

Effective action for vortex dynamics in clean d -wave superconductors

Predrag Nikolić and Subir Sachdev

Department of Physics, Harvard University, Cambridge MA 02138, USA

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We describe the influence of the gapless, nodal, fermionic quasiparticles of a two-dimensional d -wave superconductor on the motion of vortices. A continuum, functional formalism is used to obtain the effective vortex action, after the fermions have been integrated out. At zero temperature (T), the leading terms in the vortex action retain their original form, with only a finite renormalization of the vortex effective mass from the fermions. A universal “sub-Ohmic” damping of the vortex motion is also found. At $T > 0$, we find a Bardeen-Stephen viscous drag term, with a universal co-efficient which vanishes as $\sim T^2$. We present a simple scaling interpretation of our results, in which quantum-critical Dirac fermions respond to a moving point singularity. Our results appear to differ from those of the semiclassical theory, which obtains more singular corrections to a vortex mass appearing in transport equations.

I. INTRODUCTION

The cuprate superconductors behave like conventional BCS superconductors in many respects, albeit with a d -wave symmetry of the Cooper pair wavefunction. The BCS wavefunction provides a reasonable description of the ground state. The elementary excitations above the ground state are

- (i) fermionic, $S = 1/2$, Bogoliubov quasiparticles,
- (ii) plasmons (or “phase fluctuations”), and
- (iii) vortices with magnetic flux $hc/(2e)$.

For the traditional “low temperature” superconductors, the vortices are usually treated as a classical, configuration of the superconducting order parameter, which forms a background for the quantum fluctuations of the fermionic quasiparticles and the Cooper pairs: the quantum zero-point motion of the vortices themselves is negligibly small.

Quantum fluctuations of the vortices are expected to play a more fundamental role in the cuprate superconductors, requiring that the vortices be treated as bona fide quasiparticles and elementary excitations in their own right. There are a variety of experimental indications^{1,2,3,4,5,6,7} that the superconducting state is proximate to an insulating ground state. Vortices and anti-vortices are expected to proliferate in the vicinity of such a transition, and so the energy gap towards creation of vortex/anti-vortex pairs in the superconductor becomes vanishingly small as the superconductor-insulator transition is approached. Moreover, the superconductor coherence length, ξ , is at most a few lattice spacings, which suggests that the effective mass of the vortices is small.

It is the purpose of this paper to discuss some features of the effective action controlling interplay between the three classes of elementary excitations (noted above) of a two-dimensional d -wave superconductor. If we neglect the fermionic $S = 1/2$ quasiparticles, the required action can be readily deduced from the theory of boson-vortex duality^{8,9,10}, as we will review below. We will primarily be concerned here on the influence of the gapless, nodal, fermionic quasiparticles on the vortices. The interaction

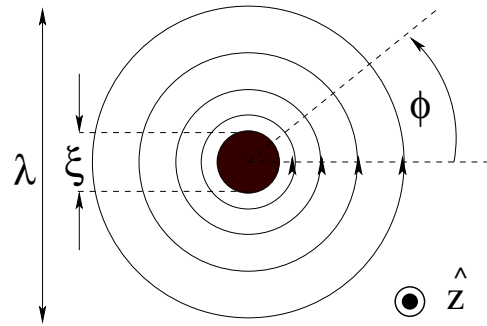


FIG. 1: Scales of a vortex. The core size is of the order of superconducting coherence length ξ , while the extent of supercurrent circulations is roughly equal to the much larger London penetration depth λ . This paper will work in the limit in which $\lambda \rightarrow \infty$.

between the fermions and the vortices is non-local, with no natural expansion parameter, and so its description poses a strong-coupling problem. Here, we will consider a single, isolated, slowly moving vortex, and exactly evaluate the functional integral over the fermionic degrees of freedom by a combination of analytic and numerical techniques; this yields a renormalization of the effective action for the vortices and the plasmons. Our primary result will be that despite the absence of an energy gap in the fermionic spectrum, the influence of the fermionic degrees of freedom is relatively innocuous, and at zero temperature (T) the leading terms in the vortex action retain the form deduced while ignoring the fermionic quasiparticles.

The interplay between fermions and moving vortices has also been examined previously in semiclassical framework by a number of workers.^{11,12,13,14} These theories are expansion in the ratio of the superconducting pairing energy to the Fermi energy, a ratio which is not small in the cuprate superconductors. We will not make any assumptions on the value of this ratio in the present paper. The results of semiclassical theory are expressed in a different framework of transport equations, and this makes direct comparison with our results difficult. An impor-

tant characteristic of the semiclassical results is that the effective mass of a vortex (as defined in their transport equations) in a d -wave superconductor diverges linearly with the spacing between vortices. We do not find any corresponding divergence in the vortex mass in our functional formalism. The reasons for this potential discrepancy between our and the semiclassical results are not completely clear to us. We speculate it may be due to the breakdown of the semiclassical approximation in describing the low energy nodal fermionic quasiparticles. Such fermionic excitations are not semiclassical, but, with their scale-invariant Dirac spectrum, better characterized as a quantum-critical system.¹⁵

We begin our presentation by reviewing the action for vortices and plasmons as derived in theories of the boson-vortex duality,^{8,9,10} while ignoring fermionic excitations of the superconductor. We use a first-quantized approach for the vortex degrees of freedom, and treat each vortex as a quantum particle at spacetime position $\mathbf{r}_v(\tau)$, where τ is imaginary time; the quantum mechanics of the vortices is described by a functional integral over $\mathbf{r}_v(\tau)$. The density or “phase” fluctuations of the superconductor are represented by a U(1) gauge field $\mathcal{A}_\mu(\mathbf{r}, \tau)$, where the spacetime index μ extends over x, y, τ : the ‘magnetic’ flux of this gauge field is proportional to the particle density of the superconductor, while the ‘electric’ field determines the local superflow velocity. A vortex (anti-vortex) is minimally coupled to the U(1) gauge fields with charge $+1$ (-1). So a single vortex and the phase fluctuations are described by the action $\mathcal{S} = \mathcal{S}_v + \mathcal{S}_A$ where

$$\mathcal{S}_v = \int d\tau \left[\frac{m_v^0}{2} \left(\frac{d\mathbf{r}_v(\tau)}{d\tau} \right)^2 + i \frac{d\mathbf{r}_v(\tau)}{d\tau} \cdot \vec{\mathcal{A}}^0(\mathbf{r}_v(\tau)) + i \int d^2\mathbf{r} d\tau \mathcal{A}_\mu(\mathbf{r}, \tau) J_{v\mu}(\mathbf{r}, \tau) \right] \quad (1)$$

is the vortex action and its coupling to the U(1) gauge field. Here m_v^0 is the bare vortex mass (which can be viewed as arising from high energy degrees of freedom which have been integrated out), $\vec{\mathcal{A}}^0$ represents the ‘background’ flux which is responsible for the Magnus force on the vortices. In a Galilean-invariant superfluid, $\nabla \times \vec{\mathcal{A}}^0 = \hat{z}2\pi(\text{density of Cooper pairs})$. In the presence of a lattice, when the superconductor is proximate to a Mott insulator, it has been argued^{16,17,18,19} that the Magnus force is reduced, and the density of pairs in the Mott insulator has to be subtracted from the density of Cooper pairs. In the last terms in \mathcal{S} , $J_{v\mu}$ is the vortex 3-current, and its temporal and spatial components are given by

$$\begin{aligned} J_{v\tau}(\mathbf{r}, \tau) &= \delta(\mathbf{r} - \mathbf{r}_v(\tau)) \\ \mathbf{J}_v(\mathbf{r}, \tau) &= \delta(\mathbf{r} - \mathbf{r}_v(\tau)) \frac{d\mathbf{r}_v(\tau)}{d\tau} \end{aligned} \quad (2)$$

The phase fluctuations of the superconductor are controlled by the action \mathcal{S}_A which, in the Coulomb gauge

$\nabla \cdot \vec{\mathcal{A}} = 0$, is

$$\mathcal{S}_A = \int \frac{d^2k d\omega}{8\pi^3} \left(\frac{1}{8\pi^2 \rho_s} [k^2 |\mathcal{A}_\tau(k, \omega)|^2 + \omega^2 |\mathcal{A}_i(k, \omega)|^2] + \frac{e^{*2} k}{4\pi} \left[\delta_{ij} - \frac{k_i k_j}{k^2} \right] \mathcal{A}_i(-k, -\omega) \mathcal{A}_j(k, \omega) \right). \quad (3)$$

We have Fourier transformed to momentum, k , and imaginary frequencies, ω , $e^* = 2e$ is the charge of a Cooper pair, and ρ_s is the superfluid stiffness in units of energy (the London penetration depth, λ , is related to ρ_s by $\rho_s = \hbar^2 c^2 d / (16\pi e^2 \lambda^2)$, where d is the interlayer spacing). The first term in \mathcal{S}_A is responsible for the log interaction between the vortices; the coefficient of this term is fixed by requiring that the log interaction have the required prefactor. The second term represents the electrical Coulomb interaction between the charged Cooper pairs; its coefficient is fixed by the knowledge that $(\nabla \times \vec{\mathcal{A}})/(2\pi)$ is the fluctuation in the number density of pairs of charge e^* . The ‘photon’ excitations of the full action \mathcal{S}_A are the plasmons of the two-dimensional superconductor with dispersion $\omega \sim \sqrt{k}$, and these plasmons can be ‘radiated’ by a moving vortex.

It is worth pointing out here a noteworthy aspect of the action for vortices and plasmons presented here in Eqs. (1) and (3). Although obtained by an approximate and uncontrolled boson-vortex duality mapping, most of the coupling constants in the final action can be fixed rather precisely. The primary unknown is the vortex mass, m_v , and we will describe microscopic contributions to its value this paper.

Clearly, the physics described by $\mathcal{S}_v + \mathcal{S}_A$ is similar to the quantum electrodynamics of a charged particle. The coupling to the phase fluctuations will renormalize the vortex mass, m_v , and lead to ‘radiation damping’ in the motion of a vortex. We review this physics in Section II. For a neutral superfluid, it has been argued^{20,21} that the phase fluctuations lead to a logarithmically divergent “dynamic mass” of the vortex; we show how the analog of this result obtains in our formalism in Section II. Section II also considers the case of a charged superfluid, with plasmon excitations as in Eq. (3): for this case, we find that the phase fluctuations are less singular and lead only to a *finite* renormalization of the vortex “dynamic mass”.

Our purpose here is to extend $\mathcal{S}_v + \mathcal{S}_A$ to include the remaining elementary excitations of the d -wave superconductor, the nodal fermionic quasiparticles. (The complete action controlling the interactions between all the elementary excitations of a d -wave superconductor is^{22,23,24,25} $\tilde{\mathcal{S}}_v + \mathcal{S}_A + \mathcal{S}_\Psi + \mathcal{S}_{cs}$ specified in Eqs. (61), (3), (63), and (64)). We will then integrate out these quasiparticles, and associated gauge fields (including \mathcal{A}_μ), in the presence of a single, slowly moving vortex, and obtain resulting renormalization of \mathcal{S}_v . For small $\mathbf{r}_v(\omega)$, the effective vortex action can be written in the form (apart from terms responsible for the Magnus force associated

with $\vec{\mathcal{A}}^0$ in Eq. (1)

$$\tilde{\mathcal{S}}_v = \int \frac{d\omega}{2\pi} F_{\parallel}(\omega) |\mathbf{r}_v(\omega)|^2 + \dots \quad (4)$$

For small ω , we find for a charged d -wave superconductor at $T = 0$ that the contribution of the nodal quasiparticles to $F_{\parallel}(\omega)$ can be written as

$$v_F^2 F_{\parallel}(\omega) = \omega^2 \Lambda F_1(v_{\Delta}/v_F) - |\omega|^3 F_2(v_{\Delta}/v_F), \quad (5)$$

where v_F and v_{Δ} are the two velocities of the nodal quasiparticles, Λ is a high energy cutoff, and $F_{1,2}$ are functions of the ratio of the velocities. Comparing with Eq. (1), we see that the first term can be interpreted as finite renormalized mass $m_v = \Lambda F_1(v_{\Delta}/v_F)/(2v_F^2)$ of the vortex. The function F_1 is not universal, and should clearly depend upon the nature of the cutoff; however, no other coupling constant appears in the value of F_1 . The second $|\omega|^3$ term in Eq. (5) represents a ‘‘sub-Ohmic’’ damping of the vortex motion. We will show that this damping has a *universal* strength *i.e.* F_2 is a universal function of v_{Δ}/v_F , and so the damping is entirely determined by the two quasiparticle velocities. (For completeness, we note that in a neutral d -wave superfluid, as we will demonstrate in Section II, there is an additional $\omega^2 \ln(\Lambda/|\omega|)$ term in $F_{\parallel}(\omega)$: this term is the manifestation, in the present formalism, of the logarithmically divergent ‘‘dynamic mass’’ of the vortex^{20,21}, noted above.)

It is interesting to note here that the expression for $F_{\parallel}(\omega)$ in Eq. (5) could have been deduced very simply and directly by dimensional arguments, after assuming that the nodal fermions are properly thought of as a ‘quantum critical’ system with dynamic exponent $z = 1$, which then responds to the moving vortex. In this scaling argument, the vortex position $\mathbf{r}_v(\tau)$ scales as a length, and so $\dim[\mathbf{r}_v(\tau)] = -1$. After a Fourier transformation, $\dim[\mathbf{r}_v(\omega)] = -2$. Counting dimensions in the action, we conclude then that $\dim[F_{\parallel}(\omega)] = 3$. Also, we know from translational invariance that $F_{\parallel}(0) = 0$. So apart from a universal $|\omega|^3$ term, $F_{\parallel}(\omega)$ can have a cutoff-dependent, analytic ω^2 term. The dependence of $F_{\parallel}(\omega)$ on the velocities follows from ensuring consistency with engineering dimensions. The success of this argument demonstrates the importance of thinking about the fermions in terms of the scale-invariant critical field theory that obeys hyperscaling properties. As we noted earlier, we believe that it is this criticality which is not properly captured by the semiclassical analysis.^{11,12,13,14}

This scaling argument, and our explicit computations, show that no $|\omega|$ ‘‘Ohmic’’ damping term is generated in $F_{\parallel}(\omega)$ at $T = 0$. If present, such a term would contribute the familiar Bardeen-Stephen damping²⁶ to the vortex equations of motion. If the vortex mass had the degree of infrared divergence claimed in the semiclassical theory^{11,12,13,14} *i.e.* a mass which diverges linearly with length scale, then this would be equivalent, in our language, to a scaling dimension $\dim[F_{\parallel}(\omega)] = 1$, and would imply that a $|\omega|$ term should have appeared in our vortex

action. However, because we find $\dim[F_{\parallel}(\omega)] = 3$, all our integrals have a degree of convergence in the infrared to allow safe expansion in powers of ω upto order ω^2 , and to prevent the appearance of a $|\omega|$ term.

In the real experimental system, our results show that any Bardeen-Stephen damping must come from other sources: a non-zero T , or from impurities which induce a non-zero density of fermionic states at zero energy. In particular, we show in Section VD that at $T > 0$, the function $F_{\parallel}(\omega)$ contains a Bardeen-Stephen term

$$F_{\parallel}(\omega) = \frac{\eta}{2} |\omega| + \dots \quad (6)$$

where the viscous drag co-efficient is universal in the clean limit:

$$\eta = \frac{(k_B T)^2}{v_F^2} F_{\eta}(v_{\Delta}/v_F), \quad (7)$$

with F_{η} a universal function of the velocity ratio. We are able to compute one contribution to η exactly: that due to the statistical phase factor acquired by the fermions upon encircling the vortex—the result appears in Eq. (58). There is a second contribution due to the Doppler shift which requires a numerical analysis.

We will begin by setting up the general formalism^{27,28,29,30} under which the effective action of a single vortex can be derived in Section III. The nature of the coupling between the fermionic quasiparticles and the vortices and the plasmons will then be reviewed in Section IV; here we will follow the framework of Tešanović and others.^{31,32,33,34,35}

With the basic formalism in place, it then remains to explicitly evaluate the functional determinant of the nodal fermions. There is no natural small parameter in this determinant, and so an exact evaluation is required. Nevertheless, it is useful to examine the structure of the perturbation theory order-by-order in the non-linearities; this will be done in Section VI. We will find that the perturbation theory has the scaling structure discussed near Eq. (5).

The non-perturbative evaluation of the fermion determinant is presented in Sections V and VII. In Section V we neglect the ‘Doppler shift’ of the fermion energy in the presence of a superflow, and retain only the ‘Berry phase’ acquired by the fermions upon encircling a vortex; we show that an analytic evaluation of the effective mass renormalization is possible in this limit. Section VII accounts for the Doppler shift in a numerical computation. These non-perturbative results are found to be entirely consistent with the expectation of the perturbation theory discussed in Section VI in that they also has the scaling structure of Eq. (5).

Finally, conclusions and implications of our results appear in Section VIII, and some technical details and supporting calculations appear in the appendices.

II. “DYNAMIC” VORTEX MASS

This section takes a slight detour. We will review earlier computations on the influence of phase fluctuations on the vortex mass. Consideration of the influence of nodal quasiparticles will begin in Section III.

Popov²⁰ and Duan²¹ have argued that vortices acquire a “dynamic” mass from their coupling to quantum fluctuations of the phase of the superconducting order, and that this contribution diverges logarithmically for a neutral superfluid. We now present the analog of this effect in our formalism, and show further that the divergence is absent in charged superfluids.

We begin with a discussion of neutral superfluids. For this case, in the action for the phase fluctuations in Eq. (3), the pre-factor k in the last term is replaced by k^2 : this ensures only short-range interaction proportional to the square of the local density fluctuation $(\nabla \times \vec{\mathcal{A}})/(2\pi)$. After a suitable rescaling of coordinates to absorb factors of the velocity of spin-waves, the resulting action can be written in the compact ‘relativistic’ form

$$\mathcal{S}_{\mathcal{A}n} = \frac{c}{2} \int \frac{d^3 p}{8\pi^3} \mathcal{A}_\mu (\delta_{\mu\nu} p^2 - p_\mu p_\nu) \mathcal{A}_\nu, \quad (8)$$

where c is a coupling constant and $p_\mu = (\omega, \mathbf{k})$ is a space-time 3-momentum. Now we can use \mathcal{S}_v in Eq. (1) and integrate out the \mathcal{A}_μ fluctuations under the action $\mathcal{S}_{\mathcal{A}n}$. In this manner, we obtain the following additional term in the vortex effective action

$$\begin{aligned} \mathcal{S}_{v\mathcal{A}} &= \frac{1}{2c} \int d\tau d\tau' \int \frac{d^2 k d\omega}{8\pi^3} \frac{e^{-i\omega(\tau-\tau')}}{(\omega^2 + k^2)} \\ &\times \left(1 + \frac{d\mathbf{r}_v(\tau)}{d\tau} \frac{d\mathbf{r}_v(\tau')}{d\tau'} \right) e^{i\mathbf{k} \cdot (\mathbf{r}_v(\tau) - \mathbf{r}_v(\tau'))} \end{aligned} \quad (9)$$

Integrating over k and ω , we obtain for large $\tau - \tau'$

$$\begin{aligned} \mathcal{S}_{v\mathcal{A}} &\sim \int d\tau d\tau' \left(1 + \frac{d\mathbf{r}_v(\tau)}{d\tau} \frac{d\mathbf{r}_v(\tau')}{d\tau'} \right) \\ &\times \frac{1}{[(\mathbf{r}_v(\tau) - \mathbf{r}_v(\tau'))^2 + (\tau - \tau')^2]^{1/2}} \end{aligned} \quad (10)$$

As we will repeatedly do in all the computations of this paper, we expand this effective action for $\mathbf{r}(\tau)$ in powers of a slowly varying $\mathbf{r}_v(\omega)$. This yields one term with the structure

$$\mathcal{S}_{v\mathcal{A}} \sim \int d\omega \omega^2 |\mathbf{r}_v(\omega)|^2 \int_{\Lambda^{-1}}^{\infty} d\tau \frac{\cos(\omega\tau)}{\tau} \quad (11)$$

(where Λ is a high energy cutoff), and another with the structure

$$\mathcal{S}_{v\mathcal{A}} \sim \int d\omega |\mathbf{r}_v(\omega)|^2 \int_{\Lambda^{-1}}^{\infty} d\tau \frac{(1 - \cos(\omega\tau))}{\tau^3}. \quad (12)$$

Both terms yield a contribution to $F_{||}(\omega)$ in Eq. (4) which is $\sim \omega^2 \ln(\Lambda/|\omega|)$. This logarithm is an indication of

the divergence of the dynamic mass, as found earlier in Refs. 20,21.

