ; \mathcal{A}_x \rightarrow \mathcal{A}_x ; \mathcal{A}_y \rightarrow \mathcal{A}_y. \quad (8)$$

Note the distinction of this from Eq. (6) for the VBS state. Monopoles also play an important role in dynamics of the sF spin liquid. [21,22,14,4] Computing their PSG remains an important open problem, and so we will perform our analysis to the extent possible, pending the eventual determination of the monopole PSG.

### 1.3 Doping the antiferromagnet

A detailed description of the theory obtained by doping the two paramagnetic Mott insulators described above appears in Sections 2 and 3 respectively. Here, we note some important common features of the two theories.

In both cases, we represent the charge carriers by *two* species [9,23,17] of spinless bosons<sup>1</sup> ('holons'),  $b_1$  and  $b_2$ . If the total hole density is  $\delta$ , then each species of boson has density  $\delta/2$ , and we will assume throughout that

$$\frac{\delta}{2} = \frac{p}{q} \quad (9)$$

where  $p$  and  $q$  are relatively prime integers. As we will see in the body of the paper, the number-theoretic properties of the integer  $q$  will play a crucial role in our analysis.

Both  $b_1$  and  $b_2$  carry physical electromagnetic charge  $e$ . However, they carry opposite charges  $\pm 1$  under the internal compact U(1) gauge field  $\mathcal{A}_\mu$ . The superconducting state is obtained when the charge  $2e$ , gauge neutral, combination  $b_1 b_2$  condenses. However, with charge  $e$  bosons present in the theory, one may wonder if there are regions of stability of flux  $hc/e$  vortices. As we review below, quite generally all stable vortices have flux  $hc/(2e)$ .

Far enough from the vortex cores, we may focus on the phases of the boson fields alone. Let us therefore write  $b_1 \sim e^{-i\theta_1}$  and  $b_2 \sim e^{-i\theta_2}$ . We denote a vortex in which the phase of  $\theta_1$  winds by  $2\pi$  by  $\psi_1$ , and a vortex in which the phase of  $\theta_2$  winds by  $2\pi$  by  $\psi_2$ . A complete dual theory of the  $\psi_{1,2}$  vortices appears in subsequent sections. Here, we continue with an analysis in the direct picture. Far from the vortex core, the free energy contains a contribution

$$F = \frac{\rho_s}{2} \int d^2r \left[ (\vec{\nabla}\theta_1 - \vec{\mathcal{A}})^2 + (\vec{\nabla}\theta_2 + \vec{\mathcal{A}})^2 \right]. \quad (10)$$

We have not included the electromagnetic magnetic field here, as its influence is important only at distances of order the London penetration depth. For a  $\psi_1$  vortex,  $\vec{\nabla}\theta_1 = \vec{e}_\theta/r$ , and  $\vec{\nabla}\theta_2 = 0$  at large  $r$  ( $\vec{e}_\theta$  is a unit vector orthogonal to the radial direction). The slow  $1/r$  decay implies a logarithmically divergent vortex energy in Eq. (10) which is cut off at the London penetration depth. Let us assume that  $\vec{\mathcal{A}} = \alpha \vec{e}_\theta/r$  at large  $r$  (which is pure gauge). Then the coefficient of the logarithmically divergent term is proportional to  $(1 - \alpha)^2 + \alpha^2$ . Minimizing w.r.t. to  $\alpha$ , we obtain  $\alpha = 1/2$ . Thus a  $\psi_1$  vortex carries a total  $\mathcal{A}_\mu$  gauge flux  $= \oint \mathcal{A}_\mu dr_\mu = \pi$ . Similarly, a  $\psi_2$  vortex carries  $\mathcal{A}_\mu$  gauge flux of  $-\pi$ . In both cases, at distances of order the London penetration depth, there must be a residual gauge flux  $\pi$  (in units of  $\hbar c/e$ ) in the physical electromagnetic gauge field to ultimately render the energy of a vortex finite. So both the  $\psi_1$  and  $\psi_2$  vortices carry flux  $hc/(2e)$ .

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<sup>1</sup> A complementary theory with spinless fermionic holons, and bosonic spinons, is discussed in Ref. [24]. Such a theory is natural when the undoped VBS state is proximate to a deconfined critical point [11,12,13] to a magnetically ordered Néel state. The approach in the present paper to doping the VBS state with bosonic holons is more closely connected to the large  $N$  limit studied in Ref. [10].

The above  $\mathcal{A}_\mu$  flux assignments also allow us to identify the connection between the vortices and the monopoles. A monopole is a tunneling event in which the  $\mathcal{A}_\mu$  flux changes by  $2\pi$ . Such a change in flux also occurs when a  $\psi_1$  vortex transforms into a  $\psi_2$  vortex. At low energies we expect the two tunneling events to always co-occur: this is especially so in the superconducting state, where the  $\mathcal{A}_\mu$  gauge flux is confined by the  $b_{1,2}$  condensate. So the operator for a monopole tunneling event is  $\psi_2^\dagger\psi_1$ .

The following Section 2 will present our theory for doping the VBS state, and the analogous theory for the sF state appears in Section 3.

## 2 Doping the Valence Bond Solid

An explicit microscopic theory of the doping of the VBS state described in Section 1.1 has already appeared in Ref. [17]. There we realized the VBS state by a quantum dimer model, and then carried out a detailed duality transformation on the doped quantum dimer model. Here, we will show how the same results can be obtained by more abstract, but also more general, symmetry arguments based upon the PSG. The present analysis makes it clear that the results of Ref. [17] are applicable to a far more general class of models, and also allows us to obtain new results on the nature of superfluid-insulator transitions.

### 2.1 Undoped insulator

The theory of the insulator was outlined in Section 1.1, and has been discussed in much detail elsewhere. Here, we focus exclusively on the crucial PSG properties. First, we complete the mappings in Eq. (6) to the full set of symmetry operations for the underlying antiferromagnet. We will consider the following operations here, and in the remainder of the paper

$$\begin{aligned}
T_x &: \text{Translation along the } x \text{ axis by one lattice site.} \\
T_y &: \text{Translation along the } y \text{ axis by one lattice site.} \\
R_{\pi/2}^{\text{dual}} &: \text{Rotation by } 90^\circ \text{ about a dual lattice site;} \\
&\quad \text{with the origin on a dual lattice site,} \\
&\quad x \rightarrow y \text{ and } y \rightarrow -x. \\
I_x^{\text{dual}} &: \text{Reflection about the } y \text{ axis of the dual lattice;} \\
&\quad \text{with the origin on a dual lattice site,} \\
&\quad x \rightarrow -x \text{ and } y \rightarrow y. \\
\mathcal{T} &: \text{Time reversal.}
\end{aligned} \tag{11}$$

The mappings of  $\mathcal{A}_\mu$  are easily determined from Eqs. (3) and (4), and we obtain

$$\begin{aligned}
T_x &: \mathcal{A}_\mu \rightarrow -\mathcal{A}_\mu \\
T_y &: \mathcal{A}_\mu \rightarrow -\mathcal{A}_\mu \\
R_{\pi/2}^{\text{dual}} &: \mathcal{A}_\tau \rightarrow -\mathcal{A}_\tau ; \mathcal{A}_x \rightarrow -\mathcal{A}_y ; \mathcal{A}_y \rightarrow \mathcal{A}_x \\
I_x^{\text{dual}} &: \mathcal{A}_\tau \rightarrow -\mathcal{A}_\tau ; \mathcal{A}_x \rightarrow \mathcal{A}_x ; \mathcal{A}_y \rightarrow -\mathcal{A}_y \\
\mathcal{T} &: \mathcal{A}_\tau \rightarrow \mathcal{A}_\tau ; \mathcal{A}_x \rightarrow -\mathcal{A}_x ; \mathcal{A}_y \rightarrow -\mathcal{A}_y.
\end{aligned} \tag{12}$$

We have not explicitly written out the transformations of the spacetime coordinates of the fields above, because they are evident in Eq. (11).

The PSG of the monopoles in  $\mathcal{A}_\mu$  requires careful consideration of the Berry phases in Eq. (5). As shown by Haldane, [15] each monopole acquires a net phase factor, which (for  $S = 1/2$ ) then leads to the following PSG transformations [12] for the monopole annihilation operator  $m$

$$\begin{aligned}
T_x &: m \rightarrow im^\dagger \\
T_y &: m \rightarrow -im^\dagger \\
R_{\pi/2}^{\text{dual}} &: m \rightarrow m^\dagger \\
I_x^{\text{dual}} &: m \rightarrow m \\
\mathcal{T} &: m \rightarrow m.
\end{aligned} \tag{13}$$

As has been discussed elsewhere, [11,12]  $e^{i\pi/4}m$  has the same transformation properties as the VBS order parameter. The condensation of  $m$  in the compact U(1) gauge theory for the VBS phase then implies the appearance of VBS order in the ground state.

## 2.2 Doped antiferromagnet

We will consider the charged excitations of the doped antiferromagnet in this subsection: the spin excitations will be considered later in Section 2.5.

As noted in Section 1.3, we represent the charge carriers in the doped antiferromagnet by  $S = 0$ , charge  $e$  holon degrees of freedom  $b_1 \sim e^{-i\theta_1}$  and  $b_2 \sim e^{-i\theta_2}$ . The boson  $b_1$  carries charge  $+1$  under  $\mathcal{A}_\mu$ , while the boson  $b_2$  carries charge  $-1$  under  $\mathcal{A}_\mu$ . The total density of holes is  $\delta$ , and the density of each species of boson is  $\delta/2$ .

We can now obtain an effective theory for these bosons, constrained by their transformations under the PSG. Interestingly, there is little arbitrariness in this PSG: it is almost entirely determined by the PSG of  $\mathcal{A}_\mu$  in Eq. (12), the  $\mathcal{A}_\mu$  charge assignments of the bosons above, and the requirements of gauge invariance. We will actually not need the explicit form of the boson action; just the PSG will suffice for our subsequent duality mapping to the vortices. A simple analysis shows that the PSG of the bosons is

$$\begin{aligned}
 T_x &: b_1 \rightarrow b_2 ; b_2 \rightarrow b_1 \\
 T_y &: b_1 \rightarrow b_2 ; b_2 \rightarrow b_1 \\
 R_{\pi/2}^{\text{dual}} &: b_1 \rightarrow ib_2 ; b_2 \rightarrow ib_1 \\
 I_x^{\text{dual}} &: b_1 \rightarrow b_2 ; b_2 \rightarrow b_1 \\
 \mathcal{T} &: b_1 \rightarrow b_1^\dagger ; b_2 \rightarrow b_2^\dagger.
 \end{aligned} \tag{14}$$

The factors of  $i$  in the transformations under  $R_{\pi/2}^{\text{dual}}$  are not determined by the requirements of gauge invariance. Instead, as discussed in some detail in Ref. [17], these phase factors depend upon microscopic details, and the nature of the short-range pairing in the superconducting ground states. The phase factors displayed above are those appropriate to  $d$ -wave pairing. In any case, such phase factors do not play any role in the duality to the vortex degrees of freedom.

