Conductivity of thermally fluctuating superconductors in two dimensions

Subir Sachdev

Department of Physics, Yale University,
P.O. Box 208120, New Haven CT 06520-8120, USA

Abstract

We review recent work on a continuum, classical theory of thermal fluctuations in two dimensional superconductors. A functional integral over a Ginzburg-Landau free energy describes the amplitude and phase fluctuations responsible for the crossover from Gaussian fluctuations of the superconducting order at high temperatures, to the vortex physics of the Kosterlitz-Thouless transition at lower temperatures. Results on the structure of this crossover are presented, including new results for corrections to the Aslamazov-Larkin fluctuation conductivity.

Key words: superonductivity, two dimensions, fluctuations

1 Introduction

The subject of fluctuation conductivity of superconductors is an old one, and many classic results can be found in a recent review article by Larkin and Varlamov [1]. Essentially all of this work is restricted to the first corrections to the BCS theory, and involving Gaussian fluctuations in the Cooper pair propagator. Such an approach is valid provided the Ginzburg fluctuation parameter is small.

In two dimensions, we know that as the superconducting critical temperature (T_c) is approached from above, there is ultimately a crossover from such a Gaussian fluctuation regime to the vortex physics of the Kosterlitz-Thouless (KT) transition. While universal characteristics of the conductivity in the latter regime are well understood [2], there is little quantitative understanding of precisely how such a crossover occurs. In the context of the small coherence

URL: `http://pantheon.yale.edu/~subir` (Subir Sachdev).

length cuprate superconductors, where there is a wide regime of temperatures where the Ginzburg fluctuation parameter is large, a quantitative analysis of such a regime is surely important in deciphering the importance of pairing fluctuations.

This paper will address the Gaussian-to-vortex crossover in the fluctuation conductivity using an approach suggested in Refs. [3,4]; the same approach will also be extended to describe the spin-wave-to-vortex crossover at low temperatures. Using the proximity of the cuprates to a superconductor-insulator quantum transition, it was argued that the fluctuation regime could be described by a universal continuum limit of the Ginzburg-Landau free energy. This is described by the following partition function for the complex pairing order parameter $\Psi(\mathbf{r})$:

$$\mathcal{Z}_{GL} = \int \mathcal{D}\Psi(\mathbf{r}) e^{-\mathcal{F}_{GL}/(k_B T)} \tag{1}$$
$$\mathcal{F}_{GL} = \int d^2 r \left[\frac{\hbar^2 |\nabla_{\mathbf{r}} \Psi(\mathbf{r})|^2}{2m^*} + a(T)|\Psi(\mathbf{r})|^2 + \frac{b}{2}|\Psi(\mathbf{r})|^4 \right]$$

Here m^*, $a(T)$ and b are parameters dependent upon the physics of the underlying electrons which have been integrated out; the function $a(T)$ vanishes at the mean-field transition temperature $a(T_c^{MF}) = 0$. Note that the previous work [1] was at the level of a mean field treatment of $\mathcal{F}_{GL}$, or its one loop fluctuations. Furthermore, the strong fluctuations near the KT transition have been invariably treated in previous work in terms of theory for the vortices alone or in terms of a phase-only XY model. Refs. [3,4] proposed that the combined amplitude and phase fluctuations implied by (1) should be taken seriously, from temperatures well above T_c, and across the vortex physics of the KT transition. This paper will present a few results on the electrical conductivity associated with the continuum theory of equal-time fluctuations described by (1). Our results for the structure of the crossover contains non-perturbative information in the form of a few universal numerical constants which were computed in recent computer simulations [3,5,6].

2 Review of static theory

This section reviews a few important static properties of $\mathcal{Z}_{GL}$ [3,7].