Let us now extend the above analysis to the charged case of interest to the cuprates, where the \mathcal{A}_μ action is in Eq. (3). Now it is convenient to work in the Coulomb gauge, when the only potentially singular contribution to the vortex action comes from the fluctuations of the spatial components $\vec{\mathcal{A}}$. In this case, Eq. (9) is replaced by

$$\begin{aligned} \mathcal{S}_{v\mathcal{A}} &= \frac{\pi}{e^*2} \int d\tau d\tau' \int \frac{d^2 k d\omega}{8\pi^3} \frac{e^{-i\omega(\tau-\tau')}}{(k + \omega^2/(2\pi\rho_s e^*2))} \\ &\times \frac{dr_{vi}(\tau)}{d\tau} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \frac{dr_{vj}(\tau')}{d\tau'} e^{i\mathbf{k} \cdot (\mathbf{r}_v(\tau) - \mathbf{r}_v(\tau'))} \end{aligned} \quad (13)$$

As above, we expand in powers of $\mathbf{r}_v(\omega)$, and perform the momentum and frequency space integrals to obtain an expression like that in Eq. (11), but with the $1/\tau$ under the τ integral replaced by $1/\tau^3$. This τ integral is convergent in the infrared, and hence we obtain a ω^2 term in $F_{||}(\omega)$ with a finite co-efficient *i.e.* a *finite* renormalization of the vortex mass.

III. EFFECTIVE VORTEX ACTION

We begin by deriving a general effective action for a single vortex.^{27,28,29,30} We will focus on the contributions of the fermionic quasiparticles, with the aim of renormalizing the action $\mathcal{S}_v + \mathcal{S}_{\mathcal{A}}$ presented in Section I. The vortex will be treated as a singular configuration of external gauge and supercurrent fields in the quasiparticle Hamiltonian. We write a path-integral for quasiparticles and immediately integrate them out, which gives us the vortex action as a functional of the time-dependent vortex location. This is done in an expansion in the vortex velocity. Formally, for this procedure to be well defined, quasiparticles must be massive. Therefore, we will initially give a finite mass to the nodal quasiparticles, and send it to zero later when it becomes safe.

The vortex action is the logarithm of the fermion determinant in the path-integral:

$$S_v = -\text{tr} \log G^{-1}. \quad (14)$$

Here, the time-ordered quasiparticle Green’s function in imaginary time satisfies:

$$\left(\frac{\partial}{\partial\tau} + H \right) G(\mathbf{r}, \tau; \mathbf{r}', \tau') = \delta(\tau - \tau') \delta^2(\mathbf{r} - \mathbf{r}') I, \quad (15)$$

where I is the unit-matrix for spin degrees of freedom. In a d -wave superconductor there are four flavors of nodal quasiparticles - their contributions must be calculated separately and added to the vortex action.

The quasiparticle Hamiltonian H , which we discuss in the next section, depends on the vortex position $\mathbf{r}_v(\tau)$.

We expand it to the second order in small vortex displacements from the origin:

$$\begin{aligned} H &= H_0 + (\mathbf{r}_v \nabla_v) H_0 + \frac{1}{2} (\mathbf{r}_v \nabla_v)^2 H_0 + \dots \quad (16) \\ &= H_0 + \Delta H(\mathbf{r}_v) , \end{aligned}$$

where $\nabla_v = \frac{\partial}{\partial \mathbf{r}_v}$. We also expand the Green's function, and write in the operator notation:

$$G^{-1} = G_0^{-1} + \Delta H , \quad (17)$$

with G_0 being the unperturbed Green's function (vortex sitting at the origin). Now expand (14):

$$S_v = -\text{tr} \log G_0^{-1} - \text{tr}(G_0 \Delta H) + \frac{1}{2} \text{tr}(G_0 \Delta H)^2 + \dots \quad (18)$$

The first term is a constant and we drop it. In order to evaluate the other terms, we substitute the solution of (15) for the unperturbed Green's function:

$$G_0(\mathbf{r}, \tau; \mathbf{r}', \tau') = \sum_n \frac{1}{\beta} \sum_\omega \frac{e^{-i\omega(\tau-\tau')}}{-i\omega + \epsilon_n} \psi_n(\mathbf{r}) \psi_n^\dagger(\mathbf{r}') . \quad (19)$$

The wavefunctions ψ_n and energies ϵ_n must be solutions of the Schrödinger equation in position representation:

$$H_0(\mathbf{r}) \psi_n(\mathbf{r}) = \epsilon_n \psi_n(\mathbf{r}) , \quad (20)$$

and frequencies ω take discrete values $(2k+1)\pi T$ at finite temperature $T = 1/\beta$. Clearly, the vortex dynamics can be derived entirely from the knowledge of quasiparticle energies and wavefunctions in presence of the vortex at rest.

The second term of (18) expanded to the second order in vortex position is:

$$-\text{tr}(G_0 \Delta H) = -\text{tr}[G_0(\mathbf{r}_v \nabla_v) H_0] - \frac{1}{2} \text{tr}[G_0(\mathbf{r}_v \nabla_v)^2 H_0] .$$

In absence of disorder, translational symmetry requires that there be no linear terms in $\mathbf{r}_v(\tau)$ in the vortex action. This is explicitly demonstrated in Appendix A. At the second order in \mathbf{r}_v we have:

$$\begin{aligned} -\frac{1}{2} \text{tr}[G_0(\mathbf{r}_v \nabla_v)^2 H_0] &= \quad (21) \\ &= - \int d\tau d^2 r \text{tr}[G_0(\mathbf{r}, \tau; \mathbf{r}, \tau) (\mathbf{r}_v(\tau) \nabla_v)^2 H_0(\mathbf{r})] . \end{aligned}$$

Since there is only one time variable in this integral, it produces a ‘‘trapping’’ potential $\sim |\mathbf{r}_v(\tau)|^2$, which seems to trap the vortex at the origin even in absence of disorder. This term will, therefore, be cancelled in the complete vortex action, as we will see below.

We are left with the third term of (18):

$$\begin{aligned} \frac{1}{2} \text{tr}(G_0 \Delta H)^2 &= \quad (22) \\ &= \int d\tau d^2 r \int d\tau' d^2 r' \text{tr} \left[G_0(\mathbf{r}, \tau; \mathbf{r}', \tau') (\mathbf{r}_v(\tau') \nabla_v H_0(\mathbf{r}')) \right. \\ &\quad \left. G_0(\mathbf{r}', \tau'; \mathbf{r}, \tau) (\mathbf{r}_v(\tau) \nabla_v H_0(\mathbf{r})) \right] . \end{aligned}$$

Let us utilize the absence of disorder and introduce a convenient notation for the following matrix elements:

$$\begin{aligned} U_{n,n'} &= - \int d^2 r \psi_n^\dagger(\mathbf{r}) \nabla_v \psi_{n'}(\mathbf{r}) \quad (23) \\ &= \int d^2 r \psi_n^\dagger(\mathbf{r}) \nabla \psi_{n'}(\mathbf{r}) . \end{aligned}$$

If we substitute (19) into the trace (22), integrate out the time variables and partially sum over frequencies (details of which are in Appendix A), we obtain a simple expression:

$$\begin{aligned} \frac{1}{2} \text{tr}(G_0 \Delta H)^2 &= \quad (24) \\ &= \frac{1}{2} \sum_{n,n'} \frac{1}{\beta} \sum_\omega \frac{f(\epsilon_n) - f(\epsilon_{n'})}{\epsilon_n - \epsilon_{n'} - i\omega} (\epsilon_n - \epsilon_{n'})^2 |\mathbf{r}_v(\omega) \mathbf{U}_{n,n'}|^2 , \end{aligned}$$

where $f(\epsilon) = 1/(1 + e^{\beta\epsilon})$ is the Fermi-Dirac distribution function. Note that in the last expression ω already describes vortex motion - it is a ‘‘bosonic’’ Matsubara frequency, taking values $2k\pi T$ at finite temperatures. This expression is not zero at $\omega = 0$, which implies its non-invariance under translations, even in absence of disorder. However, the trapping potential encountered earlier can now be exactly cancelled. A simple recipe, which is rigorously derived in Appendix A, is to subtract from (24) its zero-frequency part. This way we obtain the complete effective vortex action:

$$\begin{aligned} S_v &= \frac{1}{2} \sum_{n,n'} \frac{1}{\beta} \sum_\omega (f(\epsilon_n) - f(\epsilon_{n'})) \frac{i\omega(\epsilon_n - \epsilon_{n'})}{\epsilon_n - \epsilon_{n'} - i\omega} \times \\ &\quad |\mathbf{r}_v(\omega) \mathbf{U}_{n,n'}|^2 . \quad (25) \end{aligned}$$

The vortex action must be invariant under all lattice symmetries of the superconducting material. Translational symmetry is already guaranteed, while the 90° degree rotation symmetry of the square lattice is implemented only after the contributions from all quasiparticle nodes are included. Symmetry under 90° rotations generates the same requirements on the vortex action at the second order in vortex displacement as the symmetry under any rotation. The allowed rotationally invariant terms are therefore $|\mathbf{r}_v(\omega)|^2$ and $i\hat{z}(\mathbf{r}_v^*(\omega) \times (\mathbf{r}_v(\omega)))$, and we may write the action as:

$$S_v = \frac{1}{\beta} \sum_\omega \left[F_{\parallel}(\omega) |\mathbf{r}_v(\omega)|^2 + F_{\perp}(\omega) i\hat{z}(\mathbf{r}_v^*(\omega) \times (\mathbf{r}_v(\omega))) \right] . \quad (26)$$

The first term describes vortex mass and dissipation, while the second term breaks the time-reversal symmetry and describes possible Magnus-type forces on the vortex arising from presence of quasiparticles.

IV. QUASIPARTICLE HAMILTONIAN

In this section we derive a convenient linearized form of the quasiparticle Hamiltonian that will be used to find

the effective vortex action. While our focus will be the Bogoliubov-de Gennes (BdG) description of quasiparticles, we note that similar results can also be obtained from the “staggered-flux” model of quasiparticles in d -wave superconductors.

The BdG Hamiltonian for an extreme type-II d -wave superconductor is:

$$H_{\text{BdG}} = \begin{pmatrix} \frac{(\mathbf{p} - \frac{e}{c}\mathbf{A})^2}{2m} - E_F & \Delta \\ \Delta^* & -\frac{(\mathbf{p} + \frac{e}{c}\mathbf{A})^2}{2m} + E_F \end{pmatrix}, \quad (27)$$

where \mathbf{p} is the momentum operator, \mathbf{A} gauge field, $E_F = \hbar^2 k_F^2 / 2m$ quasiparticle Fermi energy, and Δ gap operator given by:

$$\begin{aligned} \Delta &= \frac{1}{\hbar^2 k_F^2} \{p_x, \{p_y, \Delta(\mathbf{r})\}\} - \frac{i}{4k_F^2} \Delta(\mathbf{r}) \{\partial_x, \partial_y\} \Phi(\mathbf{r}) \\ &= \frac{\Delta(\mathbf{r})}{\hbar^2 k_F^2} \left\{ p_x + \frac{\hbar \partial_x \Phi(\mathbf{r})}{2}, p_y + \frac{\hbar \partial_y \Phi(\mathbf{r})}{2} \right\}. \end{aligned} \quad (28)$$

$\Delta(\mathbf{r})$ is the superconducting center-of-mass complex gap function, with phase $\Phi(\mathbf{r})$, and amplitude Δ_0 assumed to be uniform (except at vortex cores). The braces denote anticommutators defined as $\{a, b\} = \frac{1}{2}(ab + ba)$. This Hamiltonian formally has d_{xy} symmetry, which is related to $d_{x^2-y^2}$ by a rotation.

We follow the approach from Ref. 33 and 32. It was developed for a vortex lattice, but we will use it in the limit of infinite inter-vortex separation. A vortex lattice can be partitioned into two sublattices, A and B. Let Φ_A and Φ_B be superfluid phases that originate only from the vortices on A and B sublattices respectively, so that $\Phi(\mathbf{r}) = \Phi_A(\mathbf{r}) + \Phi_B(\mathbf{r})$. Define:

$$\begin{aligned} m\mathbf{v}_A(\mathbf{r}) &= \left(\hbar \nabla \Phi_A(\mathbf{r}) - \frac{e}{c} \mathbf{A}(\mathbf{r}) \right) \\ m\mathbf{v}_B(\mathbf{r}) &= \left(\hbar \nabla \Phi_B(\mathbf{r}) - \frac{e}{c} \mathbf{A}(\mathbf{r}) \right) \\ \mathbf{a}(\mathbf{r}) &= \frac{m\mathbf{v}_A(\mathbf{r}) - m\mathbf{v}_B(\mathbf{r})}{2} \\ \mathbf{v}(\mathbf{r}) &= \frac{\mathbf{v}_A(\mathbf{r}) + \mathbf{v}_B(\mathbf{r})}{2}. \end{aligned} \quad (29)$$

In the extreme type-II limit the London penetration depth is large, and the magnetic field may appear practically uniform across inter-vortex distances. In such circumstances one can neglect contribution of the gauge field to the total flux of $m\mathbf{v}$ on A and B sublattices and keep only the 2π contribution of the superfluid order parameter. Then, the fields $\mathbf{a}(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$ look like vector potentials of a π -flux vortex lattice, where the flux alters sign between A and B sublattices in $\mathbf{a}(\mathbf{r})$, but not in $\mathbf{v}(\mathbf{r})$, which is the superfluid velocity.

Now perform a unitary Franz-Tešanović transformation³¹ U on the Hamiltonian (27):

$$U = \begin{pmatrix} e^{-i\Phi_A(\mathbf{r})} & 0 \\ 0 & e^{i\Phi_B(\mathbf{r})} \end{pmatrix}, \quad (30)$$

and obtain:

$$H_{\text{BdG}} = \begin{pmatrix} \frac{(\mathbf{p} + m\mathbf{v}_A)^2}{2m} - E_F & D \\ D & -\frac{(\mathbf{p} - m\mathbf{v}_B)^2}{2m} + E_F \end{pmatrix}, \quad (31)$$

with the transformed gap operator:

$$D = \frac{\Delta_0}{\hbar^2 k_F^2} \{p_x + a_x, p_y + a_y\}. \quad (32)$$

Finally, this Hamiltonian can be linearized near any of the four nodal points:

$$\text{near } \pm \hbar k_F \hat{\mathbf{x}} : \quad (33)$$

$$H = \pm [v_F(p_x + a_x)\sigma^z + v_\Delta(p_y + a_y)\sigma^x + mv_F v_x I]$$

$$\text{near } \pm \hbar k_F \hat{\mathbf{y}} :$$

$$H = \pm [v_F(p_y + a_y)\sigma^z + v_\Delta(p_x + a_x)\sigma^x + mv_F v_y I],$$

where $\sigma^x, \sigma^y, \sigma^z$ are Pauli matrices, and I is the unit two-by-two matrix. The last term in these Hamiltonians, proportional to I , is the Doppler shift of quasiparticle energy due to the background superfluid flow.

Before we take the limit of infinite inter-vortex separation, it is important to understand how the magnetic field varies with distance from a vortex core. A more accurate expression for the superfluid velocity $\mathbf{v}(\mathbf{r})$ is obtained by solving the conventional London equation (Appendix of Ref. 32):

$$\mathbf{v}(\mathbf{r}) = \frac{2\pi\hbar}{m} \int \frac{d^2k}{(2\pi)^2} \frac{i\mathbf{k} \times \hat{\mathbf{z}}}{k^2} \left(1 - \frac{1}{1 + \lambda^2 k^2} \right) C_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}}, \quad (34)$$

where λ is the London penetration depth, and:

$$C_{\mathbf{k}} = \sum_i \frac{e^{-i\mathbf{k}\mathbf{r}_i^A} + e^{-i\mathbf{k}\mathbf{r}_i^B}}{2}. \quad (35)$$

Note that the field $\mathbf{a}(\mathbf{r})$ does not depend on λ ; all of λ dependence is contained in the Doppler shift of (33). The superfluid velocity $\mathbf{v}(\mathbf{r})$ decays as $1/r$ with distance from the vortex core when $\lambda \rightarrow \infty$, and faster for finite λ .

V. ANALYTIC RESULTS IN ABSENCE OF THE DOPPLER SHIFT

The expression (25) for the effective vortex action, and the description of quasiparticles given by (33) provide us with all ingredients we need to study vortex dynamics.

We will follow two routes to the study of this dynamics, and the reader may examine them in either order.

A perturbative computation is presented in Section VI, and the reader can jump ahead to that section now. There we begin with an alternative formulation^{22,23,24,25} of the same theory using auxiliary Chern-Simons gauge fields. This allows a straightforward consideration of

the structure of the (uncontrolled) perturbation theory, which leads to the results in Eq. (5).

In the second route, we use a non-perturbative analysis, and show that certain physical effects can be evaluated exactly. This route proceeds in two stages. First, we will ignore the Doppler shift of quasiparticle energies in the present section, and this will allow us to analytically derive the vortex action. Such analytic results will elucidate interesting physics, and serve as a framework for the second stage (section VII) in which we include the Doppler shift. Calculations of the Doppler shift effects must be done numerically, but fortunately, no qualitative modifications of the analytic partial results are found. All these results are also consistent with Eq. (5).

A. Transition matrix elements

The main problem is to diagonalize the linearized BdG Hamiltonian (33) and calculate the matrix elements $U_{n,n'}$ given by (23), which need to be substituted into the vortex action (25). The matrix elements $U_{n,n'}$ can be evaluated exactly in absence of the Doppler shift. For now we focus on the node $\mathbf{k} = \hbar k_F \hat{\mathbf{x}}$, and postpone discussion of the other nodes for later. We start by solving the Schrödinger equation (20) for the Hamiltonian:

$$H = \begin{pmatrix} v_F(p_x + a_x) & v_\Delta(p_y + a_y) \\ v_\Delta(p_y + a_y) & -v_F(p_x + a_x) \end{pmatrix}. \quad (36)$$

We will approximate the “gauge” field $\mathbf{a}(\mathbf{r})$ to that of a π -flux vortex completely localized at the origin. Even though the vortex core will appear infinitely small, the approximation is validated by the fact that there are no strictly bound quasiparticle states in a d -wave vortex, so that the vortex dynamics is determined only by extended states.