We proceed to apply the boson-vortex duality separately to both species of bosons  $b_{1,2}$ . Then boson  $b_1$  dualizes to a vortex  $\psi_1 \sim e^{-i\phi_1}$  and a non-compact U(1) gauge field  $A_{1\mu}$ , while  $b_2$  dualizes to a vortex  $\psi_2 \sim e^{-i\phi_2}$  and a non-compact U(1) gauge field  $A_{2\mu}$ . Again, it is not necessary to write down the explicit form of the dual action as the PSG will suffice in determining the low energy continuum limit below. We can obtain the PSG of the dual fields by the requirements of gauge invariance, the facts that the boson currents  $\Delta_\mu\theta_1 - \mathcal{A}_\mu$  and  $\Delta_\mu\theta_2 + \mathcal{A}_\mu$  must transform like the dual fluxes  $\epsilon_{\mu\nu\lambda}\Delta_\nu A_{1\lambda}$  and  $\epsilon_{\mu\nu\lambda}\Delta_\nu A_{2\lambda}$  respectively, and conversely the dual vortex currents  $\Delta_\mu\phi_1 - A_{1\mu}$  and  $-\Delta_\mu\phi_2 + A_{2\mu}$  must transform like the gauge flux  $\epsilon_{\mu\nu\lambda}\Delta_\nu \mathcal{A}_\lambda$ . From these

requirements, it is not difficult to determine the PSG of the dual vortex matter fields  $\psi_{1,2}$

$$\begin{aligned}
T_x &: \psi_1 \rightarrow \psi_2 ; \psi_2 \rightarrow \psi_1 \\
T_y &: \psi_1 \rightarrow \psi_2 ; \psi_2 \rightarrow \psi_1 \\
R_{\pi/2}^{\text{dual}} &: \psi_1 \rightarrow \psi_2 ; \psi_2 \rightarrow \psi_1 \\
I_x^{\text{dual}} &: \psi_1 \rightarrow \psi_2^\dagger ; \psi_2 \rightarrow \psi_1^\dagger \\
\mathcal{T} &: \psi_1 \rightarrow \psi_1 ; \psi_2 \rightarrow \psi_2,
\end{aligned} \tag{15}$$

and of the dual U(1) gauge fields  $A_{1,2}$

$$\begin{aligned}
T_x &: A_{1\mu} \rightarrow A_{2\mu} ; A_{2\mu} \rightarrow A_{1\mu} \\
T_x &: A_{1\mu} \rightarrow A_{2\mu} ; A_{2\mu} \rightarrow A_{1\mu} \\
R_{\pi/2}^{\text{dual}} &: A_{1\tau} \rightarrow A_{2\tau} ; A_{1x} \rightarrow A_{2y} ; A_{1y} \rightarrow -A_{2x} ; \\
&A_{2\tau} \rightarrow A_{1\tau} ; A_{2x} \rightarrow A_{1y} ; A_{2y} \rightarrow -A_{1x} \\
I_x^{\text{dual}} &: A_{1\tau} \rightarrow -A_{2\tau} ; A_{1x} \rightarrow A_{2x} ; A_{1y} \rightarrow -A_{2y} ; \\
&A_{2\tau} \rightarrow -A_{1\tau} ; A_{2x} \rightarrow A_{1x} ; A_{2y} \rightarrow -A_{1y} \\
\mathcal{T} &: A_{1\tau} \rightarrow -A_{1\tau} ; A_{1x} \rightarrow A_{1x} ; A_{1y} \rightarrow A_{1y} ; \\
&A_{2\tau} \rightarrow -A_{2\tau} ; A_{2x} \rightarrow A_{2x} ; A_{2y} \rightarrow A_{2y}.
\end{aligned} \tag{16}$$

The PSG transformations in Eqs. (13), (15), and (16) contain almost all the information needed to obtain the vortex/monopole theory of the doped antiferromagnet. However, there is an crucial ingredient that has been missing from our discussion so far: the influence of the background density of  $\delta/2$  of each species of boson. In the dual vortex picture, this emerges as an average background flux of both  $A_{1,2}$  of  $2\pi(\delta/2)$  per plaquette. The consequences of such a background flux on the vortex PSG were studied in some detail in Ref. [16], and we can simply transfer those results here separately to  $\psi_1$  and  $\psi_2$ . At a rational hole density in Eq. (9), the  $\psi_{1,2}$  vortices each generate  $q$  degenerate low energy fields. Following Ref. [16], we denote these fields as  $\varphi_{1\ell}$  and  $\varphi_{2\ell}$  where  $\ell = 0, 1 \dots q$ . So at the moment there are a total of  $2q$  vortex fields that constitute the degrees of freedom of our low energy theory. The PSG of the these fields follow from Eq. (15) and Ref. [16]

$$\begin{aligned}
T_x &: \varphi_{1\ell} \rightarrow \varphi_{2,\ell+1}; \varphi_{2\ell} \rightarrow \varphi_{1,\ell+1} \\
T_y &: \varphi_{1\ell} \rightarrow \varphi_{2\ell}\omega^{-\ell}; \varphi_{2\ell} \rightarrow \varphi_{1\ell}\omega^{-\ell} \\
R_{\pi/2}^{\text{dual}} &: \varphi_{1\ell} \rightarrow \frac{1}{\sqrt{q}} \sum_{m=0}^{q-1} \varphi_{2m}\omega^{-\ell m}; \\
&\varphi_{2\ell} \rightarrow \frac{1}{\sqrt{q}} \sum_{m=0}^{q-1} \varphi_{1m}\omega^{-\ell m} \\
I_x^{\text{dual}} &: \varphi_{1\ell} \rightarrow \varphi_{2\ell}^\dagger; \varphi_{2\ell} \rightarrow \varphi_{1\ell}^\dagger \\
\mathcal{T} &: \varphi_{1\ell} \rightarrow \varphi_{1\ell}; \varphi_{2\ell} \rightarrow \varphi_{2\ell},
\end{aligned} \tag{17}$$

where the  $\ell, m$  indices are all implicitly determined modulo  $q$ , and

$$\omega \equiv e^{2\pi i p/q}. \tag{18}$$

In the gauge field sector, the low energy theory is most easily expressed in terms of the uniform and staggered dual gauge fields defined by

$$\begin{aligned}
A_{1\mu} &= B_{s\mu} + B_{a\mu} \\
A_{2\mu} &= B_{s\mu} - B_{a\mu}.
\end{aligned} \tag{19}$$

We are now ready to present the promised unification of the vortex and monopole PSGs. Indeed, all results follow from the PSGs in Eqs. (13) and (17), coupled with the requirements of gauge invariance. Before proceeding, let us briefly discuss the latter requirements. The  $\varphi_{1\ell}$  fields have charges of +1 and +1 under  $B_{s\mu}$  and  $B_{a\mu}$  respectively, while the  $\varphi_{2\ell}$  fields have charges of +1 and -1. The monopole changes the  $\mathcal{A}_\mu$  gauge flux by  $2\pi$ , and we argued earlier in Section 1.3 this implied that a monopole transformed like  $\psi_2^\dagger\psi_1$ ; in other words, the monopole operator  $m^\dagger$  has charges of 0 and +2 under  $B_{s\mu}$  and  $B_{a\mu}$  respectively. These gauge charges are summarized in Table 1.

### 2.3 Low energy theory

We are now ready to begin presentation of the effective action. This has the structure

$$\mathcal{S} = \int d^2x d\tau \left[ \mathcal{L}_f[\mathcal{A}_\mu] + \mathcal{L}_{U(1)} + \mathcal{L}_{2\varphi} + \mathcal{L}_{4\varphi} + \mathcal{L}_m \right] \tag{20}$$

The first term,  $\mathcal{L}_f[\mathcal{A}_\mu]$ , is the action for the  $S = 1/2$  fermionic spinons; we defer consideration of this term to Section 2.5.

The second term,  $\mathcal{L}_{U(1)}$  is the action for the U(1) gauge fields,  $\mathcal{A}_\mu$ ,  $B_{s\mu}$ , and  $B_{a\mu}$ . Apart from the conventional Maxwell terms, there is also a Chern-Simons

	Vortex	Vortex	Monopole	Holon	Holon	Spinon
	$\varphi_1$	$\varphi_2$	$m$	$b_1$	$b_2$	$f_s$
Dual Gauge $B_{s\mu}$	1	1	0	—	—	—
Dual Gauge $B_{a\mu}$	1	-1	2	—	—	—
Direct Gauge $\mathcal{A}_\mu$	—	—	—	1	-1	$\eta_j$ (VBS) and 1 (sF)

Table 1

Assignments of charges under the various gauge field. The direct gauge field  $\mathcal{A}_\mu$  is related to spin singlet fluctuations in the insulating RVB state, as discussed in Section 1. The flux of the dual gauge field  $B_{s\mu}$  measures the electrical supercurrent. The dual gauge field  $B_{a\mu}$  is the Chern-Simons dual of  $\mathcal{A}_\mu$  *i.e.* the flux of  $B_{a\mu}$  measures the current associated with  $\mathcal{A}_\mu$ , and vice versa.

term which ensures that the  $\mathcal{A}_\mu$  flux is equal to the vortex current associated with  $A_{1\mu} - A_{2\mu}$ : this is demanded by the discussion in Section 1.3 which showed that there is  $\pm\pi$   $\mathcal{A}_\mu$  flux associated with the  $\psi_{1,2}$  vortices.

$$\begin{aligned}
\mathcal{L}_{U(1)} = & \frac{K}{2} (\epsilon_{\mu\nu\lambda} \partial_\nu \mathcal{A}_\lambda)^2 + \frac{K_s}{2} (\epsilon_{\mu\nu\lambda} \partial_\nu B_{s\lambda})^2 \\
& + \frac{K_a}{2} (\epsilon_{\mu\nu\lambda} \partial_\nu B_{a\lambda})^2 + \frac{i}{\pi} \epsilon_{\mu\nu\lambda} \partial_\mu B_{a\nu} \mathcal{A}_\lambda.
\end{aligned} \tag{21}$$

Further  $\mathcal{A}_\mu$  dependence in the action appears only in the fermionic term  $\mathcal{L}_f[\mathcal{A}_\mu]$ . If the fermions are gapped, they can be safely integrated out, at the cost of a renormalization of the coupling  $K$ . In this case, we can also integrate out  $\mathcal{A}_\mu$  from Eq. (21), and conclude that the  $B_{a\mu}$  gauge field is gapped and can be dropped from further considerations. The situation with gapless fermions is far more complicated, and requires a full analysis of the coupled  $\mathcal{A}_\mu$ ,  $B_{a\mu}$ , and the fermionic excitations.

The action  $\mathcal{L}_{2\varphi}$  is the quadratic term for the vortex fields  $\varphi_{1,2}$  consistent with the gauge charge assignments and the PSG. After appropriate rescalings of the  $\varphi_{1,2}$  fields, and rescaling of time and space co-ordinates, this has the form

$$\mathcal{L}_{2\varphi} = \sum_{\ell=0}^{q-1} \left\{ |(\partial_\mu - iB_{s\mu} - iB_{a\mu}) \varphi_{1\ell}|^2 + |(\partial_\mu - iB_{s\mu} + iB_{a\mu}) \varphi_{2\ell}|^2 + s (|\varphi_{1\ell}|^2 + |\varphi_{2\ell}|^2) \right\} + \tilde{\mathcal{L}}_{2\varphi}. \quad (22)$$

Here we displayed the standard kinetic terms which identifies the  $2q$   $\varphi_{1,2\ell}$  fields as relativistic complex scalars. The “mass”  $s$  is the tuning parameter which accesses various phases of the doped antiferromagnet. For large positive  $s$ , the vortices are gapped, and the ground state is superconducting. Condensates of vortices or vortex-anti-vortex pairs can form a lower doping, leading to insulating or supersolid phases.

There is an additional quadratic term, denoted  $\tilde{\mathcal{L}}_{2\varphi}$  above, invariant under all PSG transformations in Eq. (17), which is allowed in some cases. For the special values  $q = 4n + 2$  ( $n$  integer), it is easy to check that

$$\tilde{\mathcal{L}}_{2\varphi} = i\Xi \sum_{\ell=0}^{q-1} (-1)^\ell \left( \varphi_{1\ell}^* \varphi_{1,\ell+q/2} - \varphi_{2\ell}^* \varphi_{2,\ell+q/2} \right) \quad (23)$$

is allowed, where  $\Xi$  is a real coupling constant. More physically, this term has its origin on a staggered potential that can act on the  $b_{1,2}$  holons. As derived explicitly in Ref. [17], or by the requirements of the PSG in Eq. (14), the boson Hamiltonian can contain the term

$$\sum_j \eta_j \left( b_{1j}^\dagger b_{1j} - b_{2j}^\dagger b_{2j} \right).$$

Upon dualizing to the vortices, this term appears as a staggered flux acting on the vortices, which in the continuum limit generates Eq. (23).

The term  $\mathcal{L}_{4\varphi}$  is a quartic polynomial in the vortex fields  $\varphi_{1,2}$ . This polynomial is constrained by the PSG transformations in Eq. (17), in a manner that has been discussed at some length in Ref. [16]. These quartic terms control the precise structure of the density-wave order in the supersolid and insulating phases.