A key first point is that the theory (1) cannot predict actual T_c on its own, as its value is dependent upon a short distance cutoff. We treat T_c as an input parameter, and then find that all subsequent cutoff dependence can be safely neglected. Knowing T_c, it is useful to define the dimensionless coupling g

$$g \equiv \frac{\hbar^2}{m^* b}\left[\frac{a(T)}{k_B T} - \frac{a(T_c)}{k_B T_c}\right], \tag{2}$$

which should be a monotonically increasing function of T crossing zero at exactly $T = T_c$. If we take $a(T)$ to be a simple linearly increasing function of temperature ($a(T) = a_0(T - T_c^{MF})$), then

$$g = \frac{\rho_s(0)}{k_B T_c}\left(1 - \frac{T_c}{T}\right), \tag{3}$$

where $\rho_s(T)$ is the helicity modulus [1] of $\mathcal{Z}_{GL}$ (the London penetration depth is $\lambda_L^{-2} = 4\pi e^{*2}\rho_s(T)/(\hbar^2 c^2)$, where $e^* = 2e$, the Cooper pair charge). In this framework, the values of $\rho_s(0)$ and T_c are the only two needed inputs to predict most physical properties of $\mathcal{Z}_{GL}$ (including the full T dependence of $\rho_s(T)$). Note that $\rho_s(0)$ and T_c are, in general, independent of each other, and the Nelson-Kosterlitz relation [8] only constraints $\rho_s(T_c)/T_c = 2/\pi$.

It was shown in Ref. [3,7] that the structure of expansions away from simple limiting mean-field regimes of $\mathcal{Z}_{GL}$ are best understood in terms of two new dimensionless couplings, $\mathcal{G}$ and $\mathcal{G}_D$. These couplings are most useful in high and low T limits, respectively.

For $T \geq T_c$, we work with the coupling $\mathcal{G} > 0$ given by

$$g = \frac{6}{\mathcal{G}} + \frac{1}{\pi}\ln\left(\frac{\mathcal{C}T}{\mathcal{G}T_c}\right), \tag{4}$$

where $\mathcal{C}$ is a universal numerical constant. Its value is determined by the condition that at the critical point with $g = 0$, $\mathcal{G} = \mathcal{G}_c = 96.9 \pm 3$ a universal critical value that has been determined in numerical simulations [3,5,6]. Using these results, we obtain [2]

$$\mathcal{C} = 79.8 \pm 2.4. \tag{5}$$

A key fact behind our main results here is that for large g (when $\mathcal{G}$ is small), all physical properties of $\mathcal{Z}_{GL}$ can be expanded in a series that involves *only*

[1] Of course, we do not expect the classical theory in (1) to be valid as $T \to 0$ because quantum effects will become important; by $\rho_s(0) \equiv -\hbar^2 a(0)/(m^* b)$ we mean the energy obtained by extrapolating the present approximation for $a(T)$ to $T = 0$.

[2] $\mathcal{C}$ is related to the number ξ_μ computed in Refs. [5,6] by $\mathcal{C} = 6\xi_\mu$.

integer powers of $\mathcal{G}$. Re-expressing any such series in terms of $1/g$ by (4) then implies a number of logarithmic terms in g: the coefficients and arguments of such logarithms are therefore entirely specified by (4). We should also note here that, as discussed in Ref. [7], the combination of (4) and (3) implies that the continuum theory $\mathcal{Z}_{GL}$ breaks down at a T so large ($T > \pi\rho_s(0)/k_B$ for the T dependence assumed in (3)) that $\mathcal{G}$ ceases to be a monotonically decreasing function of increasing T.

The equation (4) also defines $\mathcal{G}$ for $T < T_c$, but its value becomes exponentially large as $T \to 0$. Instead, for $T \leq T_c$, the useful coupling turns out to be $\mathcal{G}_D > 0$ defined by

$$g = -\frac{3}{\mathcal{G}_D} + \frac{1}{\pi} \ln \left(\frac{\mathcal{C}T}{\mathcal{G}_D T_c} \right) \tag{6}$$

The coupling $\mathcal{G}_D$ vanishes linearly as $T \to 0$, and is an increasing function of T. At low T we can expand physical properties of $\mathcal{Z}_{GL}$ in $\mathcal{G}_D$, and again, this series involves *only* integer powers of $\mathcal{G}_D$. The relationship (6) then implies logarithms in the expansion in $1/|g|$.