We will determine the spectrum of H following the analysis of Ref. 32. As they showed, microscopic details of the vortex core can be captured in a single parameter. The Hamiltonian (36) could be easily diagonalized

if it were isotropic. Therefore, we make it isotropic by applying a gauge transformation that deforms the vortex gauge field $\mathbf{a}(\mathbf{r})$ to a suitable “elliptical” shape, and then rescale the coordinates in such a way that v_F and v_Δ become equal to unity and the “elliptical” gauge field is reshaped to the isotropic form:

$$\mathbf{a}(\mathbf{r}) = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{2(x^2 + y^2)}. \quad (37)$$

The coordinate rescaling goes as follows (the original coordinates are “primed”):

$$\begin{aligned} w_x &= v_F w'_x, w_y = v_\Delta w'_y && \dots \text{ vectors} \\ d^2 r' &= v_F v_\Delta d^2 r && \dots \text{ Jacobian} \\ \nabla' &= \frac{\hat{\mathbf{x}}}{v_F} \frac{\partial}{\partial x} + \frac{\hat{\mathbf{y}}}{v_\Delta} \frac{\partial}{\partial y} && \dots \text{ gradient} \\ \psi'(x', y') &= \frac{1}{\sqrt{v_F v_\Delta}} \psi\left(\frac{x}{v_F}, \frac{y}{v_\Delta}\right) && \dots \text{ wavefunctions} \\ H &= (p_x + a_x)\sigma^z + (p_y + a_y)\sigma^x && \dots \text{ Hamiltonian} \end{aligned}$$

It is convenient to apply a unitary transformation to the rescaled Hamiltonian and write it as:

$$H = \begin{pmatrix} -m & (p_x + a_x) - i(p_y + a_y) \\ (p_x + a_x) + i(p_y + a_y) & m \end{pmatrix}, \quad (38)$$

where a quasiparticle mass m has been added “by hand”. In a d -wave superconductor $m = 0$, and eventually we set m to zero, but formally a non-zero mass is needed to integrate out quasiparticles in the path-integral and perform summations over frequencies such as (A3). Switching to the polar coordinates:

$$\begin{aligned} p_x &= \cos \phi \left(-i \frac{\partial}{\partial r} \right) - \frac{\sin \phi}{r} \left(-i \frac{\partial}{\partial \phi} \right) && a_x = -\frac{\sin \phi}{2r} \\ p_y &= \sin \phi \left(-i \frac{\partial}{\partial r} \right) + \frac{\cos \phi}{r} \left(-i \frac{\partial}{\partial \phi} \right) && a_y = \frac{\cos \phi}{2r} \end{aligned} \quad (39)$$

the Schrödinger equation takes form:

$$\begin{pmatrix} -m & e^{-i\phi} \left(-i \frac{\partial}{\partial r} - \frac{i}{r} \left(-i \frac{\partial}{\partial \phi} + \frac{1}{2} \right) \right) \\ e^{i\phi} \left(-i \frac{\partial}{\partial r} + \frac{i}{r} \left(-i \frac{\partial}{\partial \phi} + \frac{1}{2} \right) \right) & m \end{pmatrix} \begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix} = E \begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix}. \quad (40)$$

By substituting $u(\mathbf{r}) = e^{i(l-1)\phi} u(r)$ and $v(\mathbf{r}) = e^{il\phi} v(r)$

the solutions are found to be:

$$\begin{aligned} \psi_{q,l,k}(r, \phi) &= \left(\frac{k}{4\pi|E|} \right)^{\frac{1}{2}} \times && (41) \\ \left\{ \begin{aligned} &\begin{pmatrix} \left(\sqrt{E-m} \cdot J_{-l+\frac{1}{2}}(kr) e^{i(l-1)\phi} \right) \\ \left(-iq\sqrt{E+m} \cdot J_{-l-\frac{1}{2}}(kr) e^{il\phi} \right) \end{pmatrix}, && l < 0 \\ &\begin{pmatrix} \left(\sqrt{E-m} \cdot J_{l-\frac{1}{2}}(kr) e^{i(l-1)\phi} \right) \\ \left(iq\sqrt{E+m} \cdot J_{l+\frac{1}{2}}(kr) e^{il\phi} \right) \end{pmatrix}, && l > 0 \end{aligned} \right\}. \end{aligned}$$

States are characterized by the quantum numbers $n = (q, l, k)$: “charge” $q = \pm 1$ (distinguishes particles-like and hole-like states), angular momentum $l \in \mathbb{Z}$ and radial wavevector $k > 0$. Energy is $E = q\sqrt{k^2 + m^2}$. $J_l(kr)$ are Bessel functions of the first kind. These wavefunctions are normalized as:

$$\int d^2r \psi_{q_1, l_1, k_1}^\dagger(\mathbf{r}) \psi_{q_2, l_2, k_2}(\mathbf{r}) = \delta_{q_1, q_2} \delta_{l_1, l_2} \delta(k_1 - k_2).$$

The zero angular momentum channel requires special attention. Square integrability of the wavefunctions requires only that:

$$\begin{aligned} \psi_{q,0,k}(r, \phi) &= \left(\frac{k}{4\pi|E|} \right)^{\frac{1}{2}} \times \\ &\left[\sin \theta \begin{pmatrix} \sqrt{E-m} \cdot J_{-\frac{1}{2}}(kr) e^{-i\phi} \\ iq\sqrt{E+m} \cdot J_{\frac{1}{2}}(kr) \end{pmatrix} + \right. \\ &\left. \cos \theta \begin{pmatrix} \sqrt{E-m} \cdot J_{\frac{1}{2}}(kr) e^{-i\phi} \\ -iq\sqrt{E+m} \cdot J_{-\frac{1}{2}}(kr) \end{pmatrix} \right] \quad \dots \quad l = 0 \end{aligned} \quad (42)$$

where θ is a parameter with arbitrary value within $[0, \pi)$. This parameter is physically obtained from boundary conditions at the vortex core only when the core is not idealized by a point, but regarded as a finite region in space. Microscopic details of the vortex core enter vortex dynamics only through this parameter. In the following we assume that all of the flux is contained in a small cylindrical shell: this yields $\theta = 0$ for a vortex, and $\theta = \frac{\pi}{2}$ for an anti-vortex³³. Other choices do not introduce qualitative changes in the results that follow, but they may give rise to some quantitative corrections, to the vortex mass for example, which we briefly discuss in the Appendix E.

The presence of a vortex significantly affects quantum motion of quasiparticles, as is clearly seen from the local density of states (LDOS) expressed in the rescaled coordinates (for $m = 0$):

$$\begin{aligned} \rho(\epsilon; r, \theta) &= \frac{\cos(2|\epsilon|r)}{2\pi^2 r} + \frac{|\epsilon|}{\pi} \sum_{l=0}^{\infty} J_{l+\frac{1}{2}}^2(|\epsilon|r) \\ &\rightarrow \left\{ \begin{array}{ll} \frac{1}{2\pi^2 r} & , \quad |\epsilon|r \ll 1 \\ \frac{|\epsilon|}{2\pi} & , \quad |\epsilon|r \gg 1 \end{array} \right\}. \end{aligned} \quad (43)$$

At small distances from the vortex core LDOS diverges as $1/r$ due to the zero angular momentum state (42), and there are also certain oscillations. This behavior was studied earlier, and with addition of the Doppler shift it only acquires anisotropy, but does not qualitatively deviate from $1/r$ dependence³⁶. In realistic circumstances this divergence is cut-off by the finite core size. Note that the LDOS peak at $r = 0$ does not correspond to bound states in purely d -wave vortex cores (bound states could emerge only if a secondary order parameter, such as s -wave, were present in vicinity of the vortex). In fact, a careful analysis shows that the Hamiltonian (36) does not support even a bound state exactly at $E = 0$ ³⁷. Far away from the vortex core LDOS becomes constant and linear in energy as expected.

We can now calculate the matrix elements $\mathbf{U}_{n,n'}$ in (23). This calculation can be done directly in the rescaled coordinate system, with the states (41) and (42) that were obtained after a gauge transformation, rescaling and a unitary transformation:

$$\mathbf{U}_{n_1, n_2} = \int d^2r \psi_{n_1}^\dagger(\mathbf{r}) \left(\frac{\hat{\mathbf{x}}}{v_F} \frac{\partial}{\partial x} + \frac{\hat{\mathbf{y}}}{v_\Delta} \frac{\partial}{\partial y} \right) \psi_{n_2}(\mathbf{r}). \quad (44)$$

Here n_i denotes (q_i, l_i, k_i) .

The calculation of (44) is pretty tedious, so that we only sketch it in the Appendix B. A closed form for arbitrary quasiparticle mass m is obtained by algebraic manipulations and elementary integrations, with help of Bessel function identities. It turns out that no singularities are introduced in the end by taking the $m \rightarrow 0$ limit, so that in the following we will consider only the case of interest $m = 0$. Since \mathbf{U}_{n_1, n_2} are essentially transition matrix elements of the momentum operator, the only allowed transitions are between states whose angular momentum is different by one. Defining $\sigma = l_2 - l_1 = \pm 1$, we obtain for the node $\hbar k_F \hat{\mathbf{x}}$ and zero quasiparticle mass:

$$\mathbf{U}_{n_1, n_2} = \frac{1}{8} e^{\frac{i\pi}{4}(q_2 - q_1)} \left(\sigma \frac{\hat{\mathbf{x}}}{v_F} + i \frac{\hat{\mathbf{y}}}{v_\Delta} \right) U_{(q_1, l-\sigma, k_1), (q_2, l, k_2)}, \quad (45)$$

where

$$U_{(q_1, l-\sigma, k_1), (q_2, l, k_2)} = \left\{ \begin{array}{ll} 4\sqrt{k_1 k_2} \delta(E_2 - E_1) - C_{\sigma; q_1, q_2} \left(\frac{k_1}{k_2} \right)^{\sigma l - \frac{1}{2}} \frac{E_1 + E_2}{\sqrt{k_1 k_2}} \Theta(\sigma(k_2 - k_1)) & , \quad l > \frac{\sigma+1}{2} \\ -4\sqrt{k_1 k_2} \delta(E_1 - E_2) - C_{\sigma; q_1, q_2} \left(\frac{k_1}{k_2} \right)^{\sigma l - \frac{1}{2}} \frac{E_1 + E_2}{\sqrt{k_1 k_2}} \Theta(\sigma(k_1 - k_2)) & , \quad l < \frac{\sigma+1}{2} \\ \frac{2\sigma q_1 q_2}{\pi} \frac{E_1 + E_2}{E_1 - E_2} + \frac{1}{2\pi} C_{\sigma; q_1, q_2} \left(\frac{k_1}{k_2} \right)^{\sigma l - \frac{1}{2}} \frac{E_1 + E_2}{\sqrt{k_1 k_2}} \log \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 & , \quad l = \frac{\sigma+1}{2} \end{array} \right\}. \quad (46)$$

Here, $\delta(x)$ is the Dirac delta-function, and:

$$\Theta(x) = \left\{ \begin{array}{ll} 1 & , \quad x > 0 \\ \frac{1}{2} & , \quad x = 0 \\ 0 & , \quad x < 0 \end{array} \right\}; \quad C_{\sigma; q_1, q_2} = \left\{ \begin{array}{ll} q_2 & , \quad \sigma = 1 \\ q_1 & , \quad \sigma = -1 \end{array} \right\}.$$

Quasiparticle energies are now $E_i = q_i k_i$. The expression (46) defines infinities at $E_1 = E_2$ completely, except when $l = (\sigma + 1)/2$. In that case, however, it can be shown by finite-size regularization that U at $E_1 = E_2$ diverges only logarithmically with the system size, which does not affect the important integrals over energies in the limit of infinite system size beyond what is given by the expression (46).

B. Summation over quasiparticle nodes and angular momenta

The effective vortex action (25) depends on \mathbf{U}_{n_1, n_2} through:

$$|\mathbf{r}_v(\omega)\mathbf{U}_{n_1, n_2}|^2 = \left| \frac{U_{(q_1, l-\sigma, k_1), (q_2, l, k_2)}}{8} \right|^2 \times \quad (47)$$

$$\left[\left(\frac{|\hat{\mathbf{x}}\mathbf{r}_v(\omega)|^2}{v_F^2} + \frac{|\hat{\mathbf{y}}\mathbf{r}_v(\omega)|^2}{v_\Delta^2} \right) + \frac{i\sigma}{v_F v_\Delta} \hat{\mathbf{z}} (\mathbf{r}_v^*(\omega) \times \mathbf{r}_v(\omega)) \right],$$

where we have used (45), relevant for the quasiparticle node at $\mathbf{p} = \hbar k_F \hat{\mathbf{x}}$. Note that this dependence on vortex coordinates is not isotropic; only final expressions with all quasiparticle nodes included become isotropic. It is easily checked from (33) that the linearized Hamiltonians at different nodes are related by unitary transformations and an exchange $v_F \leftrightarrow v_\Delta$ as needed, in absence of the Doppler shift. Thus, in order to add contributions at all nodes we only need to symmetrize with respect to v_F and v_Δ and multiply by the number of nodes:

$$\sum_{\text{nodes}} |\mathbf{r}_v(\omega)\mathbf{U}_{n_1, n_2}|^2 = \frac{|U_{(q_1, l-\sigma, k_1), (q_2, l, k_2)}|^2}{32} \times \quad (48)$$

$$\left[\left(\frac{1}{v_F^2} + \frac{1}{v_\Delta^2} \right) |\mathbf{r}_v(\omega)|^2 + \frac{2i\sigma}{v_F v_\Delta} \hat{\mathbf{z}} (\mathbf{r}_v^*(\omega) \times \mathbf{r}_v(\omega)) \right].$$

Note that for our purposes we do not need to worry about inter-node scattering of quasiparticles. If we went back to the non-linear quasiparticle Hamiltonian and calculated \mathbf{U}_{n_1, n_2} for states that belong to different nodes, we could obtain singular behavior only when the momentum transfer is $\sim \hbar k_F$. Such transitions are not stimulated by a slowly moving vortex, and cannot play a significant role in low-energy vortex dynamics.

Owing to the degeneracy of the quasiparticle energies with respect to angular momentum, it is possible to analytically carry out summation over angular momentum channels in (25), that is summation over l and σ . Vortex mass and possible dissipation are determined by the following sum, which is just a geometric series that we

evaluate from (46):

$$\sum_{l, \sigma} |U_{(q_1, l-\sigma, k_1), (q_2, l, k_2)}|^2 = \frac{(E_1 + E_2)^2 (E_1^2 + E_2^2)}{|E_1^2 - E_2^2| \max(E_1^2, E_2^2)}$$

$$+ \left(\frac{E_1 + E_2}{\pi} \right)^2 \left[\frac{8}{(E_1 - E_2)^2} + \frac{2}{k_1 k_2} \log^2 \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 \right.$$

$$\left. + \frac{1}{4} \left(\frac{1}{E_1^2} + \frac{1}{E_2^2} \right) \log^2 \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 \right]. \quad (49)$$

Here, we have dropped the Dirac delta-functions in (46), since in (25) there is a factor that becomes zero when $E_1 = E_2$ (safe to do only in absence of the Doppler shift). For the same reason the other possible divergences at $k_1 = k_2$, in (46) or in the sum over angular momenta turn out not be dangerous: they actually appear only when $q_1 = q_2$ as well, that is only when $E_1 = E_2$ (particle-particle or hole-hole transitions).

Similarly, we find the sum that determines ‘‘transversal dynamics’’:

$$\sum_{l, \sigma} \sigma |U_{(q_1, l-\sigma, k_1), (q_2, l, k_2)}|^2 =$$

$$\frac{(E_1 + E_2)^2}{\max(E_1^2, E_2^2)} \text{sign}(|E_1| - |E_2|)$$

$$+ \left(\frac{E_1 + E_2}{\pi} \right)^2 \left[\frac{2q_1 q_2}{E_1 - E_2} \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \log^2 \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 \right.$$

$$\left. + \frac{1}{4} \left(\frac{1}{E_2^2} - \frac{1}{E_1^2} \right) \log^2 \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 \right]. \quad (50)$$

C. Vortex mass at zero temperature

This subsection collects our results above and computes the contribution of the π -flux to the vortex mass. We will find that this contribution is finite.

The vortex mass is determined by the ‘‘longitudinal’’ part of the vortex action (26). Thus we will restrict our attention to the sum (49) and substitute it into the vortex action (25). At zero temperature the difference between Fermi-Dirac functions $f(\epsilon_{n_1}) - f(\epsilon_{n_2})$ allows only transitions between states whose energies have opposite signs. This means that only particle-hole transitions will be allowed. Using the invariance of (49) under exchange $E_1 \leftrightarrow E_2$, and symmetrizing the ‘‘longitudinal’’ part of (25) with respect to its complex conjugate, we have:

$$S_{v\parallel} = \int \frac{d\omega}{2\pi} \int_{-\Lambda}^0 dE_1 \int_0^\Lambda dE_2 \frac{\omega^2 (E_2 - E_1)}{(E_2 - E_1)^2 + \omega^2} \times$$

$$\sum_{\text{nodes}} \sum_{l, \sigma} |\mathbf{r}_v(\omega)\mathbf{U}_{n_1, n_2}|^2. \quad (51)$$

Energy cut-off is Λ . If slow vortex dynamics reduces to vortex inertia, then the leading low-frequency term in the

“longitudinal” vortex action is:

$$S_{v\parallel} = \int \frac{d\omega}{2\pi} \frac{m_v \omega^2}{2} |\mathbf{r}_v(\omega)|^2, \quad (52)$$

where m_v is the vortex mass:

$$m_v = \frac{1}{16} \left(\frac{1}{v_F^2} + \frac{1}{v_\Delta^2} \right) \int_{-\Lambda}^0 dE_1 \int_0^\Lambda dE_2 \quad (53)$$

$$\frac{1}{E_2 - E_1} \sum_{l,\sigma} |U_{(q_1, l-\sigma, k_1), (q_2, l, k_2)}|^2.$$

Note that this expression is obtained by simply setting $\omega = 0$ in the denominator of (51), which is justified only if m_v turns out to be finite. In absence of thermal fluctuations m_v could in principle be infra-red divergent only due to transitions at $E_1 = E_2 = 0$. However, only the bulk density of states, which shapes such transitions, can be responsible for infra-red divergences. In the bulk, far away from the vortex core, LDOS is linear in energy (see (43)) and hence suppresses transitions at $E = 0$. This is formally reflected in the sum (49) by absence of the singularity at $E_1 = 0^+$, $E_2 = 0^-$. Then, without help from the transition matrix elements, the extra factor of $E_2 - E_1$ in (53) is innocuous and fully integrable. Note that the $1/r$ behavior of LDOS close to the vortex core matters, but not enough to cause infra-red divergency of m_v ; it produces the second term in (49), including the logarithms, but no singularity at $E = 0$.

Careful integration, presented in Appendix C, shows that indeed the vortex mass is finite:

$$m_v \approx 0.05 \left(\frac{1}{v_F^2} + \frac{1}{v_\Delta^2} \right) \Lambda. \quad (54)$$

Therefore, vortex dynamics at zero temperature is entirely specified by inertia. The cut-off scale Λ is roughly given by the maximum superconducting gap Δ_0 . We can write:

$$m_v \sim 0.05 \left(\alpha_D + \frac{1}{\alpha_D} \right) m_e, \quad (55)$$

where m_e is the electron mass, and $\alpha_D = v_F/v_\Delta$. In d -wave superconductors $\alpha_D \sim 1 \div 10^2$, so that the purely quantum effects contribute roughly an electron mass to the mass of a vortex.