Finally, we turn to the main term being introduced in this paper: the  $\mathcal{L}_m$  which couples the vortices and the monopoles. In obtaining this term, we simply have to search for invariants under the PSGs in Eq. (13) and (17). A straightforward analysis shows that the lowest order invariant, present only for  $q = 2n$  is

$$\mathcal{L}_m = \lambda m^\dagger \sum_{\ell=0}^{q-1} \left[ e^{-i\pi/4} (-1)^\ell \varphi_{1\ell}^* \varphi_{2\ell} + e^{i\pi/4} \varphi_{1\ell}^* \varphi_{2,\ell+q/2} \right] + \text{c.c.}, \quad (24)$$

where  $\lambda$  is a complex coupling constant. Note that this is the only term which couples together the  $\varphi_1$  and  $\varphi_2$  vortices, and so places a crucial role in determining the vortex spectrum.

#### 2.4 Vortex spectrum in the superconductor

The vortices are gapped in the superconductor, and here we are interested in the nature of the single vortex excited state. (Strictly speaking, a single vortex has a logarithmically divergent energy, and so the excitations are vortex-anti-vortex pairs—we imagine there is an anti-vortex far away, and examine the motion of the vortex). Once a  $\varphi_1$  vortex has been created (say), it can transmute into a  $\varphi_2$  vortex by the monopole tunnelling process in Eq. (24). In the superconductor, we expect that the flux associated with this monopole will be largely confined to the vortex cores, and so the action for the monopole will be finite. Under these circumstances, it is legitimate to simply treat the field  $m$  as a constant [12] (in the absence of gapless fermions, it is also valid to replace  $m$  by a constant across the transition to the insulator).

The spectrum of a single vortex is then simply determined by diagonalizing the quadratic action  $\mathcal{L}_{2\varphi} + \mathcal{L}_m$  and ignoring the gauge fields. We describe the results of this procedure below for different values of  $q$  in the following subsections.

##### 2.4.1 $q = 1, 3 \pmod{4}$

For  $q$  odd,  $\mathcal{L}_m = 0$  and  $\tilde{\mathcal{L}}_{2\varphi} = 0$ , and so the  $\varphi_{1\ell}$  and  $\varphi_{2\ell}$  remain vortex eigenstates. The degeneracy of vortex states for this case is therefore  $q' = 2q$ .

##### 2.4.2 $q = 0 \pmod{4}$

Next, we consider the case where  $q$  is a multiple of 4. Now  $\mathcal{L}_m$  in Eq. (24) is non-zero, and it splits the  $2q$  dimensional vortex space into  $4 \times 4$  blocks in which  $\varphi_{1\ell}$ ,  $\varphi_{1,\ell+q/2}$ ,  $\varphi_{2\ell}$ , and  $\varphi_{2,\ell+q/2}$  are coupled to each other. In each block, the action has the structure

$$\Omega^2 + k^2 + s + \begin{pmatrix} 0 & 0 & \alpha & \beta \\ 0 & 0 & \beta & \alpha \\ \alpha^* & \beta^* & 0 & 0 \\ \beta^* & \alpha^* & 0 & 0 \end{pmatrix} \quad (25)$$

$q$	$q'$
$2n + 1$	$2q$
$4n$	$q$
$4n + 2$	$q/2$

Table 2

Degeneracy of the vortex states,  $q'$ , at hole density  $\delta$ . Here  $\delta$  is related to  $q$  as in Eq. (9), and  $n$  is a positive integer.

where  $\alpha \equiv \lambda(-1)^\ell m^\dagger e^{-i\pi/4}$ ,  $\beta \equiv \lambda m^\dagger e^{i\pi/4}$ ,  $\Omega$  is the frequency, and  $k$  is the wavevector. The eigenvalues of this action are  $\pm (|\alpha|^2 + |\beta|^2 \pm (\alpha^* \beta + \beta^* \alpha))^{1/2} + \Omega^2 + k^2 + s$ . Now note that  $\alpha^* \beta + \beta^* \alpha = 0$ , and hence there are 2 pairs of doubly degenerate eigenvalues. Consequently, the  $2q$  vortex states have split into 2 blocks of  $q$  degenerate states. The lowest energy vortex states therefore have the degeneracy  $q' = q$ .

### 2.4.3 $q = 2 \pmod{4}$

Finally, consider the remaining case  $q = 4n + 2$ , where  $n$  is a positive integer. Now  $\mathcal{L}_{2\varphi}$  in Eq. (23) is also non-zero, but the action still has a  $4 \times 4$  block structure. The matrix in Eq. (25) is replaced by

$$\Omega^2 + k^2 + s + \begin{pmatrix} 0 & i\gamma & \alpha & \beta \\ -i\gamma & 0 & \beta & -\alpha \\ \alpha^* & \beta^* & 0 & -i\gamma \\ \beta^* & -\alpha^* & i\gamma & 0 \end{pmatrix} \quad (26)$$

where  $\gamma \equiv (-1)^\ell \Xi$ . Now the 4 eigenvalues are  $\rho_1 (|\alpha|^2 + |\beta|^2 - i\rho_2(\alpha^* \beta - \beta^* \alpha))^{1/2} + \Omega^2 + k^2 + s + \rho_2 \gamma$ , where  $\rho_{1,2} = \pm 1$ . All 4 eigenvalues are distinct, and so the  $2q$  vortex eigenstates have now split into blocks of  $q/2$  degenerate states. The lowest energy vortex states therefore have the degeneracy  $q' = q/2$ .

### 2.4.4 General discussion

We summarize the above results on the degeneracy of the vortex states in Table 2. The numerology of the vortex degeneracy seems rather mysterious, but actually has a simple physical interpretation. The results in Table 2 can be reproduced by the simple formula

$$\frac{p'}{q'} = \frac{1}{2} - \frac{p}{q}, \quad (27)$$

where  $p'$  and  $q'$  are relatively prime integers (as are  $p$  and  $q$ ). Recalling that  $p/q$  is half the density of holes (Eq. (9)) away from the half-filled Mott insulator, we now see that  $p'/q'$  is half the density of electrons *i.e.*  $p'/q'$  is the total density of Cooper pairs per unit cell. So the degeneracy of the vortex spectrum is always equal to that of a model of elementary bosons with the boson density equal to the density of Cooper pairs. This is one of the main results of this paper.

The above result can also be understood directly from the PSG. Consider the action of the PSG on field combinations which are neutral under  $B_{a\mu}$  (only such combinations appear in the action above). For this, we define

$$\bar{\varphi}_{1\ell} = \varphi_{1\ell} m^{1/2} \quad ; \quad \bar{\varphi}_{2\ell} = \varphi_{2\ell} (m^\dagger)^{1/2} \quad (28)$$

The action of the PSG on these fields is

$$\begin{aligned} T_x : \bar{\varphi}_{1\ell} &\rightarrow e^{i\pi/4} \bar{\varphi}_{2,\ell+1} \quad ; \quad \bar{\varphi}_{2\ell} \rightarrow e^{-i\pi/4} \bar{\varphi}_{1,\ell+1} \\ T_y : \bar{\varphi}_{1\ell} &\rightarrow e^{-i\pi/4} \bar{\varphi}_{2\ell} \omega^{-\ell} \quad ; \quad \bar{\varphi}_{2\ell} \rightarrow e^{i\pi/4} \bar{\varphi}_{1\ell} \omega^{-\ell} \\ R_{\pi/2}^{\text{dual}} : \bar{\varphi}_{1\ell} &\rightarrow \frac{1}{\sqrt{q}} \sum_{m=0}^{q-1} \bar{\varphi}_{2m} \omega^{-\ell m} \quad ; \\ \bar{\varphi}_{2\ell} &\rightarrow \frac{1}{\sqrt{q}} \sum_{m=0}^{q-1} \bar{\varphi}_{1m} \omega^{-\ell m} \end{aligned} \quad (29)$$

Note that the transformations in Eq. (29) obey

$$T_x T_y = -\omega T_y T_x = \exp\left(-\frac{2\pi i p'}{q'}\right) T_y T_x, \quad (30)$$

thus explaining the  $q'$ -fold degeneracy of vortex states.

## 2.5 Fermionic excitations

The analysis of Ref. [17] of spin  $S = 1/2$ , neutral excitations of the quantum dimer and related models applies directly to our theory of the doped VBS phase. These excitations are represented by Fermi operators  $f_{j\sigma}$  on the sites,  $j$ , of the square lattice, with  $\sigma = \uparrow, \downarrow$  a spin index. This fermion carries  $\mathcal{A}_\mu$  gauge charge  $\eta_j$ , as indicated in Table 1. Also crucial are its transformation properties under the PSG

$$\begin{aligned}
T_x &: f_\sigma \rightarrow f_\sigma \\
T_y &: f_\sigma \rightarrow f_\sigma \\
R_{\pi/2}^{\text{dual}} &: f_\sigma \rightarrow i f_\sigma \\
I_x^{\text{dual}} &: f_\sigma \rightarrow f_\sigma \\
\mathcal{T} &: f_\sigma \rightarrow \epsilon_{\sigma\sigma'} f_{\sigma'}^\dagger \quad ; \quad f_\sigma^\dagger \rightarrow -\epsilon_{\sigma\sigma'} f_{\sigma'},
\end{aligned} \tag{31}$$

where  $\epsilon_{\sigma\sigma'}$  is the antisymmetric tensor. The transformations of  $f_\sigma^\dagger$  are the Hermitian conjugates of those of  $f_\sigma$ , except for the case of time-reversal. Time-reversal does not correspond to a canonical unitary transformation, and we have chosen to describe time-reversal as a symmetry of the Grassman coherent-state path integral. The minus sign in the time-reversal transformation of  $f^\dagger$  above is allowed since the  $f^\dagger$  and  $f$  fields are *not* complex conjugates but actually independent in the Grassman integral. The factor  $i$  in the transformation under  $R_{\pi/2}^{\text{dual}}$  is related to the corresponding factor in the PSG for the holons in Eq. (14). It corresponds to an assumption of  $d$ -wave pairing.

With the gauge charges and the PSG at hand, we can write down allowed terms in the fermionic contribution,  $\mathcal{L}_f[\mathcal{A}_\mu]$ , to the action:

$$\begin{aligned}
\mathcal{L}_f[\mathcal{A}_\mu] &= \sum_j f_{j\sigma}^\dagger \left( \frac{\partial}{\partial \tau} - i\eta_j \mathcal{A}_{j\tau} \right) f_{j\sigma} + v \sum_j f_{j\sigma}^\dagger f_{j\sigma} \\
&\quad + \sum_{j\alpha} \Delta_\alpha e^{-i\eta_j \mathcal{A}_{j\alpha}} f_{j\sigma}^\dagger \epsilon_{\sigma\sigma'} f_{j+\alpha,\sigma'}^\dagger + \text{H.c.} \\
&\quad - t \sum_{j,\alpha=x,y} b_{j1} b_{j+\alpha,2}^\dagger f_{j\sigma}^\dagger f_{j+\alpha,\sigma} + \text{H.c.}
\end{aligned} \tag{32}$$

Clearly,  $\Delta_\alpha$  is a pairing amplitude, and we have  $\Delta_x = -\Delta_y$ .

In principle, it is now possible to take the continuum limit of  $\mathcal{L}_f$  and then examine the properties of Eq. (20). However, with the expected proliferation of monopoles in the insulator, we expect that  $\mathcal{A}_\mu$  fluctuations are very strong, and it may be more appropriate to use fields that are neutral under  $\mathcal{A}_\mu$ . The quadratic terms in the fermion dispersion in Eq. (32) have a gapped fermion spectrum, and so no low energy fermions are available to suppress monopole events.

Rather, it seems more reasonable to assume that strong  $\mathcal{A}_\mu$  fluctuations bind the  $b_{1,2}$  holons to the  $f_\sigma$  fermions. We can then express the fermionic Hamiltonian in terms of charge  $e$  physical electron operators  $c_{j\sigma}$ :

$$\begin{aligned}
c_{j\sigma} &= e^{i\theta_{j1}} f_{j\sigma} \quad \text{for } \eta_j = 1 \\
c_{j\sigma} &= e^{i\theta_{j2}} f_{j\sigma} \quad \text{for } \eta_j = -1
\end{aligned} \tag{33}$$

At the same time, we can work with the  $\bar{\varphi}_{1,2}$  fields defined in Eq. (28), and then all degrees of freedom are neutral under the  $\mathcal{A}_\mu$  and  $B_{a\mu}$  gauge fields: the action is expressed entirely in terms of the  $c_{j\sigma}$ ,  $\bar{\varphi}_{1,2}$ , and  $B_{s\mu}$  fields.

The Hamiltonian for the  $c_{j\sigma}$  fermions is essentially identical to the BCS Hamiltonian

$$\begin{aligned} \mathcal{L}_c = & \sum_j c_{j\sigma}^\dagger \frac{\partial c_{j\sigma}}{\partial \tau} + v \sum_j c_{j\sigma}^\dagger c_{j\sigma} \\ & + \sum_{j\alpha} \Delta_\alpha e^{-i(\theta_{j1} + \theta_{j+\alpha,2} + \mathcal{A}_{j\alpha})} c_{j\sigma}^\dagger \epsilon_{\sigma\sigma'} c_{j+\alpha,\sigma'}^\dagger + \text{H.c.} - t \sum_{j,\alpha=x,y} c_{j\sigma}^\dagger c_{j+\alpha,\sigma} + \text{H.c.} \end{aligned} \quad (34)$$

Notice that the exponential in the second line containing a gauge-invariant combination of fields which carries physical electrical charge 2. So the  $c_\sigma$  fermions couple only to the physical BCS order parameter. Further, just as in the BCS theory, the  $c_\sigma$  fermions display the usual gapless nodal fermion excitations of a  $d$ -wave superconductor.

At this point, the coupling of the  $c_\sigma$  fermions to the vortices can be described following the methods discussed in a variety of papers in the literature [25,26,27,28,29,30]: it yields a theory for neutral fermions valid in a regime where the fermionic excitations can be gapless. After a singular gauge transformation which eliminates the phase of the pairing amplitude [25,26], the vortices have two important effects of the quasiparticle motion: the fermions acquire a ‘statistical’ phase of  $\pi$  upon encircling a  $\bar{\varphi}_{1,2}$ , and also acquire a ‘Doppler shift’ proportional to the local superflow velocity. The statistical phase is implemented by coupling both the vortices and the fermions to U(1) gauge fields,  $\alpha_\mu$ , and  $a_\mu$  respectively, along with a mutual Chern-Simons term

$$\mathcal{L}_{cs}[\alpha_\mu, a_\mu] = \frac{i}{\pi} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu \alpha_\lambda. \quad (35)$$

These gauge fields have to couple to conserved currents, and for the fermions a convenient choice [25,26,31] is the  $z$  component of the spin. We denote the resulting fermionic Lagrangian by  $\mathcal{L}_c[a_\mu]$ , and refer the reader to these earlier works for the explicit form. For our purposes, we need the transformations of the  $a_\mu$  gauge field under the square lattice symmetry operations. Using the usual transformations of the electron operator  $c_\sigma$  under the square lattice symmetry operations, and with requirements of gauge invariance, we can easily deduce the following PSG for  $a_\mu$ :

$$\begin{aligned}
T_x &: a_\mu \rightarrow a_\mu \\
T_y &: a_\mu \rightarrow a_\mu \\
R_{\pi/2}^{\text{dual}} &: a_\tau \rightarrow a_\tau ; a_x \rightarrow a_y ; a_y \rightarrow -a_x \\
I_x^{\text{dual}} &: a_\tau \rightarrow a_\tau ; a_x \rightarrow -a_x ; a_y \rightarrow a_y \\
\mathcal{T} &: a_\tau \rightarrow -a_\tau ; a_x \rightarrow a_x ; a_y \rightarrow a_y.
\end{aligned} \tag{36}$$

For the vortex sector, we couple  $\alpha_\mu$  to the vortex current. We assume that a diagonalization of the vortex spectrum has been carried out as discussed in Section 2.4, and focus exclusively on the  $q'$  degenerate low energy vortex modes. We choose these modes as eigenvectors of the  $T_y$  operator, and denote these modes simply as  $\bar{\varphi}_m$ , with the index  $m = 0, 1, \dots, (q'-1)$ . Collecting these terms, our theory for the vicinity of the superconductor-insulator transition is then

$$\begin{aligned}
\mathcal{L}_{\text{dSC-VBS}, \bar{\varphi}} &= \frac{K_s}{2} (\epsilon_{\mu\nu\lambda} \partial_\nu B_{s\lambda})^2 + \mathcal{L}_{cs}[\alpha_\mu, a_\mu] \\
&+ \sum_{m=0}^{q'-1} \left\{ |(\partial_\mu - iB_{s\mu} - i\zeta_m \alpha_\mu) \bar{\varphi}_m|^2 + s |\bar{\varphi}_m|^2 \right\} + \mathcal{L}_{4\bar{\varphi}} + \mathcal{L}_c[a_\mu].
\end{aligned} \tag{37}$$

The action for the  $B_{s\mu}$  field has been written in a schematic relativistic form, which is appropriate for short-range interactions between the bosons—the Coulomb interactions lead to modifications presented in Ref. [30]. Note that the  $\mathcal{A}_\mu$ ,  $B_{a\mu}$ , and  $m$  fields have dropped out. The term  $\mathcal{L}_{4\bar{\varphi}}$  now includes quartic invariants in the  $\bar{\varphi}_m$  which are invariant under Eq. (29).

The  $\bar{\varphi}_m$  fields have charges  $\zeta_m$  under the gauge field  $\alpha_\mu$ . If  $\alpha_\mu$  was coupling to the total vortex current, then we should choose all  $\zeta_m = 1$ . However, this gauge field implements only a statistical phase factor of -1, and this can be obtained by choosing arbitrary odd integer  $\zeta_m$ .

An action closely related to  $\mathcal{L}_{\text{dSC-VBS}, \bar{\varphi}}$  has been examined previously by Lannert *et al.* [32] for the case of Dirac spectrum in  $\mathcal{L}_c[a_\mu]$ . They considered a simplified model at half-filling with  $q' = 2$ . As discussed by Lannert *et al.*, the Chern-Simons term in  $\mathcal{L}_{\text{dSC-VBS}, \bar{\varphi}}$  drives confinement in the insulating phase with  $\langle \bar{\varphi} \rangle \neq 0$ : each  $\Psi$  fermion has a  $\alpha_\mu$  flux tube attached to it, and this acquires an additional  $B_{s\mu}$  flux tube when the total flux coupled to the  $\bar{\varphi}$  is expelled; the latter implies attachment of charge  $e$  to the spinons, and a likely gapping of the fermion spectrum in the insulator. A notable feature of this theory is that the nodal fermions remain gapless all the way up to the critical point. A full PSG analysis on the direct lattice seems necessary to verify such a scenario (as in Section III.B of Ref. [16]), but below we present some plausible constraints under which such a critical point may obtain.

We close this subsection by discussing some issues related to the PSG proper-

ties of  $\mathcal{L}_{\text{dSC-VBS},\bar{\varphi}}$ . While its properties should be invariant under arbitrary choices for  $\zeta_m$ , our approximate analysis in the following subsection will find dependence on the values of  $\zeta_m$ . It is therefore useful to find a choice of  $\zeta_m$  in which the invariance under the PSG is as explicit as possible, without using non-perturbative properties of the Chern-Simons term. We will see in the following subsection that it is useful to satisfy the constraint  $\sum_m \zeta_m = 0$ . This is clearly not possible for  $q'$  odd (which corresponds to  $q = 4n + 2$ ), and so we will not consider this case further. For other values of  $q$ , we divide the vortex fields into 2 sets, one with  $\zeta_m = 1$  and the other with  $\zeta_m = -1$ . Invariance under the PSG is then possible provided the set of fields with  $\zeta_m = 1$  either transform only among themselves, or all transform into fields with  $\zeta_m = -1$  (and conversely for the fields with  $\zeta_m = -1$ ). For  $q$  odd (with  $q' = 2q$ ), there is no monopole-induced mixing between the two sets of  $\bar{\varphi}_{1\ell}$  and the  $\bar{\varphi}_{2\ell}$  fields, and we can simply assign  $\zeta_m = 1$  for the first set, and  $\zeta_m = -1$  for the second set. For  $q$  a multiple of 4 (with  $q' = q$ ), we consider explicitly the *permutative* PSG for  $q = 4$ . This is specified in Eq. (C5) of Ref. [16], and we see that, in this permutative vortex basis, the needed conditions are satisfied for  $\zeta_0 = \zeta_2 = 1$  and  $\zeta_1 = \zeta_3 = -1$ . We suspect a similar choice is possible for other  $q$  multiples of 4. In all the cases for which suitable  $\zeta_m$  are possible, the PSG of the  $\bar{\varphi}_m$  implies a corresponding PSG for the  $\alpha_\mu$ , which we will not write explicitly. Combining this with the PSG for the  $a_\mu$  in Eq. (36), we finally have to test the invariance of the Chern-Simons term  $\mathcal{L}_{cs}[\alpha_\mu, a_\mu]$  in Eq. (35): we find that  $\mathcal{L}_{cs}[\alpha_\mu, a_\mu]$  changes sign under most PSG transformations. However, this sign is clearly not physically significant because statistical phases of  $\pm\pi$  are equivalent.

## 2.6 ‘Undualizing’ to fractionally charged bosons

We now examine the critical theory in Eq. (37), associated with the condensation of the  $\bar{\varphi}$  vortices, by undoing the duality into direct lattice degrees of freedom. Here it is possible to apply recent ideas [11,12] on ‘deconfined criticality’ to ‘undualize’  $\mathcal{L}_{\text{dSC-VBS},\bar{\varphi}}$  and obtain a theory expressed in terms of fractionalized direct lattice degrees of freedom. The procedure for doing this was discussed at length in Ref. [16] for boson models at arbitrary rational filling on the square lattice. Here, we can apply exactly the same procedure to the  $q'$  vortices  $\bar{\varphi}_m$ . As discussed in Section III.A of Ref. [16], such a transformation to direct lattice bosons is possible only if it is possible to find a ‘permutative representation’ of the PSG. In the following, we assume that it is possible to transform the relevant terms in  $\mathcal{L}_{4\bar{\varphi}}$  so that such a permutative representation exists. Then, as shown in Section III.A of Ref. [16], the  $q'$  vortex fields  $\bar{\varphi}_m$  and the U(1) gauge field  $B_{s\mu}$  can be ‘undualized’ into  $q'$  boson fields  $\xi_m$  and  $q'$  non-compact U(1) gauge fields  $\tilde{\mathcal{A}}_\mu^m$ . The  $\xi_m$  boson fields each carry physical electromagnetic charge  $2e/q'$ , and so each Cooper pair has fractionalized into

$q'$  elementary bosons. One contribution to the direct formulation of Eq. (37) is the theory of these fields presented in Eq. (3.5) of Ref. [16]. Combining these terms with the contribution of the Dirac fermions, we can tentatively propose the following ‘undualized’ formulation of  $\mathcal{L}_{\text{dSC-VBS},\bar{\varphi}}$ :

$$\begin{aligned} \mathcal{L}_{\text{deconfined}} = & \sum_{m=0}^{q'-1} \left[ \left| (\partial_\mu - i\tilde{\mathcal{A}}_\mu^m) \xi_m \right|^2 + \tilde{s} |\xi_m|^2 \right] \\ & + \sum_{m,n} K_{mn} \left( \epsilon_{\mu\nu\lambda} \partial_\nu \tilde{\mathcal{A}}_\lambda^m \right) \left( \epsilon_{\mu\rho\sigma} \partial_\rho \tilde{\mathcal{A}}_\sigma^n \right) + \sum_{m,n} \tilde{v}_{mn} |\xi_m|^2 |\xi_n|^2 + \mathcal{L}_c[a_\mu], \end{aligned} \quad (38)$$

where the couplings  $K_{mn}$  and  $v_{mn}$  can have the most general form consistent with the permutative PSG. Now  $\tilde{s}$  is the tuning parameter, which yields superconducting ground states for negative values of  $\tilde{s}$  where the  $\xi_m$  bosons are condensed, and insulating states for positive  $\tilde{s}$  where the bosons are gapped. A specific duality transformation applied to  $\mathcal{L}_{\text{dSC-VBS},\bar{\varphi}}$ , along the lines of Ref. [16], does indeed yield precisely the action in Eq. (38), with no residual Chern-Simons term—this absence is one of the principal advantages of the direct formulation. This duality transformation also shows that the  $q'$  gauge fields  $\tilde{\mathcal{A}}_\mu^m$  and the  $a_\mu$  gauge field are not all independent, but certain linear combinations are ‘Higgsed’ out and so become gapped. We assume that we are working at energies much lower than this energy gap, and so impose constraints projecting out these linear combinations. In this manner we find the constraints

$$\sum_{m=0}^{q'-1} \tilde{\mathcal{A}}_\mu^m = 0 \quad (39)$$

$$a_\mu = \frac{1}{2} \sum_{m=0}^{q'-1} \zeta_m \tilde{\mathcal{A}}_\mu^m. \quad (40)$$

The theory in Eq. (38) of  $q'$ , charge  $2e/q'$ , relativistic complex scalars  $\xi_m$  and the Dirac fermions  $\Psi$ , coupled to the U(1) gauge field  $a_\mu$  and the  $q'$  U(1) gauge fields  $\tilde{\mathcal{A}}_\mu^m$  (subject to the constraints in Eqs. (39) and (40)), is then our final theory for the deconfined criticality between a  $d$ -wave superconductor and VBS insulators at Cooper pair density  $p'/q'$ .