In various circumstances, which are not the focus of this paper, the bulk LDOS may not vanish at zero energy. For example, this can be expected in presence of disorder, or Zeeman splitting in magnetic field. In such situations the vortex mass calculated from the expression (53) could become infra-red divergent, and it would be more appropriate to go back to the full action (51).

D. Dissipation at finite temperatures

At finite temperatures the “longitudinal” part of the vortex action is:

$$S_{v\parallel} = \frac{1}{2\beta} \sum_\omega \int_{-\Lambda}^\Lambda dE_1 \int_{-\Lambda}^\Lambda dE_2 (f(E_1) - f(E_2)) \times$$

$$\frac{\omega^2 (E_2 - E_1)}{(E_2 - E_1)^2 + \omega^2} \sum_{\text{nodes}} \sum_{l,\sigma} |\mathbf{r}_v(\omega) \mathbf{U}_{n_1, n_2}|^2. \quad (56)$$

The difference between Fermi-Dirac functions $f(E_1) - f(E_2)$ does not strictly prohibit transitions between two particle-like or two hole-like states at $T > 0$. Therefore, singularities in which the factors $E_2 - E_1$ become zero at finite energies will be included in the vortex action. We can no longer naively neglect ω^2 in the denominator, even for small frequencies.

Consider the “longitudinal correlation” $F_{\parallel}(\omega)$ defined by (26). At small frequencies we expect that small values of $E_2 - E_1$ will matter the most, so that we approximate:

$$f(E_1) - f(E_2) \approx (E_1 - E_2) \frac{\partial f}{\partial E} \Big|_{\frac{E_1 + E_2}{2}}, \quad |E_1 - E_2| \ll T.$$

With a change of variables $\Delta E = E_2 - E_1$, $2E = E_2 + E_1$ we have at small frequencies:

$$F_{\parallel}(\omega) \approx \frac{1}{64} \left(\frac{1}{v_F^2} + \frac{1}{v_\Delta^2} \right) \int_{-T}^T d\Delta E \int_{-\Lambda}^\Lambda dE \left(-\frac{\partial f(E)}{\partial E} \right) \times$$

$$\frac{\omega^2 \Delta E^2}{\Delta E^2 + \omega^2} \sum_{l,\sigma} |U_{(q_1, l-\sigma, k_1), (q_2, l, k_2)}|^2. \quad (57)$$

We do not care about $|\Delta E| > T$, because for sufficiently large and finite ΔE we can safely expand in powers of ω^2 and obtain only the high-frequency corrections to vortex inertia. Our goal here is to elucidate sub-quadratic powers of frequency. Similarly, it will be clear shortly that the precise boundaries of ΔE do not matter after analytic continuation. The sum over angular momenta is given by (49). Its “most divergent” part (2nd term), which comes from the particle-particle and hole-hole transitions that involve the zero angular momentum channel, dominates the small frequency behavior:

$$F_{\parallel}(\omega) \approx \frac{1}{64} \left(\frac{1}{v_F^2} + \frac{1}{v_\Delta^2} \right) \int_{-T}^T d\Delta E \int_{-\Lambda}^\Lambda dE \left(-\frac{\partial f(E)}{\partial E} \right) \times$$

$$\frac{\omega^2 \Delta E^2}{\Delta E^2 + \omega^2} \left(\frac{32}{\pi^2} \frac{E^2}{\Delta E^2} + \dots \right) \quad (58)$$

$$\approx \frac{\pi}{6} \left(\frac{1}{v_F^2} + \frac{1}{v_\Delta^2} \right) T^2 |\omega| + \dots$$

After analytic continuation to real frequency, this has the “Ohmic” dissipation form of the Bardeen-Stephen

viscous drag. The ellipses denote corrections that become more noticeable as the frequency grows. The first such correction is also interesting and comes from particle-particle and hole-hole transitions that do not involve the zero angular momentum channel (1st term of (49)):

$$\begin{aligned} \Delta F_{\parallel}(\omega) &\approx \frac{1}{64} \left(\frac{1}{v_F^2} + \frac{1}{v_{\Delta}^2} \right) \int_{-T}^T d\Delta E \int_{-\Lambda}^{\Lambda} dE \left(-\frac{\partial f(E)}{\partial E} \right) \\ &\times \frac{\omega^2 \Delta E^2}{\Delta E^2 + \omega^2} \frac{4E^2(2E^2 + \frac{\Delta E^2}{2})}{|2E\Delta E|(E^2 + \frac{\Delta E^2}{4} + |E\Delta E|)} \\ &\approx \frac{\log(2)}{8} \left(\frac{1}{v_F^2} + \frac{1}{v_{\Delta}^2} \right) T\omega^2 \log \left(1 + \frac{T^2}{\omega^2} \right) + \dots \end{aligned}$$

This could be interpreted as a frequency-dependent correction to the vortex mass. With such an interpretation the vortex mass at rest would be logarithmically divergent. However, we feel that it is more natural to associate only the strictly quadratic frequency behavior to the vortex mass, and regard everything that dominates it at small frequencies as various forms of dissipation. All other corrections are $\mathcal{O}(\omega^2)$, and hence contribute the normal vortex mass.

In conclusion, dissipation occurs at finite temperatures, its strength being a power law in temperature due to the gapless quasiparticle nodes.

E. Transversal dynamics

The ‘‘transversal correlation’’ $F_{\perp}(\omega)$ defined in (26) describes aspects of vortex dynamics that are associated with the Magnus force. We argue below that the nodal quasiparticles do not give rise to transversal forces on a vortex at any temperature.

The expression for ‘‘transversal correlations’’ is:

$$\begin{aligned} F_{\perp}(\omega) &= \frac{1}{32v_F v_{\Delta}} \int_{-\Lambda}^{\Lambda} dE_1 \int_{-\Lambda}^{\Lambda} dE_2 (f(E_1) - f(E_2)) \times \\ &\frac{-i\omega(E_2 - E_1)}{E_2 - E_1 - i\omega} \sum_{l,\sigma} \sigma |U_{(q_1, l-\sigma, k_1), (q_2, l, k_2)}|^2. \quad (59) \end{aligned}$$

The sum over angular momenta is given by (50); its crucial property as a function of E_1 and E_2 is that it changes sign under the change of variables $E_1 \rightarrow -E_2$ and $E_2 \rightarrow -E_1$. If we apply this change of variables to $F_{\perp}(\omega)$, and add the result to the original expression above, we obtain:

$$\begin{aligned} 2F_{\perp}(\omega) &\propto \int_{-\Lambda}^{\Lambda} dE_1 \int_{-\Lambda}^{\Lambda} dE_2 \sum_{l,\sigma} \sigma |U_{(q_1, l-\sigma, k_1), (q_2, l, k_2)}|^2 \times \\ &(f(E_1) + f(-E_1) - f(E_2) - f(-E_2)) \frac{-i\omega(E_2 - E_1)}{E_2 - E_1 - i\omega} \\ &= 0. \end{aligned}$$

The final conclusion follows from the identity $f(E) + f(-E) = 1$.

The physical reason for complete absence of quasiparticle-induced transversal forces is related to the fact that there are no circulating currents of quasiparticles in thermal equilibrium. This connection has been made at small frequencies in Ref. 29. At small frequencies it is safe to neglect ω in the denominator of (59), and then cancel out the factors of $E_2 - E_1$. What remains is an integral that can be represented as a pure trace of a quantum operator, and this trace can be evaluated in real space. Using the definition of transition matrix elements (23), one finds:

$$F_{\perp}(\omega) \propto -i\omega \int d^2r \hat{z}(\nabla \times \mathbf{j}) = -i\omega \oint \mathbf{j} ds, \quad (60)$$

where \mathbf{j} is the net current density of quasiparticles, determined by wavefunctions (41), (42), and temperature. The contour integral above is taken on a loop of infinite radius. In thermal equilibrium quasiparticles are at rest with respect to the substrate, so that $\mathbf{j} = 0$.

VI. PERTURBATIVE COMPUTATION

This section reviews an alternative formulation^{22,23,24,25} of the quantum theory of vortices, fermionic quasiparticles, and plasmons which appears in Sections I and IV. In this approach, the complete action for the vortices, plasmons, and nodal quasiparticles is $\tilde{\mathcal{S}}_v + \mathcal{S}_{\mathcal{A}} + \mathcal{S}_{\Psi} + \mathcal{S}_{cs}$ specified in Eqs. (61), (3), (63), and (64). This theory is closely related to a continuum limit of the theory considered by Balents and Fisher.²³

The new feature of this approach is that the Berry phase terms are mediated by auxiliary Chern-Simons gauge fields. Such a formalism readily allows a perturbative analysis of the couplings between the elementary excitations using standard techniques. It must be kept in mind, however, that there is no small parameter which justifies such a perturbative analysis.

We will see below that the perturbative analysis of the Berry phase terms lead to conclusions consistent with the exact evaluation already presented in Section V. Empowered by this success, we will examine the influence of the Doppler shift term using perturbation theory in Section VI A.

We begin by extending the vortex action \mathcal{S}_v in Eq. (1) by coupling it in addition to a second U(1) gauge field α_{μ} :

$$\begin{aligned} \tilde{\mathcal{S}}_v &= \int d\tau \left[\frac{m_v^0}{2} \left(\frac{d\mathbf{r}_v(\tau)}{d\tau} \right)^2 + i \frac{d\mathbf{r}_v(\tau)}{d\tau} \cdot \vec{\mathcal{A}}^0(\mathbf{r}_v(\tau)) \right. \\ &\left. + i \int d^2\mathbf{r} d\tau \{ \mathcal{A}_{\mu}(\mathbf{r}, \tau) + \alpha_{\mu}(\mathbf{r}, \tau) \} J_{v\mu}(\mathbf{r}, \tau) \right] \quad (61) \end{aligned}$$

This is to facilitate the Berry phase coupling to the fermionic quasiparticles, as we now describe.

The fermionic quasiparticle Hamiltonian is discussed in Sections I and IV, and we work with a single node appearing in the first equation in Eq. (33). Let this Hamiltonian act on the Nambu spinor Ψ . We define $\bar{\Psi} = -i\Psi^\dagger\sigma^y$, introduce the Dirac gamma matrices $\gamma^\mu = (-\sigma^y, \sigma^x, \sigma^z)$, rescale coordinates and gauge fields to obtain isotropic velocities as discussed in Section V, and thence obtain the continuum action for the fermions

$$\mathcal{S}_\Psi = \int d^2\mathbf{r}d\tau \left[-i\bar{\Psi}\gamma^\mu(\partial_\mu - ia_\mu)\Psi + \frac{iv_F}{2}\bar{\Psi}\gamma^0\Psi\partial_x\Phi \right]. \quad (62)$$

The last Doppler-shift term is a coupling to the superflow fluctuations, which we have represented in this paper by the U(1) gauge field \mathcal{A}_μ . Extracting the explicit connection between the gradient of the phase of the order parameter and the ‘electric’ field associated with \mathcal{A}_μ , we may rewrite this term to obtain the fermionic action in its final form

$$\mathcal{S}_\Psi = \int d^2\mathbf{r}d\tau \left[-i\bar{\Psi}\gamma^\mu(\partial_\mu - ia_\mu)\Psi + \frac{iv_F}{4\pi\rho_s}\bar{\Psi}\gamma^0\Psi(\partial_y\mathcal{A}_\tau - \partial_\tau\mathcal{A}_y) \right]. \quad (63)$$

Finally, we need to tie the gauge field a_μ to the vortices, as was done explicitly in Eq. (37). In the present formalism, this is done conveniently with a Chern-Simons term:

$$\mathcal{S}_{cs} = \frac{i}{\pi} \int d^2\mathbf{r}d\tau \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu \alpha_\lambda \quad (64)$$

We have now written all the required terms coupling together the vortices, fermionic quasiparticles, and plasmons in a d -wave superconductor. The complete action is $\tilde{\mathcal{S}}_v + \mathcal{S}_A + \mathcal{S}_\Psi + \mathcal{S}_{cs}$ specified in Eqs. (61), (3), (63), and (64).

At this point, it is useful to describe how the results of Sections IV, V, and VII relate to the present formalism. The Hamiltonian in Eq. (33) is obtained by exactly integrating out the gauge fields a_μ , α_μ , and only the scalar potential \mathcal{A}_τ in the Coulomb gauge $\nabla \cdot \vec{\mathcal{A}} = 0$. The retardation effects contained in $\vec{\mathcal{A}}$ fluctuations are neglected. The computations of Section V and VII describe the consequences of subsequently exactly integrating out the fermions.

The present section proceeds by integrating out degrees of freedom in the *opposite* order. We first integrate out the fermionic fields Ψ exactly, and then study the perturbative consequences of subsequently integrating out the a_μ , α_μ and \mathcal{A}_μ . As we have already seen in Section II, such an approach readily allows accounting for the $\vec{\mathcal{A}}$ fluctuations.

We will first integrate out the fermions Ψ , and study its influence on the a_μ and α_μ gauge fields, and then investigate the resulting modifications to the vortex/plasmon

action in Eqs. (1) and (3). We defer consideration of the Doppler shift term, and the resulting modification of plasmon (*i.e.* \mathcal{A}_μ) fluctuations to the following Section VIA.

Integrating out Ψ in Eq. (63) also yields an effective action $\text{Tr}[\gamma^\mu(\partial_\mu - ia_\mu)]$ for a_μ . This involves terms to all orders in a_μ and is not easy to work with. To obtain a first understanding of its influence, we expand the functional determinant to second order in a_μ . Subsequently we integrate out the a_μ , while accounting for the Chern-Simons term in \mathcal{S}_{cs} . This yields an action for the gauge field α_μ of the form

$$\mathcal{S}_\alpha = \frac{\tilde{c}}{2} \int \frac{d^3p}{8\pi^3} |p| \alpha_\mu \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \alpha_\nu, \quad (65)$$

where \tilde{c} is a coupling constant. Now we can examine the effect of α_μ fluctuations on the vortices just as in Section II. The action \mathcal{S}_α is similar to that in Eq. (8) but with an extra power of p . Consequently, we obtain results for the effective vortex action as in Eqs. (11) and (12) but with an extra power of $1/\tau$ in the integrand. The τ integrals are now linearly convergent in the infrared, and we obtain then a *finite* renormalization of the vortex mass. This structure is entirely consistent with the exact results of Section V where we found vortex mass contributions expressed in terms of integrals which were also linearly convergent in the infrared.

More explicitly, in frequency and momentum space we obtain an expression which is just as in Eq. (9), but with the $(\omega^2 + k^2)$ in the denominator replaced by $\sqrt{\omega^2 + k^2}$. Expanding the resulting expression in powers of $\mathbf{r}_v(\omega)$ we obtain a result in the form of Eq. (4), with a contribution

$$F_{\parallel}(\omega) = \frac{1}{2\tilde{c}} \int \frac{d^2k}{4\pi^2} \frac{(k^2 - k\sqrt{k^2 + \omega^2} + 2\omega^2)}{2\sqrt{k^2 + \omega^2}} \quad (66)$$

It is now easily seen that the result of the momentum integral (and after reverting to physical units by reinserting factors of the velocities) has the form of Eq. (5), with F_2 a universal function of the velocity ratio. In other words, we obtain a finite non-universal renormalization of m_v , and a universal ‘sub-Ohmic’ dissipation.

A. Doppler shift

We are interested here in the influence of the last Doppler-shift coupling in Eq. (63) between the nodal fermions and the superflow (which is the dual ‘electric field’). At first glance, it appears that this term is innocuous: integrating out the fermions leads to an ultraviolet finite fermion loop, and this contributes a term proportional to the square of the \mathcal{A}_μ ‘electric’ field with a finite co-efficient. Comparing with Eq. (3) we conclude that this is merely a renormalization of the superfluid stiffness ρ_s which controls the plasmon fluctuations.

However, for completeness, we do have to also consider the subleading non-analytic momentum and frequency dependence of the fermion loop, which could lead to singular corrections to the vortex action. We will see below that this does not happen, and that such terms lead only to a finite renormalization of the vortex mass.

Let us write the electric field $\mathcal{E}_x = \partial_\tau \mathcal{A}_x - \partial_x \mathcal{A}_\tau$, and similarly for \mathcal{E}_y . Then the term

$$\frac{1}{8\pi^2 \rho_s} \int d^2 \mathbf{r} \int d\tau (\mathcal{E}_x^2 + \mathcal{E}_y^2) \quad (67)$$

in Eq. (3) is replaced by

$$\begin{aligned} & \frac{1}{8\pi^2 \rho_s} \int \frac{d^2 k}{4\pi^2} \int \frac{d\omega}{2\pi} \left[\left(1 + \frac{\mathcal{K}(k_x, k_y, \omega)}{\rho_s} \right) |\mathcal{E}_y(\mathbf{k}, \omega)|^2 \right. \\ & \left. + \left(1 + \frac{\mathcal{K}(k_y, k_x, \omega)}{\rho_s} \right) |\mathcal{E}_x(\mathbf{k}, \omega)|^2 \right]. \quad (68) \end{aligned}$$

We compute the universal correction \mathcal{K} at one loop order, using the original unscaled units in the Hamiltonian in Eq. (33). This leads to (after including contributions from all nodes)

$$\begin{aligned} \mathcal{K}(k_x, k_y, \omega) &= \frac{v_F^2}{2} \int \frac{d^2 p}{4\pi^2} \frac{d\epsilon}{2\pi} \text{Tr} \\ & \times [-i(\epsilon + \omega) + v_F(p_x + k_x)\sigma^z + v_\Delta(p_y + k_y)\sigma^x]^{-1} \\ & \times [-i\epsilon + v_F p_x \sigma^z + v_\Delta p_y \sigma^x]^{-1} \\ &= \frac{v_F^2}{2} \int \frac{d^2 p}{4\pi^2} \frac{d\epsilon}{2\pi} [-\epsilon(\epsilon + \omega) + v_F^2(p_x + k_x)p_x \\ & + v_\Delta^2(p_y + k_y)p_y] [\epsilon^2 + v_F^2 p_x^2 + v_\Delta^2 p_y^2]^{-1} \\ & \times [(\epsilon + \omega)^2 + v_F^2(p_x + k_x)^2 + v_\Delta^2(p_y + k_y)^2]^{-1} \\ &= -\frac{v_F(\omega^2 + 3v_F^2 k_x^2 + 3v_\Delta^2 k_y^2)}{64v_\Delta [\omega^2 + v_F^2 k_x^2 + v_\Delta^2 k_y^2]^{1/2}} \quad (69) \end{aligned}$$

where dimensional regularization was used in evaluating the integrals (this is a rapid way of picking out the non-analytic piece, as the analytic contribution has already been absorbed by renormalizing ρ_s).