An important ingredient so far left unspecified is the set of odd-integer values of the  $\zeta_m$ . A full understanding of their values probably requires a direct derivation of  $\mathcal{L}_{\text{deconfined}}$  from the underlying lattice model, as discussed in Section III.B of Ref. [16], without the long detour taken here into dual vortex variables. We defer such an analysis to future work, but will now note some important restrictions on the values of the  $\zeta_m$ .

To obtain these restrictions, let us consider the structure of vortices that

can be created from the  $\xi_m$  boson fields, using arguments similar to those presented in Section 1.3. Because these boson fields descended from a duality transformation of  $q'$  vortex fields  $\bar{\varphi}_m$  each carrying physical magnetic flux  $hc/2e$ , we expect that the same vortex structure should also emerge from an analysis of vortex saddle points of Eq. (38). That this is indeed the case was explained in Ref. [16], and we now review the argument. An elementary vortex is created by inducing a  $2\pi$  winding in the phase of  $\xi_0$  (say), while keeping the remaining  $(q' - 1)$  boson fields  $\xi_{m \neq 0}$  topologically trivial. (Moving the phase winding to the other  $(q' - 1)$  fields yields a total of  $q'$  distinct elementary vortices, as expected.) Then, under the action in Eq. (38), the  $\tilde{\mathcal{A}}_\mu^m$  gauge fields will respond to minimize the total action of this vortex state. At a large distance,  $r$ , from the vortex center, we expect that these fields will be pure gauge, and oriented purely in the azimuthal (the angular co-ordinate  $\theta$ ) direction; keeping in mind the symmetry of the this vortex state and the constraint in Eq. (39) we obtain

$$\begin{aligned}\tilde{\mathcal{A}}_\theta^0(r \rightarrow \infty) &= \frac{A}{r} \\ \tilde{\mathcal{A}}_\theta^m(r \rightarrow \infty) &= -\frac{A}{(q' - 1)r}, \quad m \neq 0,\end{aligned}\tag{41}$$

where  $A$  is a constant to be determined by minimizing the action. The  $1/r$  decay above ensures that  $\oint dr_\mu \tilde{\mathcal{A}}_\mu^m$  is a constant on a contour far from the center of the vortex and measures the total  $\tilde{\mathcal{A}}_\mu^m$  flux trapped near the center of the vortex. Inserting the above configurations of  $\xi_m$  and  $\tilde{\mathcal{A}}_\mu^m$  into the action in Eq. (38), we find that the total energy of a single vortex is logarithmically divergent, and the optimal vortex configuration will minimize the co-efficient of this logarithmic divergence. The co-efficient is proportional to [16]

$$(1 - A)^2 + \sum_{m=1}^{q'-1} (A/(q' - 1))^2,\tag{42}$$

and minimizing this expression yields  $A = 1 - 1/q'$ . Upon including the physical electromagnetic field, the logarithmic divergence is cutoff by the London penetration depth, and the total magnetic flux will be that required to saturate the remaining phase winding of the  $\xi_m$  at large  $r$ ; a simple calculation from the result in Eq. (42) shows that this yields the required flux of  $hc/2e$ .

Turning to our objective of restricting the values of the  $\zeta_m$ , let us now insert the results above for the values of  $\tilde{\mathcal{A}}_\mu^m$  into Eq. (40), and so obtain the value of the total  $a_\mu$  flux associated with the vortex:

$$\begin{aligned}
\oint dr_\mu a_\mu &= \pi \left( \zeta_0 A + \sum_{m=1}^{q'-1} \zeta_m \left[ -\frac{A}{(q'-1)} \right] \right) \\
&= \pi \left( \zeta_0 - \frac{1}{q'} \sum_{m=0}^{q'-1} \zeta_m \right)
\end{aligned} \tag{43}$$

This total  $a_\mu$  flux is observed by the  $\Psi$  fermions to be trapped at each vortex. Because the  $\Psi$  fermions pick up a  $-1$  Berry phase around each vortex, and recalling the constraint that all the  $\zeta_m$  have to be odd integers, we obtain the additional constraint that

$$\frac{1}{q'} \sum_{m'=0}^{q'-1} \zeta_{m'} = \text{even integer}. \tag{44}$$

The choice  $\zeta_m = 1$  for all  $m$  does not satisfy this constraint. However, choosing an equal number of  $\zeta_m = 1$  and  $\zeta_m = -1$  so that  $\sum_m \zeta_m = 0$ , does satisfy Eq. (44); explicit choices of this type were discussed at the end of Section 2.5. It is also clear that no solution is possible for odd  $q'$ , indicating that a deconfined critical point does not exist in this case.

### 3 Doping the staggered flux spin liquid

The staggered flux (sF) spin liquid has been considered in some detail in the context of the ‘‘SU(2) slave particle’’ description of Wen, Lee and collaborators [9,20]. We will begin with the same theory in the undoped insulator, but use a vortex theory to describe the doped system. As noted earlier, aspects of our analysis will be uncomplete, because the PSG of the monopoles above the sF phase is not available. Our analysis will address a variety of transitions out of a finite doping superconducting state. The resulting quantum critical points are in a different universality class from the zero-doping quantum critical points which have been proposed elsewhere as the fixed points controlling the finite doping physics [33].

#### 3.1 Undoped insulator

The staggered flux spin liquid is described by a mean-field Hamiltonian expressed in terms of a  $S = 1/2$  fermionic spinor  $f_{j\sigma}$ :

$$\begin{aligned}
H_{sF} &= \sum_j \left\{ -t(f_{j\sigma}^\dagger f_{j+\hat{x},\sigma} + f_{j\sigma}^\dagger f_{j+\hat{y},\sigma}) \right. \\
&\quad \left. -i\eta_j t'(f_{j\sigma}^\dagger f_{j+\hat{x},\sigma} - f_{j\sigma}^\dagger f_{j+\hat{y},\sigma}) + \text{h.c.} \right\}
\end{aligned} \tag{45}$$

For  $t' \neq t$ , these fermions see a staggered flux ( $\neq \pi$ ) which apparently breaks translational symmetry. Since the sign of this flux cannot be removed by a U(1) gauge transformation, translational symmetry is implemented instead with a particle/hole transformation. The PSG transformations which leave  $H_{sF}$  invariant are

$$\begin{aligned}
T_x : f_{j\sigma} &\rightarrow i\eta_{j'}\epsilon_{\sigma\sigma'} f_{j',\sigma}^\dagger \\
T_y : f_{j\sigma} &\rightarrow i\eta_{j'}\epsilon_{\sigma\sigma'} f_{j',\sigma}^\dagger \\
R_{\pi/2}^{\text{dual}} : f_\sigma &\rightarrow i f_\sigma \\
I_x^{\text{dual}} : f_{j\sigma} &\rightarrow i\eta_{j'}\epsilon_{\sigma\sigma'} f_{j',\sigma}^\dagger \\
\mathcal{T} : f_{j\sigma} &\rightarrow i\eta_j f_{j\sigma} \quad ; \quad f_{j\sigma}^\dagger \rightarrow i\eta_j f_{j\sigma}^\dagger,
\end{aligned} \tag{46}$$

As in Eq. (31), all operations are canonical apart from time-reversal, and for this case the transformation of  $f_\sigma^\dagger$  is independent of  $f_\sigma$ . In all cases, the site  $j'$  is the image of the site  $j$  under the noted transformation *e.g.*  $j' = j + \hat{x}$  under  $T_x$ .

Moving beyond mean field theory, the  $f_\sigma$  fermions are coupled to a U(1) gauge field  $\mathcal{A}_\mu$ , under which they have a uniform charge of +1 (see Table 1), in contrast to the charge of  $\eta_j$  obtained in the VBS case. The PSG transformation of  $\mathcal{A}_\mu$  are easily deduced from those of the  $f_\sigma$  by imposing the requirement of gauge invariance. The completion of the PSG Eq. (8), as in Eq. (12), is

$$\begin{aligned}
T_x : \mathcal{A}_\mu &\rightarrow -\mathcal{A}_\mu \\
T_y : \mathcal{A}_\mu &\rightarrow -\mathcal{A}_\mu \\
R_{\pi/2}^{\text{dual}} : \mathcal{A}_\tau &\rightarrow \mathcal{A}_\tau \quad ; \quad \mathcal{A}_x \rightarrow \mathcal{A}_y \quad ; \quad \mathcal{A}_y \rightarrow -\mathcal{A}_x \\
I_x^{\text{dual}} : \mathcal{A}_\tau &\rightarrow -\mathcal{A}_\tau \quad ; \quad \mathcal{A}_x \rightarrow \mathcal{A}_x \quad ; \quad \mathcal{A}_y \rightarrow -\mathcal{A}_y \\
\mathcal{T} : \mathcal{A}_\tau &\rightarrow -\mathcal{A}_\tau \quad ; \quad \mathcal{A}_x \rightarrow \mathcal{A}_x \quad ; \quad \mathcal{A}_y \rightarrow \mathcal{A}_y.
\end{aligned} \tag{47}$$

Finally, to complete the analysis of the sF spin liquid, we need the transformations of the monopoles in  $\mathcal{A}_\mu$ , the analog of the relations in Eq. (13). These we do not present here, but our analysis below can easily be extended to include them.

### 3.2 Doped staggered flux state

As stated earlier, we dope the spin liquid state by introducing two species of holon bosons,  $b_{1,2}$ , which carry charges  $\pm 1$  under the  $\mathcal{A}_\mu$  gauge field. The PSG of these bosons can be deduced by the requirement that the physical electron

operator be invariant under all transformations. The latter is connected to the slave particles by [9,17]

$$c_{j\sigma} = b_{j1}^\dagger f_{j\sigma} + i\eta_j b_{j2}^\dagger \epsilon_{\sigma\sigma'} f_{j\sigma'}^\dagger, \quad (48)$$

From Eqs. (46) and (48) we can deduce

$$\begin{aligned} T_x &: b_1 \rightarrow b_2 ; b_2 \rightarrow b_1 \\ T_y &: b_1 \rightarrow b_2 ; b_2 \rightarrow b_1 \\ R_{\pi/2}^{\text{dual}} &: b_1 \rightarrow b_1 ; b_2 \rightarrow b_2 \\ I_x^{\text{dual}} &: b_1 \rightarrow b_2 ; b_2 \rightarrow b_1 \\ \mathcal{T} &: b_1 \rightarrow b_2^\dagger ; b_2 \rightarrow b_1^\dagger. \end{aligned} \quad (49)$$

Note that the differences from the corresponding Eq. (14) for the VBS case are restricted to  $R_{\pi/2}^{\text{dual}}$  and  $\mathcal{T}$ . In Eq. (14), the 1, 2 vortex flavors are interchanged under  $R_{\pi/2}^{\text{dual}}$  but not under  $\mathcal{T}$ , while in Eq. (49) above the opposite is true.