Now we can proceed to a computation of the renormalization of the vortex action as in Section II except that the substitution in Eq. (68) has to be performed in the gauge field action in Eq. (3). As before, we work in the Coulomb gauge. We focus on the \mathcal{A}_τ fluctuations, and the expression in Eq. (13) is now replaced by

$$\begin{aligned} \tilde{\mathcal{S}}_{vA} &= 2\pi^2 \rho_s \int d\tau d\tau' \int \frac{d^2 k d\omega}{8\pi^3} e^{-i\omega(\tau - \tau')} \\ & \times \left[k_y^2 \left(1 + \frac{\mathcal{K}(k_x, k_y, \omega)}{\rho_s} \right) + k_x^2 \left(1 + \frac{\mathcal{K}(k_y, k_x, \omega)}{\rho_s} \right) \right]^{-1} \\ & \times e^{i\mathbf{k} \cdot (\mathbf{r}_v(\tau) - \mathbf{r}_v(\tau'))} \quad (70) \end{aligned}$$

Expanding this result as usual in small $\mathbf{r}_v(\omega)$, and also

to leading order in \mathcal{K} , we obtain the form in Eq. (4) with

$$\begin{aligned} F_{\parallel}(\omega) &= -\pi^2 \int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2} [k_y^2 (\mathcal{K}(k_x, k_y, \omega) - \mathcal{K}(k_x, k_y, 0)) \\ & + k_x^2 (\mathcal{K}(k_y, k_x, \omega) - \mathcal{K}(k_y, k_x, 0))] \quad (71) \end{aligned}$$

It is interesting to note that all factors of ρ_s have dropped out of the expression for $F_{\parallel}(\omega)$, and a universal expression dependent only upon the velocities v_F and v_Δ has been obtained. The momentum integral in Eq. (71) is also free of infrared divergences. However, an ultraviolet divergence is present, and then the result has the form claimed earlier in Eq. (5) with F_2 a universal function of the velocity ratio.

It can be verified in a similar manner that the fluctuations of the spatial component of the gauge field, \vec{A} lead to a non-universal correction to the vortex mass.

VII. INFLUENCE OF THE DOPPLER SHIFT: BEYOND PERTURBATION THEORY

Section V has provided insight into the vortex dynamics with neglected Doppler shift. Then we included the effect of the Doppler shift in Section VIA, but only in perturbation theory. Now we go back to the full linearized quasiparticle Hamiltonian (33) and describe a computation which includes the Doppler shift beyond perturbation theory. We will only focus on an isolated vortex at zero temperature (some interesting effects in presence of other vortices are discussed in the Appendix D). In such circumstances, only the ‘‘longitudinal’’ dynamics is non-trivial. The main questions that we ask are how the vortex mass is modified, and whether dissipation or some anomalous dynamics emerges at small frequencies due to the Doppler shift. We will find that even after accounting for the Doppler shift, the vortex mass remains finite, as anticipated in the perturbation theory in the previous section.

It was hinted in Section V that infra-red divergent vortex mass, or various forms of dissipation, could emerge at zero temperature if the bulk LDOS were finite at zero energy. Generally, one would expect that the presence of a vortex does not alter the bulk quasiparticle LDOS. This was found to be true in absence of the Doppler shift, and should remain true when the Doppler shift is included, since it is a localized perturbation (at any finite London penetration depth). Then, the vanishing zero-energy bulk LDOS of an ideal d -wave superconductor leads to a finite quasiparticle contribution to the vortex mass. This scenario is intuitive, and needs to be verified by a rigorous calculation. We will also verify in this section whether the Doppler shift introduces dangerous changes in LDOS near the vortex core. The concern about possible anomalous vortex dynamics also comes from a different point of view. The anomalous dynamics was found in Section V whenever particle-particle or hole-hole transitions were allowed. With the Doppler shift viewed as a perturbation, the exact eigenstates will be superpositions of states

(41) and (42), so that the exact particle-like states will have “hole tails”. Even the particle-hole transitions between exact states, allowed at zero temperature, will incorporate hole-hole transitions between the unperturbed states.

In the following we will exactly diagonalize the Hamiltonian (33) on a sample with finite size R , and at the node $\mathbf{p} = \hbar k_F \hat{\mathbf{x}}$; the other nodes are easily included on symmetry grounds. Only the worst case scenario of infinite London penetration depth will be considered. We will use the states (41) and (42) as the basis states for our representation. The main reason for choosing these basis states is that the transition matrix elements between them are analytically known. This way all singularities, which are difficult to handle numerically, will be captured inside analytical expressions. Therefore, even a rough information about wavefunctions in the thermodynamic limit $R \rightarrow \infty$, obtained from numerical finite size scaling, might suffice to establish properties of the vortex action.

At zero temperature we adapt the expression (25) and write the longitudinal part of the vortex action:

$$S_{v\parallel} = \int \frac{d\omega}{2\pi} \int_0^{2\Lambda} d\Delta \epsilon \frac{\omega^2 \Delta \epsilon}{\Delta \epsilon^2 + \omega^2} \mathcal{R}(\Delta \epsilon) |\mathbf{r}_v(\omega)|^2 \quad (72)$$

$$\mathcal{R}(\Delta \epsilon) \propto \sum_{n,n'} \Theta(\epsilon_n) \Theta(-\epsilon_{n'}) \delta(\epsilon_n - \epsilon_{n'} - \Delta \epsilon) |\mathbf{U}_{n,n'}|^2.$$

We will use labels n for the exact eigenstates, and ϵ_n for exact energies. The basis states will be labeled by their angular momentum l and energy E , with $q = \text{sign}(E)$ and $k = |E|$ understood. The exact transition matrix elements $\mathbf{U}_{n,n'}$, defined by (23), take the following form in the chosen representation:

$$\mathbf{U}_{n_1, n_2} = \sum_{l_1, l_2, -\Lambda}^{\Lambda} \int dE_1 \int_{-\Lambda}^{\Lambda} dE_2 \langle E_1, l_1 | n_1 \rangle \langle n_2 | E_2, l_2 \rangle \times \mathbf{U}_{(E_1, l_1), (E_2, l_2)}, \quad (73)$$

where the matrix elements $\mathbf{U}_{(E_1, l_1), (E_2, l_2)}$ between the basis states are given by (45).

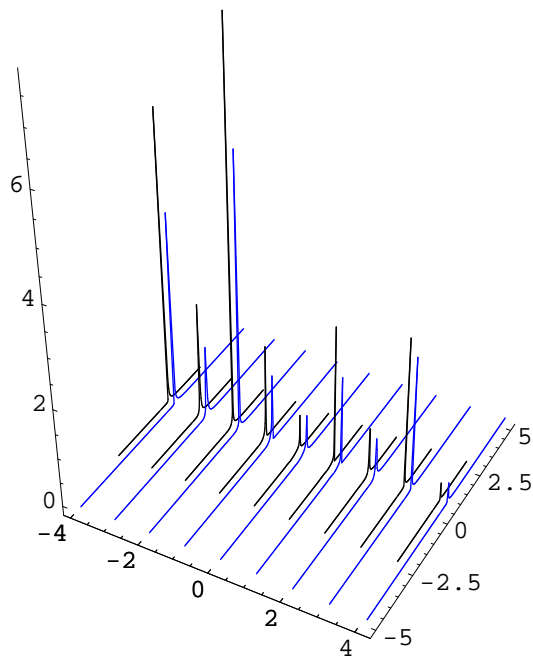
A. Exact quasiparticle spectrum and states

We carried out numerical diagonalizations with typically $N_k = 50 \div 100$ discrete values for the radial wavevector k . This merely sets the energy cut-off to $\Lambda \sim N_k R^{-1}$, and provides a plenty of states with $kR \gg 1$ that we are interested in. The number of angular momentum channels l used in calculations was varied between $N_l = 9 \div 29$. Ideally, $N_l \sim \Lambda R$ is needed in order to include all possible states with $|E| < \Lambda$ in the Hamiltonian representation, but this would be too formidable. Fortunately, even though we worked with small N_l , the final results showed

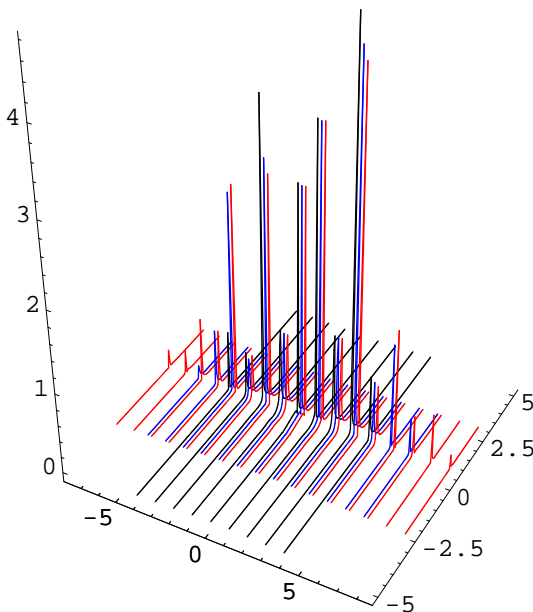
virtually no dependence on N_l , meaning that all interesting physics at low temperatures is set by small angular momenta. The finite size scaling can be done in two ways, by changing either R , or Λ . This is because there is no specific scale in the linearized Hamiltonian and thermodynamic limit (all coordinate dependence comes from the products kr). Details of the Hamiltonian representation are shown in the Appendix F.

The numerically obtained spectrum appears to be gapless. It also appears symmetric under “particle-hole” exchange: for every state with energy ϵ there is exactly one state with energy $-\epsilon$. This property is, actually, protected by a symmetry of the full Hamiltonian. There is a unitary transformation that transforms the Hamiltonian H in (33) at any quasiparticle node to $-H$, and this transformation is simply a coordinate system rotation by 180° degrees ($\phi \rightarrow \phi + \pi$ in (33) after substitutions like (39)). Therefore, the Doppler shift does not cause “spectral flow”, that is a shift of some hole-like energy levels ($\epsilon < 0$) into particle-like regions ($\epsilon > 0$) or the other way round. The situation would have been different if we considered a system of vortices, say a vortex lattice, that is not inversion symmetric. If the 180° rotation symmetry were violated, “spectral flow” could happen. As analyzed in Appendix D, this could lead to dissipation and anomalous dynamics at zero temperature.

An example of numerically obtained eigenstates is plotted in Figures 2(a) and 2(b). These figures show the amplitude of $\Psi_n(E, l) = \langle E, l | n \rangle$ for a randomly chosen eigenstate $|n\rangle$ of the full Hamiltonian as a function of E and l (which characterize the basis states - eigenstates in absence of the Doppler shift). In order to gain insight about what happens in the thermodynamic limit, the wavefunctions $\Psi_n(E, l)$ were normalized to the system radius $\frac{R}{\pi}$ instead of usual unity. The Figure 2(a) compares two different system sizes R . It can be noticed that roughly at $\epsilon \sim E$ there is a single point where the amplitude is $|\Psi|^2 \propto R$. This indicates that a Dirac peak may be developing in the thermodynamic limit at $\epsilon = E$. At all other points, the amplitudes $|\Psi|^2$ coincide for the different system sizes, and there are “tails” of non-zero amplitude that spread away from $\epsilon \sim E$, describable by finite analytic functions. Possible survival of the Dirac peak can be understood in a perturbative point of view where the strength of the Doppler shift H_D is parametrized and the full Hamiltonian written as $H = H_0 + \lambda H_D$. For $\lambda = 1$ we would obtain the Hamiltonian (33) that is being numerically diagonalized. If λ were zero, then the wavefunction Ψ would be only the Dirac peak by definition: $\Psi_{\epsilon, i}(E, l) = \delta_{i, l} \delta(\epsilon - E)$, while for $\lambda > 0$ the amplitude would start leaking to other energies $\epsilon \neq E$, as was found numerically. From the scattering theory we would expect that even for $\lambda = 1$ the Dirac peak survives at $\epsilon = E$; the Doppler shift is a sufficiently localized perturbation to the Hamiltonian that still has a singular gauge field.



(a)



(b)

FIG. 2: (a) Typical wavefunction amplitude as a function of energy and angular momentum for two system sizes, $R \approx 63$ and $R \approx 125$. (b) Typical wavefunction amplitude as a function of energy and angular momentum for three numbers of angular momentum channels, $N_l = 9$, $N_l = 13$, and $N_l = 17$. Normalization is $\frac{R}{\pi}$, and $\alpha_D = 0.4$.

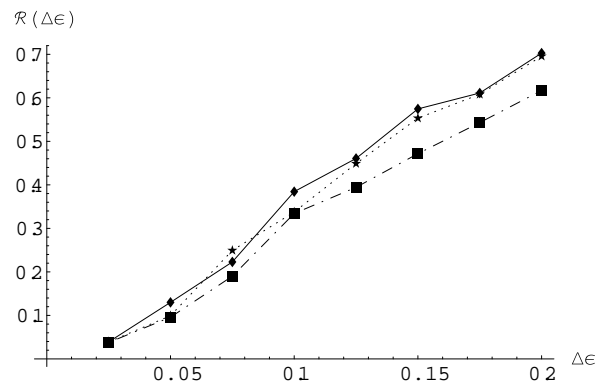


FIG. 3: Function $\mathcal{R}(\Delta\epsilon)$ for $\alpha_D = 10$, calculated with different numbers $l_{\max} \in \{4, 6, 8\}$ of included angular momentum channels: $l \in \{-l_{\max}, \dots, l_{\max}\}$. Both the horizontal and vertical axes are expressed in units that scale as $1/R$ with system radius R .

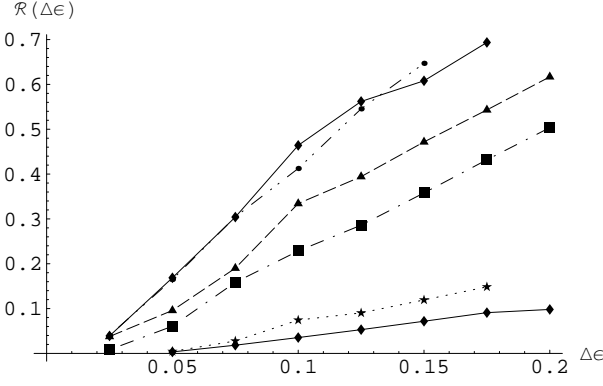
B. Vortex dynamics at small frequencies

In order to obtain Ohmic dissipation, the function $\mathcal{R}(\Delta\epsilon)$ in (72) should depend on $\Delta\epsilon$ as $\Delta\epsilon^{-1}$. Actually, $\Delta\epsilon^\nu$ with any $\nu \leq 0$ is a candidate for anomalous behavior that dominates over finite vortex inertia at small frequencies. The plot of $\mathcal{R}(\Delta\epsilon)$ evaluated numerically is shown in the Figure 3. There are no indications that any anomalous dynamics could occur. The situation resembles very much the one that was encountered in absence of the Doppler shift at zero temperature. The function $\mathcal{R}(\Delta\epsilon)$ appears to be linear, so that its slope s from the Figure 3 determines the vortex mass:

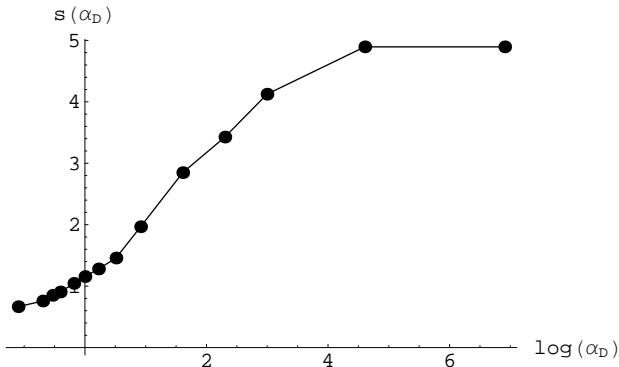
$$m_v = \frac{s(\alpha_D)}{8} \left(\alpha_D + \frac{1}{\alpha_D} \right) m_e. \quad (74)$$

For $\alpha_D = 10$ this roughly equals 3.3 electron masses, which is a factor of 6.59 enhancement of the vortex mass to that obtained in absence of the Doppler shift. Clearly the Doppler shift alone provides most of the vortex mass, but otherwise seems not to alter the qualitative physics that was uncovered in Section V. In Figure 4 we compare the functions $\mathcal{R}(\Delta\epsilon)$ for several values of the anisotropy factor α_D , and plot the slope $s(\alpha_D)$. The slope $s(\alpha_D)$ does not change significantly above $\alpha_D \geq 10$, so that in this range the vortex mass is roughly proportional to the anisotropy factor α_D . In the limit $\alpha_D \rightarrow 0$ the matrix elements of the Doppler shift vanish (see Appendix F), so that the slope $s(\alpha_D)$ slowly converges to $s \approx 8 * 0.05 = 0.4$ given by (55).

Another indication that there should be no anomalous dynamics is found in the quasiparticle density of states. The total density of states at low energies in Figure 5 is easily obtained from the exact spectra, and it is indicative of the bulk LDOS. The bulk LDOS remains a linear function of energy, and roughly equal to LDOS in absence of the Doppler shift, as expected. Close to the vortex core LDOS acquires anisotropy in the rescaled coordi-



(a)



(b)

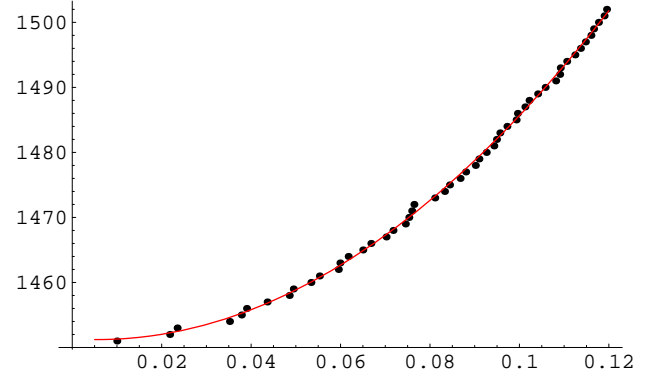
FIG. 4: (a) Function $\mathcal{R}(\Delta\epsilon)$ for $\alpha_D \in \{0.33, 1, 5, 10, 100, 1000\}$ (slope grows with α_D), calculated with $l_{\max} = 4$. Both the horizontal and vertical axes are expressed in units that scale as $1/R$ with system radius R . (b) Slope of $\mathcal{R}(\Delta\epsilon)$ (linear fit) as a function of the anisotropy ratio α_D .

nates, but does not deviate from the $1/r$ behavior. An asymptotic solution³⁶ for the wavefunctions that dominate LDOS at $r \rightarrow 0$ in presence of the Doppler shift (at the node $\hbar k_F \hat{x}$) is:

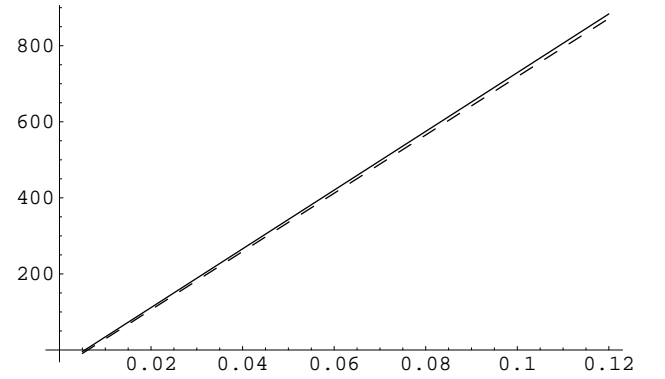
$$\psi(\mathbf{r}) \propto \frac{1}{\sqrt{r}} \begin{pmatrix} c_1 \cos\left(\frac{\cos\phi}{2}\right) - c_2 \sin\left(\frac{\cos\phi}{2}\right) \\ -e^{-i\phi} \left(c_1 \sin\left(\frac{\cos\phi}{2}\right) + c_2 \cos\left(\frac{\cos\phi}{2}\right) \right) \end{pmatrix}, \quad (75)$$

where c_1 and c_2 are constants fixed by matching to the regions of large r . The transition matrix elements (23) that involve these states and other normalizable states are always finite; the only possible concern are the “diagonal” matrix elements, but they must be finite as well because the states at finite energies carry finite momenta.