Proceeding with the duality from bosons  $b_{1,2}$  to vortices  $\psi_{1,2}$  and U(1) gauge fields  $A_{1\mu}$  and  $A_{2\mu}$  as in Section 2.2, the PSG of the vortices in Eq. (15) now becomes

$$\begin{aligned} T_x &: \psi_1 \rightarrow \psi_2 ; \psi_2 \rightarrow \psi_1 \\ T_y &: \psi_1 \rightarrow \psi_2 ; \psi_2 \rightarrow \psi_1 \\ R_{\pi/2}^{\text{dual}} &: \psi_1 \rightarrow \psi_1 ; \psi_2 \rightarrow \psi_2 \\ I_x^{\text{dual}} &: \psi_1 \rightarrow \psi_2^\dagger ; \psi_2 \rightarrow \psi_1^\dagger \\ \mathcal{T} &: \psi_1 \rightarrow \psi_2 ; \psi_2 \rightarrow \psi_1, \end{aligned} \quad (50)$$

while the PSG of the dual U(1) gauge fields  $A_{1,2}$  in Eq. (16) is replaced here by

$$\begin{aligned} T_x &: A_{1\mu} \rightarrow A_{2\mu} ; A_{2\mu} \rightarrow A_{1\mu} \\ T_x &: A_{1\mu} \rightarrow A_{2\mu} ; A_{2\mu} \rightarrow A_{1\mu} \\ R_{\pi/2}^{\text{dual}} &: A_{1\tau} \rightarrow A_{1\tau} ; A_{1x} \rightarrow A_{1y} ; A_{1y} \rightarrow -A_{1x} ; \\ &A_{2\tau} \rightarrow A_{2\tau} ; A_{2x} \rightarrow A_{2y} ; A_{2y} \rightarrow -A_{2x} \\ I_x^{\text{dual}} &: A_{1\tau} \rightarrow -A_{2\tau} ; A_{1x} \rightarrow A_{2x} ; A_{1y} \rightarrow -A_{2y} ; \\ &A_{2\tau} \rightarrow -A_{1\tau} ; A_{2x} \rightarrow A_{1x} ; A_{2y} \rightarrow -A_{1y} \\ \mathcal{T} &: A_{1\tau} \rightarrow -A_{2\tau} ; A_{1x} \rightarrow A_{2x} ; A_{1y} \rightarrow A_{2y} ; \\ &A_{2\tau} \rightarrow -A_{1\tau} ; A_{2x} \rightarrow A_{1x} ; A_{2y} \rightarrow A_{1y}. \end{aligned} \quad (51)$$

Again, note that the differences from Eqs. (15,16) for the VBS case are restricted to  $R_{\pi/2}^{\text{dual}}$  and  $\mathcal{T}$ . In Eqs. (15,16), the 1, 2 vortex flavors are interchanged under  $R_{\pi/2}^{\text{dual}}$  but not under  $\mathcal{T}$ , while in Eqs. (50,51) above the opposite is true.

Just as in Section 2.2, we now account for the influence of the mean hole density of  $\delta$  (with  $\delta/2 = p/q$ ) on the vortices by introducing  $2q$  flavors of vortices  $\varphi_{1\ell}$  and  $\varphi_{2\ell}$ . The PSG analog of the transformations of these vortices in Eq. (17) is now

$$\begin{aligned}
T_x &: \varphi_{1\ell} \rightarrow \varphi_{2,\ell+1} ; \varphi_{2\ell} \rightarrow \varphi_{1,\ell+1} \\
T_y &: \varphi_{1\ell} \rightarrow \varphi_{2\ell}\omega^{-\ell} ; \varphi_{2\ell} \rightarrow \varphi_{1\ell}\omega^{-\ell} \\
R_{\pi/2}^{\text{dual}} &: \varphi_{1\ell} \rightarrow \frac{1}{\sqrt{q}} \sum_{m=0}^{q-1} \varphi_{1m}\omega^{-\ell m} ; \\
&\quad \varphi_{2\ell} \rightarrow \frac{1}{\sqrt{q}} \sum_{m=0}^{q-1} \varphi_{2m}\omega^{-\ell m} \\
I_x^{\text{dual}} &: \varphi_{1\ell} \rightarrow \varphi_{2\ell}^\dagger ; \varphi_{2\ell} \rightarrow \varphi_{1\ell}^\dagger \\
\mathcal{T} &: \varphi_{1\ell} \rightarrow \varphi_{2\ell} ; \varphi_{2\ell} \rightarrow \varphi_{1\ell},
\end{aligned} \tag{52}$$

Once again, the differences from Eq. (17) for the VBS case are the opposite treatment of the 1, 2 vortex flavors between  $R_{\pi/2}^{\text{dual}}$  and  $\mathcal{T}$ .

### 3.3 Low energy theory

Following the analysis in Section 2.3, we now need to write down the most general effective action consistent with the PSG and gauge charge assignments obtained in Sections 3.1 and 3.2. This action can then describe transitions out of a  $d$ -wave superconductor into supersolid and insulating phases affiliated with the staggered flux phase.

The required action continues to have the form in Eq. (20), with the contributions  $\mathcal{L}_{U(1)}$  and  $\mathcal{L}_{2\varphi}$  retaining their forms in Eqs. (21) and (22). The quartic term  $\mathcal{L}_{4\varphi}$  will remain unspecified as the most general quartic polynomial in  $\varphi$  which is invariant under Eq. (52).

The fermionic contribution  $\mathcal{L}_f[\mathcal{A}_\mu]$  is obtained by taking the continuum limit of Eq. (45) and yields the familiar Dirac form at the four nodal points, which was schematically indicated in Eq. (7).

The additional quadratic invariant,  $\tilde{\mathcal{L}}_{2\varphi}$ , in Eq. (22) is now no longer given by Eq. (23). Instead, now there is a term whose origin is the staggered flux acting on the holons. Upon dualizing to vortices, this flux becomes equivalent to a staggered “chemical potential” acting on the vortices. A search for such terms reveals the following contribution which is present *only* when  $q$  is a multiple of 4:

$$\begin{aligned} \tilde{\mathcal{L}}_{2\varphi} = h_s \sum_{\ell=0}^{q-1} (-1)^\ell & \left[ \varphi_{1,\ell+q/2}^* \left( \frac{\partial}{\partial\tau} - iB_{s\tau} - iB_{a\tau} \right) \varphi_{1\ell} \right. \\ & \left. - \varphi_{2,\ell+q/2}^* \left( \frac{\partial}{\partial\tau} - iB_{s\tau} + iB_{a\tau} \right) \varphi_{2\ell} \right]. \end{aligned} \quad (53)$$

The PSG transformation associated with time-reversal allows only Eq. (53) in the present case, and only Eq. (23) in the doped VBS case.

The final ingredient are the monopole terms  $\mathcal{L}_m$  which couple monopoles to vortex bilinears. As in the VBS case, the vortex bilinears must carry dual staggered gauge ( $B_{a\mu}$ ) charge  $\pm 2$ , and so must mix the 1,2 vortex types. We will leave such terms undetermined here, but expect their influence can be easily included in the analysis below.

### 3.4 Vortex spectrum and quantum phase transitions out of the superconductor

As in the VBS case, it is useful to divide the discussion into various classes of values of  $q \pmod{4}$ .

#### 3.4.1 $q = 1, 3 \pmod{4}$

Although we do not have the explicit form of the monopole terms,  $\mathcal{L}_m$ , available, it appears a safe assumption that such terms will not contribute for  $q = 1, 3 \pmod{4}$ . The reason for this is similar to that for the VBS case: the wavevectors associated with the vortex fields  $\varphi_{1,2\ell}$  are not expected to match those of the monopole transformations.

The ‘staggered flux’ term in Eq. (53) also does not contribute. So there are  $2q$  vortex species, just as in the VBS case. The low energy theory is given by the sum of Eqs. (21), (22), (7) and  $\mathcal{L}_{4\varphi}$ . This theory will describe transitions from the  $d$ -wave superconductor into proximate insulating or supersolid phases.

The present theory is closely related to the critical theory in Eq. (37) for the doped VBS case. Here the mutual statistics between the vortices and the Dirac fermions is implemented by  $B_{a\mu}$  gauge field, and  $\varphi_1$  and  $\varphi_2$  vortices have opposite charges under  $B_{a\mu}$ . In contrast, the mutual statistics in Eq. (37) is implemented by  $\alpha_\mu$ , and the vortices had charges  $\zeta_m$ . So the present case corresponds to choosing  $\zeta_m = 1$  for the  $\varphi_1$  vortices and  $\zeta_m = -1$  for the  $\varphi_2$  vortices. The only remaining difference between the theories then are the differences between the PSGs of the vortices in Eqs. (17) and (52). This will lead to minor differences in the range of the competing orders which can appear in the insulating phases.

While the above derivation of the critical theory of fermions and vortices does have the advantage of preserving the lattice PSG at all stages, it does have the unphysical feature that the nodal points are pinned at the wavevectors  $(\pm\pi/2, \pm\pi/2)$  even in the finite doped superconducting case. A different continuum limit, along the lines of that discussed in Section 2.5 is needed to rectify this defect.

We can undualize the critical theory to obtain the theory for the deconfined critical point as in Eq. (38): the only changes are that the values of  $\zeta_m$  are as specified above, the  $a_\mu$  gauge field is replaced by  $\mathcal{A}_\mu$  with  $\mathcal{A}_\mu$  obeying the constraint Eq. (40), and the fermion term  $\mathcal{L}_c[a_\mu]$  takes the Dirac form  $\mathcal{L}_\Psi[\mathcal{A}_\mu]$  in Eq. (7).

### 3.4.2 $q = 2 \pmod{4}$

The ‘staggered flux’ term in Eq. (53) does not contribute for this case either. However, we expect non-trivial contributions from the monopole terms in  $\mathcal{L}_m$  now. It is a plausible hypothesis that such terms will reduce the degeneracy of the vortex spectrum from  $2q$  to  $q/2$ , as was the case for the VBS state, reviewed in Table 2.

### 3.4.3 $q = 0 \pmod{4}$

This is the case most relevant for application to the cuprates, and displays some rather interesting features, not encountered in any of the cases considered so far.

Now we have to include the ‘staggered flux’ term in Eq. (53). This term has first order time derivatives, and so raises the possibility that the vortices will have a ‘non-relativistic’ dispersion spectrum.

We also have to consider the possible influence of monopole terms,  $\mathcal{L}_m$  here. However, we will see below that the ‘staggered flux’ term in Eq. (53) is already sufficient to reduce the degeneracy of the vortex species from  $2q$  to  $q$ , as required by the degeneracy spectrum for this value of  $q$  from Table 2. Therefore, it is not an unreasonable expectation that the monopole terms will not contribute here. In any case, one can view the following analysis as a diagonalization of the existing terms in the vortex action, which can be useful basis for considering the subsequent possible influence of monopole terms.