The last argument against dissipation and anomalous dynamics is presented in the Appendix G. There we study in detail analytic properties of the exact transi-



(a)



(b)

FIG. 5: (a) Total number of quasiparticle states in presence of a vortex. Also shown is a quadratic fit to the data. (b) Total density of states of the nodal quasiparticles in presence of a vortex. Dashed line shows the density of states in absence of the Doppler shift. Both plots have parameters $\alpha_D = 10$, $l_{\max} = 14$, and both the horizontal and vertical axes are expressed in units that scale as $1/R$ with system radius R .

tion matrix elements U_{n_1, n_2} . Making use of the known transition matrix elements (45) between the basis states in absence of the Doppler shift, we ask the following general question: what should the exact eigenstates look like in this basis in order for dissipation to occur. The general conclusion is that dissipation would require from the exact Hamiltonian to either systematically shift or shuffle the energy levels of the basis states in vicinity of the zero energy. The first scenario is equivalent to an effective chemical potential for the linearized quasiparticles, and it was discussed in the Appendix (D), while the second scenario could happen due to Zeeman splitting or disorder. In both cases the bulk LDOS would be finite. Neither of these scenarios seem to be realized in our situation of interest, when we consider the quantum-coherent Doppler shift due to the circulating supercurrents of a

single vortex.

VIII. DISCUSSION AND CONCLUSIONS

We studied various effects introduced by nodal quasiparticles in clean d -wave superconductors on dynamics of isolated vortices in thermal equilibrium. These effects generally give dominant contributions to vortex mass and dissipation in fermionic superfluids, over the corresponding contributions of the superfluid order parameter. In d -wave superconductors the quasiparticles have gapless nodes, and this makes quantum-mechanical phenomena fundamentally important for vortex dynamics at low temperatures. Therefore, we used methods that carefully took into account the phase change of quasiparticle wavefunction upon encircling a vortex, resulting with the proper determination of quasiparticle density of states. Classical effects of the superfluid flow in the vicinity of a vortex were taken into account through the Doppler shift of quasiparticle Hamiltonian, and it was shown that they do not lead to qualitative changes, but give rise to important quantitative corrections to the vortex mass. Vortex dynamics was studied both perturbatively and non-perturbatively, and the two approaches yielded the same results. The perturbation theory was formulated in a dual description of d -wave superconductors, where the Cooper-pair fluctuations are handled as a $U(1)$ gauge field that mediates interactions between “charged” particle-like vortices. The non-perturbative approach was focused on the small-amplitude response of a vortex to a weak oscillating force. Our findings are also in agreement with simple scaling arguments that regard the d -wave quasiparticles as a quantum critical system.

Our main conclusions are the following: (1) at zero temperature the vortex mass is finite and completely specifies vortex dynamics, apart from a universal “sub-Ohmic” damping; (2) at finite temperatures vortex motion becomes dissipated in an Ohmic fashion, with certain logarithmic corrections. Furthermore, quasiparticles mediate new interactions between vortices (assumed to be far enough apart so that quasiparticle motion between vortices is incoherent), leading to both longitudinal and transversal forces in addition to the usual “hydrodynamic” forces. All parameters that characterize vortex dynamics depend quantitatively (and not qualitatively) on the boundary conditions at the vortex core. We reached these conclusions under several assumptions. First, the vortex core was idealized by a point in the continuum limit. This was primarily motivated by the fact that in good d -wave superconductors the coherence length ξ is not much larger than the Fermi wavelength k_F^{-1} , so that the core is indeed small. Furthermore, the superconducting gap vanishes in the nodal momentum directions, leaving a route for quasiparticles inside the core to leak out. Both of these facts conspire against bound core states, which is the situation that our idealization of the core models well, and various other studies sup-

port. The vortex mass is, then, determined by scattering of free quasiparticles from a moving vortex. Smallness of the core renders quantum phenomena very important. At finite temperatures we ignored thermal fluctuations of the superfluid order parameter, but kept the quasiparticles in thermal equilibrium. Presumably, as long as temperatures are low and quasiparticle contributions to vortex dynamics dominant, such an approximation is valid. Effects of thermal non-equilibrium have not been taken into account, but they are expected not to be important in the circumstances that we consider (slow oscillatory vortex motion with very small amplitude).

As we noted in Section I, there is a possible disagreement between our results and the semiclassical approach^{11,12,13,14}. The latter works find that the nodal quasiparticles induce an effective mass which diverges linearly with the separation between vortices. In particular, it does appear that direct application of our theory and the semiclassical theory leads to distinct predictions for a certain class of experiments. It has recently been argued^{16,18} that the zero point quantum motion of vortices leads to modulations in the local density of states, and this was proposed as the explanation of recent STM experiments on the cuprate superconductors^{38,39}. The spatial extent of the zero point motion, and hence density modulations, is determined¹⁸ by the value of m_v . (Indeed, this fact was the initial motivation for our study.) Consequently, m_v can be determined, in principle, from STM measurements of the modulations. Our theory predicts that such an m_v should be independent of the field H , at small H , while the semiclassical theory predicts a divergence in m_v as $\sim 1/\sqrt{H}$.

In the future, it is also possible that dynamic (THz scale) measurements of vortex dynamics will lead to more direct quantitative determinations of m_v , and these should allow tests of the theoretical proposals.

At the end, we make some comments about the incoherent quasiparticle mediated interactions between vortices, explored in the Appendix D. They were modeled by an effective chemical potential μ in the linearized quasiparticle Hamiltonian, which describes the Doppler shift due to the supercurrents from a second nearby vortex. The interactions that we discussed in Appendix D are specific to d -wave superconductors (for similar effects to be observed in s -wave superconductors, μ would have to be larger than the gap, that is the two vortices practically overlapping). The effects of these interactions would be visible in vortex-vortex scattering events: the vortices would lose some of their initial kinetic energy during scattering, and excite some low-energy extended quasiparticle states. Anisotropy of d -wave superconductors would be reflected in these interactions, so that the differential scattering cross-section, and the amount of kinetic energy lost, would depend on the initial direction of approach toward the point of collision. We also speculate that these interactions might play an important role in shaping the properties of vortex lattices (especially if pinning disorder ruins the vortex lattice inversion symmetry). They could

lead to anisotropic stiffness and pinning of the vortex lattice orientation to that of the substrate. The discovered dissipation-looking response of a single vortex in presence of another vortex could transform into significant and magnetic field dependent correction to the vortex mass in the equilibrium conditions of the vortex lattice. We leave these interesting issues for future work.

IX. ACKNOWLEDGEMENTS

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APPENDIX A: DETAILS OF THE VORTEX ACTION DERIVATION

In this appendix we provide support for the section III by explicitly evaluating traces from the expansion (18). At the first order in vortex displacement we have:

$$\begin{aligned}
& - \operatorname{tr} [G_0(\mathbf{r}_v \nabla_v) H_0] = \\
& = - \int d\tau d^2 r \operatorname{tr} [G_0(\mathbf{r}, \tau; \mathbf{r}, \tau) (\mathbf{r}_v(\tau) \nabla_v) H_0(\mathbf{r})] \\
& = \int d\tau d^2 r \sum_n \frac{1}{\beta} \sum_\omega \frac{e^{i\omega 0^+}}{-i\omega + \epsilon_n} \times \\
& \quad \operatorname{tr} [\psi_n(\mathbf{r}) \psi_n^\dagger(\mathbf{r}) \mathbf{r}_v(\tau) \nabla H_0(\mathbf{r})] \quad (\text{A1}) \\
& = \int d\tau d^2 r \sum_n \frac{1}{\beta} \sum_\omega \frac{e^{i\omega 0^+}}{-i\omega + \epsilon_n} (\epsilon_n - \epsilon_n) \times \\
& \quad \operatorname{tr} [(\mathbf{r}_v(\tau) \nabla \psi_n(\mathbf{r})) \psi_n^\dagger(\mathbf{r})] = 0,
\end{aligned}$$

where we have used (19), (20), and integration by parts. For integration purposes, in absence of disorder, we may replace the vortex position gradient ∇_v with the negative normal gradient: $\nabla_v \rightarrow -\nabla$. This is just a shift of the coordinate system that puts a displaced vortex back to the origin, and simplifies derivations.

At the second order in vortex displacement we have two contributions. One comes from the second term in (18):

$$\begin{aligned}
& -\frac{1}{2} \operatorname{tr} [G_0(\mathbf{r}_v \nabla_v)^2 H_0] = \quad (\text{A2}) \\
& -\frac{1}{2} \int d\tau d^2 r d^2 r' \sum_{n,n'} \frac{1}{\beta} \sum_\omega \frac{e^{i\omega 0^+}}{-i\omega + \epsilon_n} \times \\
& \quad \operatorname{tr} \left[\psi_n(\mathbf{r}) \psi_n^\dagger(\mathbf{r}') (\mathbf{r}_v(\tau) \nabla)^2 \left(\epsilon_{n'} \psi_{n'}(\mathbf{r}') \psi_{n'}^\dagger(\mathbf{r}) \right) \right].
\end{aligned}$$

In order to evaluate it, we sum over the frequencies:

$$\frac{1}{\beta} \sum_\omega \frac{e^{-i\omega\tau}}{-i\omega + \epsilon} = e^{-\epsilon\tau} \left\{ \begin{array}{ll} -f(\epsilon) & ; \quad \tau < 0 \\ 1 - f(\epsilon) & ; \quad \tau > 0 \end{array} \right\}, \quad (\text{A3})$$

Then, after integration by parts, some algebraic manipulation and making use of the fact that the wavefunctions $\psi_n(\mathbf{r})$ form a complete set of states, the expression (A2) reduces to:

$$\frac{1}{2} \sum_{n,n'} (f(\epsilon_n) - f(\epsilon_{n'})) (\epsilon_{n'} - \epsilon_n) \int d\tau |\mathbf{r}_v(\tau) \mathbf{U}_{n,n'}|^2 \quad (\text{A4})$$

This has the form of a trapping potential.

The other contribution to the second order in vortex displacement comes from the third term in (18):

$$\begin{aligned}
& \frac{1}{2} \operatorname{tr} (G_0 \Delta H)^2 = \quad (\text{A5}) \\
& \int d\tau d^2 r \int d\tau' d^2 r' \operatorname{tr} \left[G_0(\mathbf{r}, \tau; \mathbf{r}', \tau') (\mathbf{r}_v(\tau') \nabla_v H_0(\mathbf{r}')) \right. \\
& \quad \left. G_0(\mathbf{r}', \tau'; \mathbf{r}, \tau) (\mathbf{r}_v(\tau) \nabla_v H_0(\mathbf{r})) \right] \\
& = -\frac{1}{2} \int d\tau d\tau' \sum_{n,n'} \frac{1}{\beta^2} \sum_{\omega,\omega'} \frac{e^{-i(\omega-\omega')(\tau-\tau')}}{(-i\omega + \epsilon_n)(-i\omega' + \epsilon_{n'})} \times \\
& \quad (\epsilon_n - \epsilon_{n'})^2 (\mathbf{r}_v(\tau') \mathbf{U}_{n,n'}) (\mathbf{r}_v(\tau) \mathbf{U}_{n',n}).
\end{aligned}$$

The matrix elements $\mathbf{U}_{n,n'}$ are defined by (23). If we integrate out τ and τ' and use $\mathbf{U}_{n',n} = -\mathbf{U}_{n,n'}^*$ we obtain:

$$\begin{aligned}
\frac{1}{2} \operatorname{tr} (G_0 \Delta H)^2 & = \frac{1}{2} \sum_{n,n'} \frac{1}{\beta^2} \sum_{\omega,\omega'} \frac{e^{-i(\omega-\omega')(\tau-\tau')}}{(-i\omega + \epsilon_n)(-i\omega' + \epsilon_{n'})} \times \\
& \quad (\epsilon_n - \epsilon_{n'})^2 |\mathbf{r}_v(\omega - \omega') \mathbf{U}_{n,n'}|^2. \quad (\text{A6})
\end{aligned}$$

Let $\Delta\omega = \omega - \omega'$. We can carry out summation over ω using the formula:

$$\frac{1}{\beta} \sum_\omega \frac{1}{(-i\omega + \epsilon_1)(-i\omega + \epsilon_2)} = \frac{f(\epsilon_1) - f(\epsilon_2)}{\epsilon_1 - \epsilon_2}. \quad (\text{A7})$$

The trace (A6) becomes:

$$\begin{aligned}
\frac{1}{2} \operatorname{tr} (G_0 \Delta H)^2 & = \frac{1}{2} \sum_{n,n'} \frac{1}{\beta} \sum_{\Delta\omega} \frac{f(\epsilon_n) - f(\epsilon_{n'} + i\Delta\omega)}{\epsilon_n - \epsilon_{n'} - i\Delta\omega} \times \\
& \quad (\epsilon_n - \epsilon_{n'})^2 |\mathbf{r}_v(\Delta\omega) \mathbf{U}_{n,n'}|^2. \quad (\text{A8})
\end{aligned}$$

Note that $\Delta\omega$ takes values $2\pi T \times \text{integer}$, so that $f(\epsilon_{n'} + i\Delta\omega)$ is simply $f(\epsilon_{n'})$.

The vortex action at the second order in vortex displacement is obtained by adding (A4) and (A8). In expression (A4) we perform Fourier transformation of $\mathbf{r}_v(\tau)$ and integrate over τ , while in expression (A8) we relabel $\Delta\omega$ into ω . The result is given by (25).

**APPENDIX B: NOTES ON THE DERIVATION
OF U_{n_1, n_2} IN ABSENCE OF THE DOPPLER
SHIFT**

Here we outline some steps that lead to the result (45). We calculate (44) in the rescaled coordinate system upon substitution of (41) and (42), together with the representation of gradient in the polar coordinates (39). In order to treat all cases of angular momentum on the same footing in (41) and (42), we label the indices of the Bessel functions in the upper and lower spinor component by l' and l'' respectively. Later we worry about dependence of l' and l'' on l . First we integrate out the polar angle ϕ . This prohibits all transitions between states whose angular momenta do not differ by one. Hence, let $\sigma = l_2 - l_1 = \pm 1$. Then, we use the following identities for the Bessel functions:

$$\begin{aligned} \frac{l}{kr} J_l(kr) &= \frac{1}{2} (J_{l-1}(kr) + J_{l+1}(kr)) \quad (\text{B1}) \\ \frac{1}{k} \frac{\partial}{\partial r} J_l(kr) &= \frac{1}{2} (J_{l-1}(kr) - J_{l+1}(kr)) \end{aligned}$$

in order to carry out differentiation over r and eliminate the $1/r$ factors that originate from the azimuthal gradient components. We introduce notation:

$$\gamma_{l_1, l_2}(k_1, k_2) = \int_0^\infty dr r J_{l_1}(k_1 r) J_{l_2}(k_2 r), \quad (\text{B2})$$

and

$$\begin{aligned} A' &= \frac{1}{2} \sqrt{\frac{k_1 k_2}{|E_1 E_2|}} (\sqrt{E_1 - m})^* \sqrt{E_2 - m} \quad (\text{B3}) \\ A'' &= q_1 q_2 \text{sign} \left(l_1 - \frac{1}{2} \right) \text{sign} \left(l_2 - \frac{1}{2} \right) \times \\ &\quad \frac{1}{2} \sqrt{\frac{k_1 k_2}{|E_1 E_2|}} (\sqrt{E_1 + m})^* \sqrt{E_2 + m}. \end{aligned}$$

After some algebraic manipulations, we arrive at:

$$\begin{aligned} \mathbf{U}_{n_1, n_2} &= \left(\frac{\hat{\mathbf{x}}}{v_F} + i\sigma \frac{\hat{\mathbf{y}}}{v_\Delta} \right) \frac{k_2}{4} \times \quad (\text{B4}) \\ &\quad \left[A' \left(1 + \sigma \frac{l_2 - 1}{l_2'} \right) \gamma_{l_1', l_2' - 1}(k_1, k_2) + \right. \\ &\quad A' \left(-1 + \sigma \frac{l_2 - 1}{l_2'} \right) \gamma_{l_1', l_2' + 1}(k_1, k_2) + \\ &\quad A'' \left(1 + \sigma \frac{l_2}{l_2''} \right) \gamma_{l_1'', l_2'' - 1}(k_1, k_2) + \\ &\quad \left. A'' \left(-1 + \sigma \frac{l_2}{l_2''} \right) \gamma_{l_1'', l_2'' + 1}(k_1, k_2) \right], \end{aligned}$$

for $\sigma = l_2 - l_1 = \pm 1$ (otherwise, $\mathbf{U}_{n_1, n_2} = 0$). Next, we evaluate $\gamma_{l_1, l_2}(k_1, k_2)$. Note that for the case of interest

$\theta = 0$ we can write:

$$l' = \left\{ \begin{array}{ll} -l + \frac{1}{2} & , \quad l \leq 0 \\ l - \frac{1}{2} & , \quad l > 0 \end{array} \right\} \quad (\text{B5})$$

$$l'' = \left\{ \begin{array}{ll} -l - \frac{1}{2} & , \quad l \leq 0 \\ l + \frac{1}{2} & , \quad l > 0 \end{array} \right\}.$$

If this is substituted in (B4), than it can be easily seen that the angular momentum indices of γ never differ by more than two. This allows us to evaluate the γ terms. If the angular momentum indices are equal, then:

$$\gamma_{l, l}(k_1, k_2) = \frac{1}{k_2} \delta(k_2 - k_1). \quad (\text{B6})$$

If the angular momentum indices differ by two, then use the first identity from (B1) to obtain:

$$\begin{aligned} \gamma_{l, l \pm 2}(k_1, k_2) &= \frac{2(l \pm 1)}{k_2} \int_0^\infty dr J_l(k_1 r) J_{l \pm 1}(k_2 r) \\ &\quad - \gamma_{l, l}(k_1, k_2). \quad (\text{B7}) \end{aligned}$$

The integral above is a special case of Weber-Schafheitlin formula⁴⁰, which yields simple expressions:

$$\begin{aligned} \gamma_{l, l-2}(k_1, k_2) &= \frac{2(l-1)}{k_2^2} \left(\frac{k_2}{k_1} \right)^l \Theta(k_1 - k_2) \\ &\quad - \frac{1}{k_2} \delta(k_1 - k_2) \quad (\text{B8}) \\ \gamma_{l, l+2}(k_1, k_2) &= \frac{2(l+1)}{k_1^2} \left(\frac{k_1}{k_2} \right)^{l+2} \Theta(k_2 - k_1) \\ &\quad - \frac{1}{k_1} \delta(k_2 - k_1), \end{aligned}$$

where $\Theta(x)$ is a step-function:

$$\Theta(x) = \left\{ \begin{array}{ll} 1 & , \quad x > 0 \\ \frac{1}{2} & , \quad x = 0 \\ 0 & , \quad x < 0 \end{array} \right\}. \quad (\text{B9})$$

If $l_1 = 0$ or $l_2 = 0$ then in some cases the two angular momentum indices of the γ terms may differ by one. Then, however, exact expressions for the appropriate Bessel functions are simple enough to directly integrate out γ and add their contributions to \mathbf{U}_{n_1, n_2} . Further manipulations are still quite complex, and we do not include them here. We only note that it is safe at this stage to set $m = 0$ (no singularities are introduced). The final result is (45).

**APPENDIX C: CALCULATION OF THE
VORTEX MASS AT ZERO TEMPERATURE IN
ABSENCE OF THE DOPPLER SHIFT**

In this appendix we substitute (49) into (53) term by term, and explicitly calculate the integral for the vortex

mass in absence of the Doppler shift. The goal is to show that there are no infra-red divergences, and to provide an estimate of the vortex mass in terms of the electron mass.

The first term of (49) contributes the following to the vortex mass:

$$\begin{aligned}
M_V^{(1)} &\propto \int_{-\Lambda}^0 d\epsilon_1 \int_0^\Lambda d\epsilon_2 \frac{1}{\epsilon_2 - \epsilon_1} \frac{(\epsilon_1 + \epsilon_2)^2 (\epsilon_1^2 + \epsilon_2^2)}{|\epsilon_1^2 - \epsilon_2^2| \max(\epsilon_1^2, \epsilon_2^2)} \\
&= \int_0^\Lambda dk_1 \int_0^\Lambda dk_2 \frac{1}{k_2 + k_1} \frac{(k_2 - k_1)^2 (k_1^2 + k_2^2)}{|k_2^2 - k_1^2| \max(k_1^2, k_2^2)} \\
&= 2 \int_0^\Lambda dk_1 \int_0^{k_1} dk_2 \frac{(k_1 - k_2)(k_1^2 + k_2^2)}{(k_1 + k_2)^2 k_1^2} = \dots \\
&= 3(3 - 4 \log 2) \Lambda = 0.682234 \Lambda,
\end{aligned}$$

where the constant of proportionality is:

$$C = \frac{1}{16} \left(\frac{1}{v_F^2} + \frac{1}{v_\Delta^2} \right). \quad (\text{C1})$$

This contribution to the vortex mass includes effects of all particle-hole transitions between states that do not involve the zero angular momentum channel. No infra-red divergency has been encountered. The second term of (49) includes partially the effects of transitions that involve the zero angular momentum channel. This term is the most divergent as $E_2 - E_1 \rightarrow 0$, and thus the most dangerous. However, it only gives a finite contribution to the vortex mass:

$$\begin{aligned}
M_V^{(2)} &\propto \int_{-\Lambda}^0 d\epsilon_1 \int_0^\Lambda d\epsilon_2 \frac{1}{\epsilon_2 - \epsilon_1} \frac{8}{\pi^2} \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 - \epsilon_2} \right)^2 \\
&= \frac{8}{\pi^2} \int_0^\Lambda dk_1 \int_0^\Lambda dk_2 \frac{(k_2 - k_1)^2}{(k_2 + k_1)^3} \\
&= \frac{2}{\pi^2} \int_0^\Lambda d\Delta k \int_{\Delta k/2}^{\Lambda - \Delta k/2} dk \frac{\Delta k^2}{k^3} = \dots \\
&= \frac{16}{\pi^2} \left(\log 2 - \frac{1}{2} \right) \Lambda = 0.313118 \Lambda.
\end{aligned}$$

The rest of (49) is hard to integrate exactly, but it is

easy to show that there are no infra-red divergences:

$$\begin{aligned}
M_V^{(3)} &\propto \frac{\Lambda}{\pi^2} \int_0^1 dx_1 \int_0^1 dx_2 \frac{(x_1 - x_2)^2}{x_1 + x_2} \times \\
&\left[\frac{1}{4} \left(\frac{1}{x_1^2} + \frac{1}{x_2^2} \right) \log^2 \left(\frac{x_1 - x_2}{x_1 + x_2} \right)^2 + \frac{2}{x_1 x_2} \log \left(\frac{x_1 - x_2}{x_1 + x_2} \right)^2 \right] \\
&= \frac{2\Lambda}{\pi^2} \int_0^{\pi/4} d\phi \int_0^{1/\cos\phi} r dr \frac{1}{r} \frac{1 - \sin 2\phi}{\cos\phi + \sin\phi} \times \\
&\left[\frac{4}{\sin 2\phi} \log \left(\frac{1 - \sin 2\phi}{1 + \sin 2\phi} \right) + \frac{1}{\sin^2 2\phi} \log^2 \left(\frac{1 - \sin 2\phi}{1 + \sin 2\phi} \right) \right] \\
&= \frac{2\Lambda}{\pi^2} \int_0^{\pi/4} d\phi \frac{1}{\cos\phi} \frac{1 - \sin 2\phi}{\cos\phi + \sin\phi} \times \\
&\left[\frac{4}{\sin 2\phi} \log \left(\frac{1 - \sin 2\phi}{1 + \sin 2\phi} \right) + \frac{1}{\sin^2 2\phi} \log^2 \left(\frac{1 - \sin 2\phi}{1 + \sin 2\phi} \right) \right] \\
&= \dots = -0.196462 \Lambda,
\end{aligned}$$

where $x_1 = k_1/\Lambda = r \cos \phi$, $x_2 = k_2/\Lambda = r \sin \phi$, and symmetry with respect to $x_1 \leftrightarrow x_2$ was used. It is easy to check that the function of ϕ inside the last integral is finite for every $\phi \in (0, \pi/4)$. The total vortex mass is:

$$m_v = M_V^{(1)} + M_V^{(2)} + M_V^{(3)} \approx 0.05 \left(\frac{1}{v_F^2} + \frac{1}{v_\Delta^2} \right) \Lambda. \quad (\text{C2})$$

APPENDIX D: QUASIPARTICLE MEDIATED INTERACTIONS BETWEEN VORTICES

In this appendix we will examine how the gapless quasiparticles of a d -wave superconductor mediate certain interactions between vortices. Consider placing a second vortex near the vortex whose dynamics we want to study. We assume here that the second vortex is far enough so that the fermionic quasiparticles near the two vortices can be treated independently. This requires some external source of dissipation, which will render quasiparticle motion between the vortices incoherent. We assume in the remainder of this appendix that such a source of dissipation is present, and so our conclusions here are limited to this case.

Recall the full linearized Hamiltonian for quasiparticles (33). Presence of a second vortex affects the quasiparticles that move around the first vortex in two ways. First, the ‘‘gauge’’ field $\mathbf{a}(\mathbf{r})$ is changed, but this is not crucial because we can always ‘‘gauge’’ it away. Second, the superfluid velocity $\mathbf{v}(\mathbf{r})$ is changed, which modifies the Doppler shift term. If the distance between two vortices is large, the supercurrent due to the second vortex appears practically uniform in vicinity of the first vortex. The corresponding Doppler shift formally looks like a chemical potential μ presented to the linearized quasiparticles. This effective chemical potential depends on

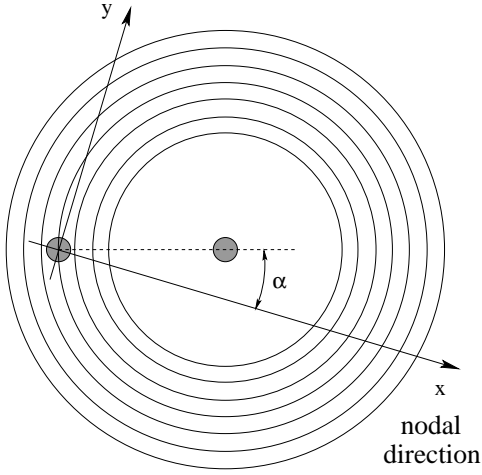


FIG. 6: One vortex in vicinity of another. Supercurrents due to the second vortex appear roughly uniform in the region occupied by the first vortex, resulting with a uniform Doppler shift of quasiparticle energies.

the distance between vortices, and their orientation with respect to the nodal directions ($\mu \propto v_x$, where v_x is the component of superfluid velocity in the nodal direction x).

In order to explore qualitative aspects of these interactions, we ignore complications arising from the vortex self-Doppler shift and non-uniformity of the supercurrents. We simply add a chemical potential μI to the Hamiltonian (36). Effect of this is that all energies are shifted by μ in the vortex action. We restrict our attention only to zero temperature, and assume $\mu > 0$ without loss of generality. Consider first the “longitudinal correlations”:

$$F_{\parallel}(\omega) \propto \int_{-\Lambda}^{\mu} dE_1 \int_{\mu}^{\Lambda} dE_2 \frac{\omega^2(E_2 - E_1)}{(E_2 - E_1)^2 + \omega^2} \times \sum_{\text{nodes}} \sum_{l,\sigma} |\mathbf{r}_v(\omega) \mathbf{U}_{n_1, n_2}|^2. \quad (\text{D1})$$

Similar to the case of finite temperatures and $\mu = 0$, certain “dangerous” particle-particle transitions are allowed when $\mu > 0$. They occur at $E_1 = E_2 = \mu$, where the bulk LDOS is not zero. Following the procedure from subsection VD, we find that the transitions involving the zero angular momentum channel dominate at small frequencies:

$$F_{\parallel}(\omega) \propto \mu^2 |\omega|, \quad |\omega| \ll \mu. \quad (\text{D2})$$

The first correction at small frequencies comes from the transitions that do not involve the zero angular momentum channel:

$$\Delta F_{\parallel}(\omega) \propto \mu \omega^2 \log\left(\frac{\mu}{|\omega|}\right), \quad |\omega| \ll \mu. \quad (\text{D3})$$

Clearly, the anomalous dynamics at small frequencies and non-zero effective chemical potential is of the same kind as that found in the case of finite temperature. Only the energy scale that controls it has been changed.

The remaining corrections are $\mathcal{O}(\omega^2)$, meaning that vortex inertia is also affected by the presence of another vortex. Physical consequences of all these effects could be directly observed in vortex-vortex scattering events as vortex kinetic energy loss to the quasiparticles. Also, dynamics of a vortex lattice should be affected. Due to quantum fluctuations of vortices the quasiparticle mediated forces would tend to pin the orientation of the vortex lattice to the substrate, and they would influence its stiffness. These effects would be even greater in absence of perfect inversion symmetry of the vortex lattice, caused for example by vortex pinning disorder.

Now consider the “transversal correlations”:

$$F_{\perp}(\omega) \propto \int_{-\Lambda}^{\mu} dE_1 \int_{\mu}^{\Lambda} dE_2 \frac{-i\omega(E_2 - E_1)}{E_2 - E_1 - i\omega} \times \sum_{l,\sigma} \sigma |U_{(q_1, l-\sigma, k_1), (q_2, l, k_2)}|^2. \quad (\text{D4})$$

As shown in subsection VE, the symmetry under $E_1 \rightarrow -E_2$ and $E_2 \rightarrow -E_1$ would set $F_{\perp}(\omega) = 0$. However, this symmetry is removed by finite μ . By substituting (50) we find:

$$F_{\perp}(\omega) \propto -i\omega (\mu\Lambda + \mathcal{O}(\mu^2 \text{sign}(\mu))) + \mathcal{O}(\omega^3). \quad (\text{D5})$$

This result can be qualitatively understood by noting that the formal particle-hole symmetry of the linearized quasiparticle Hamiltonian is broken when $\mu \neq 0$. There is a net quasiparticle current density, formed by occupied states within the energy range $0 < E < \mu$. This is possible because the exact wavefunctions (41) and (42) are mixtures of a particle and hole that move with different angular velocities.

APPENDIX E: OTHER VALUES OF THE CORE PARAMETER θ

All calculations in Section V were restricted to boundary condition at the vortex core $\theta = 0$, which physi-

cally corresponds to a complete enclosure of the vortex flux within a finite-sized cylinder. Here we briefly discuss other boundary conditions. For general value of θ the transition matrix elements (45) are modified at $l = (\sigma \pm 1)/2$:

$$\begin{aligned}
 U_{(q_1, l-\sigma, k_1), (q_2, l, k_2)} &= \cos(\theta) \left[\frac{2\sigma q_1 q_2}{\pi} \frac{E_1 + E_2}{E_1 - E_2} + \frac{C_\sigma}{2\pi} \left(\frac{k_1}{k_2} \right)^{\sigma l - \frac{1}{2}} \frac{E_1 + E_2}{\sqrt{k_1 k_2}} \log \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 \right] + \\
 &\quad \sin(\theta) \left[4\sqrt{k_1 k_2} \delta(E_2 - E_1) - C_\sigma \left(\frac{k_1}{k_2} \right)^{\sigma l - \frac{1}{2}} \frac{E_1 + E_2}{\sqrt{k_1 k_2}} \Theta(\sigma(k_2 - k_1)) \right], \quad l = \frac{\sigma + 1}{2} \\
 U_{(q_1, l-\sigma, k_1), (q_2, l, k_2)} &= -\sin(\theta) \left[\frac{2\sigma q_1 q_2}{\pi} \left(q_1 q_2 \frac{E_1 + E_2}{E_1 - E_2} + 2q_1 q_2 \sigma \right) + \frac{C_\sigma}{2\pi} \left(\frac{k_1}{k_2} \right)^{\sigma l - \frac{1}{2}} \frac{E_1 + E_2}{\sqrt{k_1 k_2}} \log \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 \right] + \\
 &\quad \cos(\theta) \left[-4\sqrt{k_1 k_2} \delta(E_1 - E_2) - C_\sigma \left(\frac{k_1}{k_2} \right)^{\sigma l - \frac{1}{2}} \frac{E_1 + E_2}{\sqrt{k_1 k_2}} \Theta(\sigma(k_1 - k_2)) \right], \quad l = \frac{\sigma - 1}{2}
 \end{aligned}$$

No new singularities have been introduced, and all results from previous subsections remain qualitatively the same. However, there are quantitative changes to the vortex mass, finite temperature dissipation coefficient, etc. Therefore, the parameters that characterize vortex dynamics depend on the boundary conditions at the vortex core. We will not investigate further this dependence.

APPENDIX F: REPRESENTATION OF THE FULL QUASIPARTICLE HAMILTONIAN

Here we construct the representation of the Hamiltonian (33) in the basis of states (41), (42). This is used to numerically diagonalize the Hamiltonian on a finite sample with radius R . The basis wavefunctions (41) and (42) are easily renormalized to unity on the finite sample if multiplied by $\sqrt{\pi/R}$. With an appropriate hard-wall boundary condition momentum is quantized to zeros of the Bessel's functions:

$$k = |E| = \frac{\pi}{2R} \left(\left| l - \frac{1}{2} \right| + 2n - \frac{1}{2} \right), \quad kR \gg 1. \quad (\text{F1})$$

Note that the degeneracy with respect to angular momentum l is lifted in a finite system. In the chosen representation, the full Hamiltonian (33) is a sum of a diagonal matrix that contains discretized energies E and the Doppler shift part whose matrix elements $\langle E_1, l_1 | H_D | E_2, l_2 \rangle$ can be evaluated using the rescaled coordinate system introduced in section V A. Assuming that the supercurrent \mathbf{v} describes a π -flux localized at the origin, as discussed in

the section IV, we have (at the $\mathbf{p} = \hbar k_F \hat{\mathbf{x}}$ node):

$$\begin{aligned}
 \langle E_1, l_1 | H_D | E_2, l_2 \rangle &= -\frac{v_F v_\Delta}{2} \int_0^{2\pi} d\phi \int_0^R dr \quad (\text{F2}) \\
 &\quad \frac{\sin \phi}{(v_F \cos \phi)^2 + (v_\Delta \sin \phi)^2} \psi_{E_1, l_1}^\dagger(r, \phi) \psi_{E_2, l_2}(r, \phi) \\
 &= -\frac{v_F v_\Delta}{2} \frac{(\sqrt{E_1})^* \sqrt{E_2}}{4R} \left[\bar{\gamma}_{l'_1, l'_2}(k_1, k_2) + \right. \\
 &\quad \left. q_1 q_2 \text{sign} \left(l_1 - \frac{1}{2} \right) \text{sign} \left(l_2 - \frac{1}{2} \right) \bar{\gamma}_{l''_1, l''_2}(k_1, k_2) \right] \times \\
 &\quad \int_0^{2\pi} d\phi \frac{\sin \phi e^{i(l_2 - l_1)\phi}}{(v_F \cos \phi)^2 + (v_\Delta \sin \phi)^2}.
 \end{aligned}$$

where, like before, $q_i = \pm 1$, $E_i = q_i k_i$, and l' and l'' are indices of the Bessel's functions in the upper and lower spinor components respectively of the wavefunctions (41) and (42) (which depend on l). Recall that \mathbf{v} is gauge invariant, so that we cannot avoid its ‘‘elliptical’’ shape in the rescaled coordinate system. The integral over ϕ can be calculated exactly; for $l = l_2 - l_1 > 0$ it is equal to:

$$\frac{\pi}{v_\Delta} \frac{1}{v_F + v_\Delta} \left| \frac{v_F - v_\Delta}{v_F + v_\Delta} \right|^{\frac{l-1}{2}} \left\{ \begin{array}{ll} i^l - (-i)^l & , \quad v_F \geq v_\Delta \\ i(1 - (-1)^l) & , \quad v_F \leq v_\Delta \end{array} \right\}.$$

The new function $\bar{\gamma}$ is:

$$\bar{\gamma}_{l_1, l_2}(k_1, k_2) = \int_0^R dr J_{l_1}(k_1 r) J_{l_2}(k_2 r). \quad (\text{F3})$$

Analytic expression for $\bar{\gamma}$ is available only in the $R \rightarrow \infty$ limit (a special case of Weber-Schafheitlin formula):

$$\bar{\gamma}_{l_1, l_2}(k_1, k_2) \approx \left\{ \begin{array}{l} \frac{k_2^{l_2}}{k_1^{l_2+1}} \frac{\Gamma(\frac{l_1+l_2+1}{2})}{\Gamma(l_2+1)\Gamma(\frac{l_1-l_2+1}{2})} {}_2F_1\left(\frac{l_1+l_2+1}{2}, \frac{l_2-l_1+1}{2}; l_2+1; \left(\frac{k_2}{k_1}\right)^2\right) \quad , \quad k_1 > k_2 \\ \frac{k_1^{l_1}}{k_2^{l_1+1}} \frac{\Gamma(\frac{l_1+l_2+1}{2})}{\Gamma(l_1+1)\Gamma(\frac{l_2-l_1+1}{2})} {}_2F_1\left(\frac{l_1+l_2+1}{2}, \frac{l_1-l_2+1}{2}; l_1+1; \left(\frac{k_1}{k_2}\right)^2\right) \quad , \quad k_2 > k_1 \end{array} \right\} , \quad (\text{F4})$$

$$\bar{\gamma}_{l_1, l_2}(k, k) \approx \left\{ \begin{array}{l} -\frac{1}{2k} \text{sign}(l_1 + l_2 + \frac{1}{2}) \text{sign}(l_1 - l_2 + \frac{1}{2}) (-1)^{\frac{l_1-l_2+1}{2}} \quad , \quad l_1 - l_2 \text{ is an odd integer} \\ \frac{1}{\pi k} \cos(\frac{l_2-l_1}{2}\pi) \log(kR) \quad , \quad \text{otherwise in the limit } kR \gg 1 \end{array} \right\} .$$

It turns out that the formulas (F1) and (F4) provide a very good approximation for our purposes. This approximation scheme uses the fact that for all states of interest $kR \gg 1$ in the thermodynamic limit. The states with $kR \sim 1$ are not treated accurately when R is finite, but all such states collapse to an infinitely small range of energies when $R \rightarrow \infty$. Then, the Hamiltonian representation can be constructed relatively fast. The results of diagonalization and vortex dynamics obtained within this approximation have been found virtually the same as the corresponding results with exact energy level discretization and exact numerical calculation of $\bar{\gamma}$ in finite systems.