It is useful to work with the following parameterization to diagonalize the vortex quadratic form in Eqs. (22) and (53). We work with the four complex fields  $W_m$ ,  $X_m$ ,  $Y_m$ , and  $Z_m$ , with  $m = 0 \dots q/2 - 1$ , defined by

$$\begin{aligned}
\varphi_{1m} &= \frac{W_m + X_m^*}{\sqrt{2h_s}} \\
\varphi_{1,m+q/2} &= (-1)^m \frac{W_m - X_m^*}{\sqrt{2h_s}} \\
\varphi_{2m} &= \frac{Y_m + Z_m^*}{\sqrt{2h_s}} \\
\varphi_{2,m+q/2} &= -(-1)^m \frac{Y_m - Z_m^*}{\sqrt{2h_s}}
\end{aligned} \tag{54}$$

Upon inserting this parameterization into Eqs. (22) and (53), and ignoring second-order time derivative terms which are unimportant at low energies, the quadratic vortex Lagrangian becomes

$$\begin{aligned}
\mathcal{L}_{2\varphi} &= \sum_{m=0}^{q/2-1} \left[ \right. \\
&W_m^* \left( \frac{\partial}{\partial \tau} - iB_{s\tau} - iB_{a\tau} \right) W_m + \frac{1}{h_s} |(\partial_i - iB_{si} - iB_{ai})W_m|^2 + \frac{s}{h_s} |W_m|^2 \\
&+ X_m^* \left( \frac{\partial}{\partial \tau} + iB_{s\tau} + iB_{a\tau} \right) X_m + \frac{1}{h_s} |(\partial_i + iB_{si} + iB_{ai})X_m|^2 + \frac{s}{h_s} |X_m|^2 \\
&+ Y_m^* \left( \frac{\partial}{\partial \tau} - iB_{s\tau} + iB_{a\tau} \right) Y_m + \frac{1}{h_s} |(\partial_i - iB_{si} + iB_{ai})Y_m|^2 + \frac{s}{h_s} |Y_m|^2 \\
&\left. + Z_m^* \left( \frac{\partial}{\partial \tau} + iB_{s\tau} - iB_{a\tau} \right) Z_m + \frac{1}{h_s} |(\partial_i + iB_{si} - iB_{ai})Z_m|^2 + \frac{s}{h_s} |Z_m|^2 \right]
\end{aligned} \tag{55}$$

The full low-energy theory is now the sum of Eqs. (21), (7), (55) and  $\mathcal{L}_{4\varphi}$ .

We note from Eq. (55) that the fields  $X_m$ ,  $W_m$ ,  $Y_m$ , and  $Z_m$  are canonical non-relativistic Bose fields. From their  $B_{s\mu}$  charges, we deduce that the  $W_m$  and  $Y_m$  bosons are vortices, while the  $X_m$  and  $Z_m$  bosons are anti-vortices. There are, therefore, a total of  $q$  flavors of vortices and anti-vortices. The total count of degenerate vortex/anti-vortex excitations is therefore the same as that obtained for the VBS case, where we also had  $q' = q$  for the case where  $q$  was a multiple of 4.

However, the non-relativistic nature of Eq. (55) has important and novel consequences for nature of the fluctuations in the superconductor. For  $s > 0$ , the ground state of Eq. (55) is exactly the vacuum of the  $W_m$ ,  $X_m$ ,  $Y_m$ , and  $Z_m$  bosons. Consequently, virtual quantum fluctuations of low energy vortex-anti-vortex pairs are essentially totally absent in the superconductor—this is in strong contrast to all other cases with a ‘relativistic’ action, where such fluctuations dominate and drive the superconductor-insulator transition.

In the absence of such vacuum fluctuations, the gauge-field interactions be-

tween the vortices and anti-vortices are unscreened, and so will lead to the formation of vortex-anti-vortex bound states which will remain robust as the value of  $s$  is lowered. Indeed, a glance at Eq. (55) shows that we can expect that the lowest energy bound states to form between the  $W_m$  and the  $X_m$  bosons and between the  $Y_m$  and the  $Z_m$  bosons. With the lowering of  $s$ , it is the energy of these bound states which will first cross zero. As the net vorticity of this condensing boson is zero, this is a transition from the superconductor to a *supersolid*. Thus we have reached the remarkable conclusion that the transition out of the sF-doped  $d$ -wave superconductor is necessarily into a supersolid, for the case that  $q$  is a multiple of 4.

We now ask whether the above spectra of low energy vortices and anti-vortices places any restrictions on the nature of density wave order in the supersolid. As in Ref. [16] we can define the density wave order operators  $\rho_{mn}$ , with  $m, n = 0, 1, \dots, q-1$ , at the wavevectors  $Q_{mn} = (2\pi p/q)(m, n)$  by

$$\rho_{mn} = \omega^{mn/2} \sum_{\ell=0}^{q-1} [\varphi_{1\ell}^* \varphi_{1,\ell+n} + \varphi_{2\ell}^* \varphi_{2,\ell+n}] \omega^{\ell m}. \quad (56)$$

These operators follow from the requirement that the PSG transformations in Eq. (52) lead to transformations for  $\rho_{mn}$  required for conventional density wave operators [16]. We now need to insert the parameterization (54) into (56), and thence deduce the nature of  $\rho_{mn}$  fluctuations under Eq. (55). The expressions so obtained are quite lengthy, but we can understand the general result by considering a few representative cases. So for *e.g.*  $q = 4$ , we find

$$\rho_{11} = e^{i\pi/4} (W_0^* W_1 + iW_1^* W_0 + X_1^* X_0 - iX_0^* X_1) + (W \rightarrow Y, X \rightarrow Z), \quad (57)$$

while

$$\rho_{10} = W_0 X_0 + iW_1 X_1 + W_0^* X_0^* + iW_1^* X_1^* + (W \rightarrow Y, X \rightarrow Z). \quad (58)$$

Notice, a crucial difference between the two cases considered above. In Eq. (57) we only have combinations between  $W, X, Y, Z$  creation and annihilation operators; however, it is impossible to annihilate such bosons from the superconductor vacuum, and so can expect that  $\rho_{11}$  fluctuations are strongly suppressed. In contrast,  $\rho_{10}$  involves combinations of operators which are always both creation or annihilation operators; in particular there are terms which lead to the creation of the  $W, X$  and  $Y, Z$  low energy bound states that were noted above. We therefore conclude that  $\rho_{10}$  fluctuations are strongly enhanced as  $s$  is lowered.

These observations can be extended into a simple general result. Let us define the susceptibility  $\chi_{mn}$  as the correlator of  $\rho_{mn}$  and  $\rho_{mn}^* = \rho_{-m, -n}$  at zero

external frequency and momentum. Then, evaluating one-loop  $\chi_{mn}$  under the Lagrangian in Eq. (55) we find

$$\chi_{mn} = 0, \text{ for } m + n \text{ even,} \quad (59)$$

while there is a divergent response at other values of  $m, n$  as  $s \rightarrow 0$ :

$$\begin{aligned} \chi_{mn} &\sim \int \frac{d\Omega d^2k}{8\pi^3} \frac{1}{(\Omega^2 + (k^2 + s)^2/h_s^2)} \\ &\sim \ln(\Lambda/s), \quad \text{for } m + n \text{ odd,} \end{aligned} \quad (60)$$

with  $\Lambda$  an upper cutoff. Unless the above effects are overwhelmed by anomalously large quartic couplings in  $\mathcal{L}_{4\varphi}$ , we conclude that the transition from the doped sF  $d$ -wave superconductor is into a supersolid in which the strongest density modulations are at wavevectors  $Q_{mn}$  with  $m + n$  odd.

We reiterate that while it is plausible that the monopoles do not modify the above conclusions, this has not yet been firmly established.

## 4 Conclusions

This paper has presented a new approach to the physics of doped U(1) spin liquids.

We began with the two most popular examples of U(1) spin liquids on the square lattice at a density of one electron per site. The first, dubbed the VBS state, is expressed as a pure compact U(1) gauge theory; condensation of monopoles leads to confinement of spinons and the appearance of Valence-Bond-Solid (VBS) order at low energies. The second, the staggered flux (sF) state, has 4 species of gapless Dirac fermion spinon excitations. It is possible that these fermions suppress monopole condensation in the spin liquid, so that there is no confinement, fractionalized gapless excitations survive, and there is no broken lattice symmetry. Even if the sF state is unstable, it still forms the basis for a quantum critical point, upon which to base an effective field theory in the spirit of this paper. In any case, it is only *more* stable than the VBS liquid (which definitely is unstable at low energies to long-range VBS order).

Essential characteristics of the VBS and sF states are their transformation properties (the ‘PSG’) of the monopole tunnelling events under the symmetries of the square lattice. The existence of a non-trivial PSG for the monopole led to the appearance of VBS order in phases where the monopoles condense.

Then we doped the spin liquids with holes of rational density  $\delta$ , obeying Eq. (9), which defines the crucial integer-valued parameter,  $q$ . The hole degrees of freedom are most naturally expressed in terms of charge  $e$ , spinless bosons (‘holons’), although this does not imply existence of quasiparticles with these quantum numbers in any phase. The bosons appear in two species, each with density  $\delta/2$ , carrying opposite gauge charges under the U(1) gauge field of the spin liquid. After a duality transformation, the charge degrees of freedom were encapsulated in the dynamics of vortices, each carrying magnetic flux  $hc/(2e)$ . Central to our analysis were the PSG transformations of these vortices, and the couplings between monopoles and vortices that were allowed by the PSG.

By an analysis of the low energy theories allowed by the PSG of the monopoles, vortices, and fermionic spinons, we arrived two main classes of results, which are summarized in the subsections below.

#### 4.1 *Vortex spectrum in the superconductor*

A recurring approach in condensed matter physics is the formulation of effective models in terms of “elementary” excitations. In some cases (e.g. Fermi liquids), such excitations are adiabatically connected to free electrons. Other elementary excitations are particular to specific states of matter, e.g. collective Goldstone modes resulting from symmetry breaking, or topological excitations such as domain walls in one dimension. The set of elementary excitations is usually considered to be a fundamental characteristic of a particular phase of matter. Most theories of quantum critical points begin by assuming the gap of some set of these excitations is tuned parametrically to zero, and the fields of the critical theory are in correspondence with them.

Whenever gapless excitations are present, there is potential for significant ambiguity in the identification of elementary excitations, because new “particles” can be built from collections of a large number of very low energy ones, to form a new basis. This occurs prominently in the theory of one-dimensional systems, where various forms of *bosonization* trade gapless fermionic and bosonic field variables. In this paper, we sought to obtain a description of the elementary excitations of two dimensional superconductors on the square lattice. Clearly this must include both electronic quasiparticles and the vortices. When the former are gapless, we again have potential difficulties of the type described above. This paper presents one approach to deal with this formidable problem.

The approach builds upon a previous paper, [16] in which vortices of superfluid phases of *boson* models on the square lattice were classified. A central result was that bosons of density  $p'/q'$  ( $p'$ ,  $q'$  relatively prime integers) led to vortices with a  $q'$ -fold degeneracy. Unitary transformations within this vortex

‘flavor’ space encoded various space group operators. Furthermore, a vortex with a fixed orientation in flavor space necessarily had static modulations at wavevectors  $(2\pi p'/q')(m, n)$  (with  $m, n$ , integers) in all spin-singlet observables in its vicinity. These modulations can be viewed as a strong coupling analog of Friedel oscillations around impurities in Fermi liquids. Quantum criticality from superfluid to insulating states can be formulated in the  $q$ -vortex variables.

A “direct” attack on the analogous problem in a superconductor would be to try to regard the Cooper pair as the boson of Ref. [16]. This has the difficulty that the Cooper pair field is coupled in a very strong fashion (as in the Bogoliubov-de Gennes equations) to the quasiparticles. The attempt to extricate the collective vortex excitations from the quasiparticles in this approach is quite non-trivial. Instead, we chose to break down the problem in two steps. First, we reformulated the microscopic electronic model using gauge theoretical methods to *fractionalize* the electron into a bosonic charge  $e$  holon and a neutral spin-1/2 spinon. A superconducting state is then obtained as a superfluid state of the holons. This has the advantage that the spinons, which become the quasiparticles in the superconducting state, are coupled *indirectly* to the holons through a gauge field. It is then possible to treat the holons themselves by duality techniques directly analogous to those of Ref. [16].

An interesting wrinkle in this procedure is that many different choices for the initial fractionalization procedure are possible. Although all of these are microscopically equivalent, saddle point and other approximations are inevitably required, leading different choices to more naturally describe different *insulating* states. Thus dependent upon the sort of Mott insulator reached from the superconductor, a particular form of fractionalization may be most appropriate. In the superfluid phase obtained by condensing the holons, however, we believe the same *superconducting* phase can be reached from different fractionalized variables. Therefore it would be natural to expect the same vortex excitations in the superconductor to be obtained from each of these choices. Despite the simplicity of this expectation, its correctness is by no means transparent in the actual calculations. Nevertheless, to the extent we are able to check this, we do indeed find agreement between the different gauge decouplings.