APPENDIX G: SEMI-ANALYTICAL CALCULATION OF $\mathcal{R}(\Delta\epsilon)$

Here we present a semi-analytical argument against dissipation and anomalous vortex dynamics caused by the Doppler shift. We make several assumptions about general properties of the exact wavefunctions $\Psi_n(E, l) = \langle E, l | n \rangle$ in the thermodynamic limit. First, we assume that the wavefunctions $\Psi_n(E, l)$ are analytic functions of E everywhere except perhaps at $E = g(\epsilon)$, and also at $E = 0$ and/or $\epsilon = 0$. The function $g(\epsilon)$ satisfies:

$$\begin{aligned} g(\epsilon) &= -g(-\epsilon) \\ \text{sign}(g(\epsilon)) &= \text{sign}(\epsilon) , \end{aligned} \quad (\text{G1})$$

and gives the energy at which $\Psi_n(E, l)$ may be developing a Dirac peak in the thermodynamic limit. These properties of $g(\epsilon)$ are clearly visible from the numerical spectra (the first one is protected by a symmetry). The numerics essentially suggests $g(\epsilon) = \epsilon$, and this is in agreement with scattering theory in the thermodynamic limit. No divergences occur at $E = 0$ or $\epsilon = 0$, and these possible non-analyticities do not play an important role. The remaining assumptions follow from the requirement that the function $\mathcal{R}(\Delta\epsilon)$, which determines the vortex action, be finite for every finite $\Delta\epsilon$. All wavefunction

singularities should be integrable; integrals of the wavefunction (not its modulus squared) across the point of possible non-analyticity are convergent in the thermodynamic limit:

$$\int_{g(\epsilon)-\Delta E}^{g(\epsilon)+\Delta E} dE |\Psi_{\epsilon, i}(E, l)| < \infty . \quad (\text{G2})$$

This is certainly consistent with a possible Dirac peak. Also, certain sums over angular momentum channels should be either finite or lead to integrable singularities. All of these assumptions are based on the numerically obtained wavefunctions and spectra.

We now analytically explore general properties of the function $\mathcal{R}(\Delta\epsilon)$ given by (72), and check whether any infra-red divergences might occur. A particular question is what the dependence on small $\Delta\epsilon$ is. If $\mathcal{R}(\Delta\epsilon) \sim \Delta\epsilon^\nu$ with $\nu \leq 0$, then vortex dynamics at small frequencies may be anomalous and not simply governed by the vortex inertia.

Let us introduce a convenient symbolic notation for the exact states $|n\rangle = |\epsilon, i\rangle$, where $\epsilon = \epsilon_n$ and i is a fictitious quantum number that replaces the angular momentum l of the states in absence of the Doppler shift. The purpose of this is only to explicitly reveal the role of energy ϵ , which does not completely specify the exact states. Then we symbolically write (72) as:

$$\begin{aligned} \mathcal{R}(\Delta\epsilon) &= \sum_{\text{nodes}} \sum_{i_1, i_2, -\Lambda}^0 \int d\epsilon_1 \int_0^\Lambda d\epsilon_2 \delta(\epsilon_2 - \epsilon_1 - \Delta\epsilon) |U_{1,2}|^2 \\ U_{1,2} &= \sum_{l_1, l_2, -\Lambda}^\Lambda \int dE_1 \int_{-\Lambda}^\Lambda dE_2 \\ &\quad \langle E_1, l_1 | \epsilon_1, i_1 \rangle \langle \epsilon_2, i_2 | E_2, l_2 \rangle \mathbf{r}_v(\omega) \mathbf{U}_{E_1, l_1; E_2, l_2} . \end{aligned} \quad (\text{G3})$$

Note that conventionally the summation over fictitious i_1 and i_2 would be replaced by the appropriate density

of states factors. We integrate out ϵ_1 in (G3) and write:

$$\begin{aligned} \mathcal{R}(\Delta\epsilon) &= \sum_{\text{nodes } i_1, i_2} \sum_0^{\Delta\epsilon} \int d\epsilon |U_{1,2}|^2 \\ U_{1,2} &= \sum_l \sum_{\sigma=\pm 1} \int_{-\Lambda}^{\Lambda} dE_1 \int_{-\Lambda}^{\Lambda} dE_2 \\ \Psi_{\epsilon-\Delta\epsilon, i_1}^*(E_1, l-\sigma) \Psi_{\epsilon, i_2}(E_2, l) &\times \mathbf{r}_v(\omega) \mathbf{U}_{E_1, l-\sigma; E_2, l}. \end{aligned} \quad (\text{G4})$$

Now we substitute individual terms from the expression (45) for transition matrix elements $\mathbf{U}_{E_1, l_1; E_2, l_2}$ between the basis states, and check what happens as $\Delta\epsilon \rightarrow 0$. We start with potentially the most critical transitions, which involve the zero angular momentum channel.

a. Transitions involving the zero angular momentum

The transition matrix elements that we look first at are given in (45) for $l = \frac{\sigma+1}{2}$:

$$\begin{aligned} U_{E_1, l-\sigma; E_2, l} &\propto \frac{2\sigma \text{sign}(E_1 E_2)}{\pi} \frac{E_1 + E_2}{E_1 - E_2} + \\ &\frac{C_\sigma(q_1, q_2)}{2\pi} \left| \frac{E_1}{E_2} \right|^{\frac{\sigma}{2}} \frac{E_1 + E_2}{\sqrt{|E_1 E_2|}} \log \left(\frac{|E_1| - |E_2|}{|E_1| + |E_2|} \right)^2. \end{aligned} \quad (\text{G5})$$

This expression, appropriate in the thermodynamic limit, appears formally infinite at $E_1 = E_2$, and we need to regularize it on a finite sample in order to see the exact effect of this singularity. When the transition matrix elements \mathbf{U} are calculated on a finite sample with radius R , (G5) is replaced by an oscillatory expression $U^{(R)}(E_2 - E_1)$, with period $\sim R^{-1}$ and local average exactly given by (G5). The discrete energies E (for $|E|R \gg 1$) have spacing equal to the half-period of $U^{(R)}$, so that indeed in the $R \rightarrow \infty$ limit the summation of $U^{(R)}$ over energies transforms into the integration over the ‘‘local average’’ (G5) for every $E_2 \neq E_1$. Exactly at $E_2 = E_1 = E$ it turns out that $U^{(R)} \sim \log(|E|R)$. However, this logarithmic divergence is innocuous, because its contribution to the sum over energies is $R^{-1} \log(|E|R) \rightarrow 0$ in the $R \rightarrow \infty$ limit. In the following, we will symbolically ignore it in the thermodynamic limit by writing a principal part of (G5), and focus on the divergent behavior close to $E_1 = E_2$. Note that there are no other divergences in this expression, in particular at $E_1 \rightarrow 0, E_2 \rightarrow 0$ or both.

Consider $U_{1,2}$ in (G4). We can change variables $E_1 = E - \Delta E, E_2 = E$. The transition matrix element (G5) diverges only at $\Delta E \rightarrow 0$, so let us fix a small finite $\Delta E < \Delta\epsilon$ and imagine integrating out E first in (G4). For some value of E , one of the two wavefunctions $\Psi_{\epsilon-\Delta\epsilon, i_1}^*(E_1, l-\sigma)$ and $\Psi_{\epsilon, i_2}(E_2, l)$ may become divergent, but not both - see Figure 7. Based on numerics, we assume that a single wavefunction divergence is integrable, so that the integral over E is finite. As a function of ΔE , this integral has a singularity inherited from

the expression (G5) at $\Delta E = 0$, but its nature has not been changed and no other singularity at any $\Delta E < \Delta\epsilon$ has been introduced. This is important, because if we now proceed with integrating out ΔE , the singularity at $\Delta E = 0$ behaves exactly the same way as if we were simply integrating (G5). However, it is not difficult to check that such an integral:

$$\int_{-\Delta E_0}^{\Delta E_0} d\Delta E \left(\text{const.} \frac{\mathbb{P}}{\Delta E} + \text{const.} \log(\Delta E^2) \right) \quad (\text{G6})$$

evaluates to a finite value (\mathbb{P} denotes the principal part due to finite sample regularization).

What remains to be examined is how sending $\Delta\epsilon$ to zero may affect this picture. Generally, the integration over E discussed above picks divergences of both wavefunctions in (G4), so that the resulting function of ΔE may be non-analytic or even divergent at some $\Delta E \sim \Delta\epsilon$. In particular, if the only divergences of the wavefunctions are Dirac peaks (as it seems), then the resulting function of ΔE is divergent as a Dirac peak at $\Delta E = g(\epsilon) - g(\epsilon - \Delta\epsilon) \sim \Delta\epsilon$; other finite non-analyticities in the wavefunctions do not matter. Now, integrating out ΔE is sensitive to such a divergence and the final result (which should be finite for $\Delta\epsilon \neq 0$) may end up scaling as $U(\Delta E \sim \Delta\epsilon) \sim \frac{1}{\Delta\epsilon}$ for small $\Delta\epsilon$. However, this can actually happen only if we introduce some non-zero chemical potential μ . Otherwise, we always approach the $\Delta\epsilon \sim E_2 - E_1 \rightarrow 0$ limit only together with $\epsilon \sim E_1 + E_2 \rightarrow 2\mu = 0$, and no divergence emerges from (G4) (for details, see analytical calculations in absence of the Doppler shift - this is equivalent to reducing the present exact wavefunctions to pure Dirac peaks, and such divergences do not give rise to dissipation without a non-zero chemical potential).

In conclusion, the transitions that involve the zero angular momentum, which are the most critical for appearance of anomalous vortex dynamics, seem to qualitatively follow the same behavior as if there were no Doppler shift.

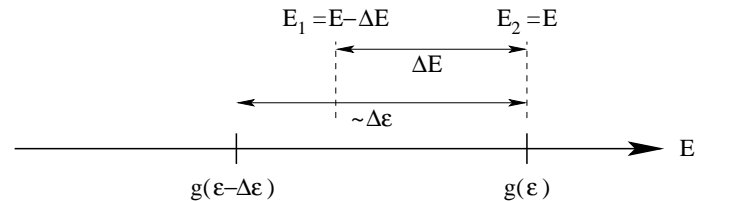


FIG. 7: Integration of $U_{1,2}$ in (G4) over E . Ticks mark the points at which the wavefunctions are possibly divergent ($\Psi_{\epsilon-\Delta\epsilon, i_1}^*(E_1, l-\sigma) = \Psi_{\epsilon-\Delta\epsilon, i_1}^*(E - \Delta E, l-\sigma)$ at the left tick, and $\Psi_{\epsilon, i_2}(E_2, l) = \Psi_{\epsilon, i_2}(E, l)$ at the right tick). As we integrate over E , for fixed ΔE , the dashed lines move from left to right in this diagram, and cross over the singularities in the wavefunctions. Only for one value of $\Delta E \sim \Delta\epsilon$ both singularities can be crossed at the same time.

b. Transitions that do not involve the zero angular momentum

Consider first the “most divergent” terms in (45) for $l \neq \frac{\sigma+1}{2}$:

$$U_{E_1, l-\sigma; E_2, l} \propto 4\sqrt{|E_1 E_2|} \delta(E_2 - E_1) \text{sign} \left(l - \frac{\sigma+1}{2} \right), \quad (\text{G7})$$

and substitute them into (G4). The sign function takes care of the signs for positive or negative l in (45). Ignoring $\mathbf{r}_v(\omega)$ and the vector structure of \mathbf{U} , we find:

$$U_{1,2} \propto \sum_{\sigma=\pm 1} \sum_{l \neq \frac{\sigma+1}{2}} \int_{-\Lambda}^{\Lambda} dE \Psi_{\epsilon-\Delta\epsilon, i_1}^*(E, l-\sigma) \Psi_{\epsilon, i_2}(E, l) |E| \text{sign} \left(l - \frac{\sigma+1}{2} \right). \quad (\text{G8})$$

According to our assumptions from the beginning of this appendix, the energies E at which the wavefunctions $\Psi_{\epsilon, i}(E, l)$ are non-analytic do not depend on angular momentum l . Therefore, we can first sum up l - the result should be finite, and have the same analytical properties as the original expression. Then, we integrate out E . As long as $\Delta\epsilon \neq 0$, the two wavefunctions in the integral cannot diverge at the same value of E , so that, according to our assumptions, the integral over E is finite. Exactly for $\Delta\epsilon = 0$, the result may be infinite, and it is $\delta(\Delta\epsilon)$ if the wavefunction divergence at $E \sim \epsilon$ is a Dirac peak.

This is essentially the same situation as in the case without the Doppler shift: $\delta(\Delta\epsilon)$ is killed by the factors of $\Delta\epsilon$ in the vortex action.

Now consider the remaining portion of (45) for $l \neq \frac{\sigma+1}{2}$ and substitute:

$$U_{E_1, l-\sigma; E_2, l} \propto \left(\frac{|E_1|}{|E_2|} \right)^{\sigma l - \frac{1}{2}} \frac{E_1 + E_2}{\sqrt{|E_1 E_2|}} \times \Theta \left(\text{sign} \left(l - \frac{\sigma+1}{2} \right) \sigma (|E_2| - |E_1|) \right)$$

into $U_{1,2}$ from (G4). The sign function selects whether the argument of the step Θ -function should be $\sigma(k_2 - k_1)$ for positive l or $\sigma(k_1 - k_2)$ for negative l (see (45), and recall $k = |E|, q = \text{sign}(E)$). We could roughly follow the previous argument, although the summation over l is not entirely trivial - it may introduce a singularity at $E_1 = E_2$. Since the wavefunction amplitudes cannot grow indefinitely with l , this new singular behavior is in the worst case:

$$\frac{E_1 + E_2}{E_1 - E_2} \quad (\text{G9})$$

(see for example (49)). After summation over l we are left with an integral over E_1 and E_2 , which has a very similar structure to the one that we considered in the previous subsection. By repeating the same argument, we arrive to the conclusion that we could obtain anomalous vortex dynamics only with a non-zero chemical potential.

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- ¹ Y. Ando, G. S. Boebinger, A. Passner, T. Kimura, and K. Kishio, *Phys. Rev. Lett.* **75**, 4662 (1995).
² A. Fang, C. Howald, N. Kaneko, M. Greven, and A. Kapitulnik, *Phys. Rev. B* **70**, 214514 (2004).
³ M. Vershinin, S. Misra, S. Ono, Y. Abe, Y. Ando, and A. Yazdani, *Science* **303**, 1995 (2004).
⁴ K. McElroy, D.-H. Lee, J. E. Hoffman, K. M. Lang, J. Lee, E. W. Hudson, H. Eisaki, S. Uchida, and J. C. Davis, *Phys. Rev. Lett.* **94**, 197005 (2005).
⁵ T. Hanaaguri, C. Lupien, Y. Kohsaka, D.-H. Lee, M. Azuma, M. Takano, H. Takagi, and J. C. Davis, *Nature* **430**, 1001 (2004).
⁶ M. A. Steiner, G. Boebinger, and A. Kapitulnik, *Phys. Rev. Lett.* **94**, 107008 (2005).
⁷ J. M. Tranquada, H. Woo, T. G. Perring, H. Goka, G. D. Gu, G. Xu, M. Fujita, K. Yamada, *Nature* **429**, 534 (2004).
⁸ C. Dasgupta and B. I. Halperin, *Phys. Rev. Lett.* **47**, 1556 (1981).
⁹ D.R. Nelson, *Phys. Rev. Lett.* **60**, 1973 (1988).
¹⁰ M. P. A. Fisher and D.-H. Lee, *Phys. Rev. B* **39**, 2756 (1989); D.-H. Lee and M. P. A. Fisher, *Phys. Rev. Lett.* **63**, 903 (1989).
¹¹ G. E. Volovik, *Pisma Zh. Eksp. Teor. Fiz.* **65** 201 (1997) [*JETP Lett.* **65** 217 (1997)].
¹² N. B. Kopnin, and V. M. Vinokur, *Phys. Rev. Lett.* **81**, 3952 (1998).
¹³ N. B. Kopnin, and V. M. Vinokur, *Phys. Rev. B* **62**, 9770 (2000).
¹⁴ N. B. Kopnin, *Theory of Nonequilibrium Superconductivity*, Cambridge University Press, Cambridge (2001).
¹⁵ S. H. Simon and P. A. Lee, *Phys. Rev. Lett.* **78**, 1548 (1997).
¹⁶ L. Balents, L. Bartosch, A. Burkov, S. Sachdev, and K. Sengupta, *Phys. Rev. B* **71**, 144508 (2005).
¹⁷ L. Balents, L. Bartosch, A. Burkov, S. Sachdev, and K. Sengupta, *Phys. Rev. B* **71**, 144509 (2005).
¹⁸ L. Bartosch, L. Balents, and S. Sachdev, *cond-mat/0502002*.
¹⁹ L. Balents, L. Bartosch, A. Burkov, S. Sachdev, and K. Sengupta, *cond-mat/0504692*.
²⁰ V. N. Popov, *Zh. Eksp. Teor. Fiz.* **64**, 672 (1973) [*Sov. Phys. JETP* **37**, 341 (1973)].
²¹ J.-M. Duan, *Phys. Rev. B* **48**, 333 (1993); J.-M. Duan and A. J. Leggett, *Phys. Rev. Lett.* **68**, 1216 (1992); J.-M. Duan, *Phys. Rev. B* **49**, 12381 (1994).
²² T. Senthil and M. P. A. Fisher, *Phys. Rev. B* **62**, 7850 (2000).
²³ L. Balents and M. P. A. Fisher, *Phys. Rev. B* **71**, 085119 (2005).
²⁴ V. M. Galitski, G. Refael, M. P. A. Fisher, and T. Senthil, *Phys. Rev. Lett.* **95**, 077002 (2005).

- ²⁵ L. Balents and S. Sachdev, to appear.
- ²⁶ J. Bardeen and M. J. Stephen, Phys. Rev. **140**, A1197 (1965); P. Nozières and W. F. Vinen, Philos. Mag. **14**, 667 (1966).
- ²⁷ E. Simanek, Phys. Lett. A **194**, 323 (1994).
- ²⁸ P. Ao, and X. M. Zhu, Physica C **282-287**, 367 (1997).
- ²⁹ P. Ao and X. M. Zhu, Phys. Rev. B **60**, 6850 (1999).
- ³⁰ J. H. Han, J. S. Kim, M. J. Kim, and P. Ao, Phys. Rev. B **71**, 125108 (2005).
- ³¹ M. Franz and Z. Tešanović, Phys. Rev. Lett. **84**, 554 (2000).
- ³² O. Vafek, A. Melikyan, M. Franz, Z. Tešanović, Phys. Rev. B **63**, 134509 (2001).
- ³³ A. Melikyan and Z. Tešanović, unpublished.
- ³⁴ L. Marinelli, B. I. Halperin, and S. H. Simon, Phys. Rev. B **62**, 3488 (2000).
- ³⁵ A. Vishwanath, Phys. Rev. B **66**, 064504 (2002).
- ³⁶ A. S. Mel'nikov, Phys. Rev. Lett. **86**, 4108 (2001).
- ³⁷ R. Jackiw, Phys. Rev. D **29**, 2375 (1984).
- ³⁸ J. E. Hoffman, E. W. Hudson, K. M. Lang, V. Madhavan, S. H. Pan, H. Eisaki, S. Uchida, and J. C. Davis, Science **295**, 466 (2002).
- ³⁹ B. W. Hoogenboom, K. Kadowaki, B. Revaz, M. Li, Ch. Renner, and Ø. Fischer, Phys. Rev. Lett. **87**, 267001 (2001); G. Levy, M. Kugler, A. A. Manuel, Ø. Fischer, and M. Li, cond-mat/0503219.
- ⁴⁰ M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover (1965).