For the case of a superconductor obtained by doping a VBS state, the intermediate steps of our analysis were quite involved, but our final result was simple. The degeneracy and PSG of the low energy vortex excitations were identical to that in pure boson models, with a density of bosons equal to the density of electron pairs *i.e.* the integer  $q'$  is determined here in Eq. (27). Consequently many of the results of Ref. [16] can be applied to this electronic model without modification.

The results for the doped sF state were tentative, pending determination of the monopole terms. However, for the cases  $q = 0, 1, 3 \pmod{4}$  we already obtained a vortex degeneracy identical to the VBS case in Table 2, plausibly suggesting that monopole terms can be ignored for these cases. It seems likely that a proper treatment of the monopoles would give complete correspondence with the VBS case, but this is left unresolved here.

## 4.2 Quantum phase transitions out of the superconductor

With the vortex spectrum in hand, we were able to address quantum phase transitions associated with the condensation of vortices (leading to a transition from the superconductor directly to an insulator) or of vortex-anti-vortex pairs (leading to transition to a supersolid). The critical theory for such a transition depends upon the fate of the fermionic  $S = 1/2$  Bogoliubov quasiparticle excitations of the superconductor. In weak-coupling BCS theory, a  $d$ -wave superconductor has gapless, nodal excitations at four points in the Brillouin zone. If these survive as gapless excitations all the way up to the quantum critical point of interest, then their influence has to be considered in the critical theory, and they could change the universality class. Alternatively, it is possible that the fermionic excitations are gapped at the quantum critical point: in this case, they can be safely integrated out and can be considered irrelevant to the critical theory.

We now list the various quantum critical points obtained in our analysis in the following subsections. We will subdivide the discussion under headers indicating the cases they apply to. We will explicitly write down the leading quadratic terms in the critical theory for each case in these subsections. Higher order couplings are also important, and are strongly constrained by the PSG: for these we refer the reader back to the body of the paper.

### 4.2.1 Superconductor-insulator transition, no gapless fermions

**Doped VBS, all  $q$ :** As parameters are changed in a  $d$ -wave superconductor, it is possible that the electron pairing becomes short ranged, and the fermionic excitation spectrum is fully gapped. Such a scenario also appears plausible from the perspective of a confining VBS insulator that is fully gapped, as it is moving towards a quantum phase transition into a superconductor. In this case, the theory for the superconductor-insulator quantum critical point was found to be identical to that discussed at some length in Ref. [16] for boson models. Note that the notation in Ref. [16] is different from ours here—the integer parameter  $q$  of Ref. [16] should be set equal to the parameter  $q'$  defined here in Eq. (27), which is related to the density of electronic Cooper pairs.

Such a quantum critical point can occur at all values of  $q$  (defined here in Eq. (9)) for the doped VBS spin liquid. The dual vortex theory is expressed in terms of  $q'$  complex scalars  $\bar{\varphi}_m$  and a U(1) gauge field  $B_{s\mu}$ :

$$\mathcal{L}_1 = \frac{K_s}{2} (\epsilon_{\mu\nu\lambda} \partial_\nu B_{s\lambda})^2 + \sum_{m=0}^{q'-1} \left\{ |(\partial_\mu - iB_{s\mu}) \bar{\varphi}_m|^2 + s |\bar{\varphi}_m|^2 \right\} \quad (61)$$

Here, and in all actions below, the action for the  $B_{s\mu}$  field has been written in a schematic relativistic form, which is appropriate for short-range interactions between the bosons—the Coulomb interactions lead to modifications presented in Ref. [30]. The superconductor-insulator transition is accessed by tuning  $s$ .

As discussed in Ref. [16], for certain values of  $q'$  which allow a “permutative” PSG, the critical point can be of the ‘deconfined’ variety [11]: in this case it has a direct formulations in terms of  $q'$  complex scalars,  $\xi_m$ , each carrying electrical charge  $2e/q'$ , and coupled to  $q' - 1$  non-compact U(1) gauge fields,  $\tilde{\mathcal{A}}_\mu^m$ . The action was given in Eq. (38), but the gapped fermionic terms can be dropped:

$$\begin{aligned} \mathcal{L}_2 = & \sum_{m=0}^{q'-1} \left[ |(\partial_\mu - i\tilde{\mathcal{A}}_\mu^m) \xi_m|^2 + \tilde{s} |\xi_m|^2 \right] \\ & + \sum_{m,n=0}^{q'-1} K_{mn} (\epsilon_{\mu\nu\lambda} \partial_\nu \tilde{\mathcal{A}}_\lambda^m) (\epsilon_{\mu\rho\sigma} \partial_\rho \tilde{\mathcal{A}}_\sigma^n), \end{aligned} \quad (62)$$

where the gauge fields obey the constraint in Eq. (39)

$$\sum_{m=0}^{q'-1} \tilde{\mathcal{A}}_\mu^m = 0. \quad (63)$$

#### 4.2.2 Superconductor-insulator transition with gapless nodal fermions

**Doped sF, odd  $q$ :** The simplest case where the nodal fermions survive all the way to a quantum critical point to an insulator was for the doped sF spin liquid, with  $q$  odd (discussed in Section 3.4.1). In this case there is no non-vanishing tri-linear coupling between the monopoles and the vortices, and the derivation of the effective action is straightforward and preserves the lattice PSG at all stages. However, such a derivation has the disadvantage that the nodal points remained pinned at  $(\pm\pi/2, \pm\pi/2)$  even in the finite doping superconductor. An analysis along the lines of Section 2.5 seems necessary to rectify this defect.

There are a total of  $2q$  vortex fields  $\varphi_{1\ell}$  and  $\varphi_{2\ell}$  coupled to 2 U(1) gauge fields,

$B_{s\mu}$  and  $B_{a\mu}$ , and 4 Dirac fermions,  $\Psi$ , coupled to a U(1) gauge field  $\mathcal{A}_\mu$ . The critical action is

$$\begin{aligned}
\mathcal{L}_3 = & \frac{K}{2} (\epsilon_{\mu\nu\lambda} \partial_\nu \mathcal{A}_\lambda)^2 + \frac{K_s}{2} (\epsilon_{\mu\nu\lambda} \partial_\nu B_{s\lambda})^2 + \frac{K_a}{2} (\epsilon_{\mu\nu\lambda} \partial_\nu B_{a\lambda})^2 \\
& + \frac{i}{\pi} \epsilon_{\mu\nu\lambda} \partial_\mu B_{a\nu} \mathcal{A}_\lambda + \sum_{\ell=0}^{q-1} \left\{ |(\partial_\mu - iB_{s\mu} - iB_{a\mu}) \varphi_{1\ell}|^2 \right. \\
& \left. + |(\partial_\mu - iB_{s\mu} + iB_{a\mu}) \varphi_{2\ell}|^2 + s (|\varphi_{1\ell}|^2 + |\varphi_{2\ell}|^2) \right\} \\
& - i\bar{\Psi} \gamma_\mu (\partial_\mu + i\mathcal{A}_\mu) \Psi
\end{aligned} \tag{64}$$

As above, under suitable conditions requiring the existence of a permutative PSG, the critical theory of Eq. (64) can be undualized into a theory of  $2q$  complex scalars  $\xi_m$ , each carrying electromagnetic charge  $e/q$ . The undualized theory is

$$\begin{aligned}
\mathcal{L}_4 = & \sum_{m=0}^{2q-1} \left[ |(\partial_\mu - i\tilde{\mathcal{A}}_\mu^m) \xi_m|^2 + \tilde{s} |\xi_m|^2 \right] + \sum_{m,n=0}^{2q-1} K_{mn} (\epsilon_{\mu\nu\lambda} \partial_\nu \tilde{\mathcal{A}}_\lambda^m) (\epsilon_{\mu\rho\sigma} \partial_\rho \tilde{\mathcal{A}}_\sigma^n) \\
& - i\bar{\Psi} \gamma_\mu \left( \partial_\mu + \frac{i}{2} \sum_{m=0}^{q-1} \tilde{\mathcal{A}}_\mu^m - \frac{i}{2} \sum_{m=q}^{2q-1} \tilde{\mathcal{A}}_\mu^m \right) \Psi,
\end{aligned} \tag{65}$$

where the gauge fields still obeys the constraint

$$\sum_{m=0}^{2q-1} \tilde{\mathcal{A}}_\mu^m = 0. \tag{66}$$

**Doped VBS,  $q \neq 2 \pmod{4}$ :** For the doped VBS case, we have already discussed a situation with the absence of gapless fermions at the quantum critical point in Section 4.2.1. However, in Sections 2.5 and 2.6, we presented plausible conditions under which the gapless nodal fermions could survive at the quantum critical point in this case too. With  $q'$  defined as in Eq. (27), it was required that  $q'$  be even, and the critical theory was the same as  $\mathcal{L}_3$  or  $\mathcal{L}_4$ , but with  $q$  replaced by  $q'/2$ . A more detailed direct lattice study of the PSG of fractionalization would be useful to firmly establish this scenario.

**Doped sF, even  $q$ :** For the doped sF case,  $q = 0 \pmod{4}$ , was found not to exhibit a superconductor-insulator transition in Section 3.4.3; rather a superconductor-supersolid transition obtains, which will be noted in the following subsection. The case  $q = 2 \pmod{4}$  is likely to have monopole terms, and so we are not able to reach any firm conclusions for this case.

### 4.2.3 Superconductor-supersolid transition

All the cases considered in this paper allow a superconductor-supersolid transition: this can happen if a vortex-anti-vortex pair condensate appears before the condensation of single vortices. This pair condensate transforms just like a conventional Landau density-wave or VBS order parameter, and the critical theory can be developed in a traditional order parameter framework. Such critical theories for the superconductor-supersolid transition were discussed at some length in Refs. [31] and [10].

**Doped sF,  $q = 0 \pmod{4}$ :** One of our surprising results was that a superconductor-supersolid transition was not merely optional for a particular case, but required as the first transition out of the translationally invariant superconductor (this conclusion is based upon an assumption on neglect of monopoles which has not been firmly established). This case was for the doped sF state with  $q = 0 \pmod{4}$ . We showed in Section 3.4.3 that the ‘non-relativistic’ nature of the vortex action in this case promoted the formation of vortex-anti-vortex bound states which would condense first. The PSG of the vortex theory also placed some unusual constraints on the nature of the ordering in the supersolid: unless some higher order couplings were anomalously large, it was found that the supersolid in the doped sF case could have density modulations only at wavevectors  $(2\pi p/q)(m, n)$  with  $m + n$  odd.

### 4.3 Discussion

To conclude, we have found a remarkable richness in the low energy spectra and quantum phase transitions of a “conventional” two-dimensional  $d$ -wave superconductor. The vortices of the superconducting state, when considered as bona fide, quantum-mechanical, quasiparticle excitations, can have surprising variety of wavefunctions on the lattice. These wavefunctions encapsulate the structure of conventional density-wave or VBS orders in proximate phases. Combining the quantum mechanics of these vortex quasiparticles with the fermionic Bogoliubov quasiparticles is a problem of considerable complexity: this paper has described such theories in two of the simplest cases, and the results are summarized above.

We conclude by briefly discussing connections to experiments. The many experimental observations [34,35,36,37,38,39,40,41,42,43] of periodic modulations in the LDOS or spin excitation spectra clearly call for attention to such competing orders: we have shown here, following Ref. [16], how such weak modulations appear naturally as an inevitable, but ancillary, consequence of quantum fluctuations of the vortices. Recent observations [44] of the elec-

tronic spectrum of such modulated states show evidence for gapless nodal fermions: this possibly points to proximity to deconfined critical points with such fermions discussed above. For the doped sF case, we also found an unexpected selection of wavevectors of the modulations, which was just noted above: for  $\delta = 1/8$ , the density modulations were dominant at wavevectors  $(\pi/8)(m, n)$  with  $m + n$  odd. This feature of the sF case appears to be in conflict with existing observations [39,42].

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