3 Conductivity

A description of electrical transport requires an extension of the partition function $\mathcal{Z}_{GL}$ to a dynamic theory. We do this in the simplest 'model A' theory, in which the time evolution of $\Psi(\mathbf{r}, t)$ is given by

$$\tau \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = -\frac{\delta \mathcal{F}_{GL}}{\delta \Psi^*(\mathbf{r}, t)} + \zeta(\mathbf{r}, t) \tag{7}$$

where τ is a damping parameter which has no singularities at $T = T_c$, and ζ is a Gaussian random noise source with the correlator

$$\langle \zeta^*(\mathbf{r}', t')\zeta(\mathbf{r}, t)\rangle = 2\tau k_B T \delta\left(\mathbf{r} - \mathbf{r}'\right) \delta\left(t - t'\right) \tag{8}$$

Wickham and Dorsey [9] have already computed the conductivity in the theory (1, 7, 8) as an expansion in powers of b, and many needed results can be obtained simply by transforming their results to spatial dimension $d = 2$ (they performed an analysis of perturbation theory in powers of $(4 - d)$ for different purposes). As the steps in the calculation are essentially identical to theirs, no details of the will be presented here.

Our results are most easily expressed in terms of a dimensionless measure of the damping parameter, which we define as

$$\overline{\tau} \equiv \frac{\hbar \tau}{m^* b} \tag{9}$$

In BCS theory, the damping arises from decay of Cooper pairs into electrons; assuming a linear dependence of $a(T)$ on T as in (3), BCS theory predicts [1]

$$\overline{\tau}_{BCS} = \frac{\pi \rho_s(0)}{8 k_B T_c}. \tag{10}$$

Of course, as $T \to 0$, the gapless quasiparticle excitations disappear, and then a simple damping term as in (7) is not expected to be valid. Our low T results for $\mathcal{Z}_{GL}$ should therefore be treated with caution.

The main new claim of this paper is that the frequency (ω) dependent conductivity can be written as

$$\sigma(\omega) = \frac{e^{*2}\overline{\tau}}{\hbar}\, \Upsilon\left(\mathcal{G}, \frac{\hbar\omega\mathcal{G}\overline{\tau}}{12 k_B T}\right) \tag{11}$$

where Υ is a universal function (the factor of 12 in the second argument is for future notational convenience). The result (11) applies for all values of g and $\mathcal{G}$, but for $g \ll 1$, a different scaling form involving $\mathcal{G}_D$ is more convenient (see Section 3.3). We discuss the structure of this function for the different limiting ranges of g in the subsections below.

3.1 $g \gg 1$

At $T \geq T_c$, where $g \gg 1$, there is an expansion for Υ in *integer powers of its first argument* $\mathcal{G}$. The results of Wickham and Dorsey [9], when mapped to $d = 2$, immediately yield an expansion of Υ to order $\mathcal{G}^3$. For the d.c. conductivity, these results are

$$\Upsilon(\mathcal{G}, 0) = \frac{\mathcal{G}}{48\pi} + \mathcal{A}\mathcal{G}^3 + \mathcal{O}(\mathcal{G}^4) \tag{12}$$

The first term in (12) is precisely the old result of Aslamazov and Larkin [1]: this can be checked by noting that for $g \gg 1$, $\mathcal{G} \approx 6/g$ from (4), and by using (3) and the value for $\overline{\tau}_{BCS}$ in (10), which yields ultimately $\sigma = e^{*2}T/(64\hbar(T - T_c))$. The numerical constant $\mathcal{A}$ involves a number of different contributions, and can be written as

$$\mathcal{A} = \frac{J}{216\pi} + \frac{2}{9}\left(\widetilde{I}_2^a(0) + \widetilde{I}_2^b(0)\right) - \frac{32}{27}\left(4\widetilde{I}_b(0) + \widetilde{I}_c^{(1)}(0) + \widetilde{I}_c^{(2)}(0)\right). \tag{13}$$

The first term in (13) is a 'mass' renormalization' correction to the leading order term in (12), and arises from (2.1) of Ref. [3] with $J = 0.014842966\ldots$; the remaining terms involving the $\widetilde{I}$'s are associated with 3-loop diagrams evaluated by Wickham and Dorsey, and are specified in Eqs. (4.26), (4.28), (5.2), (5.3), and (5.4) respectively of Ref. [9]. The momentum integrals in all the $\widetilde{I}$'s have to be evaluated in $d = 2$. This was done numerically, and we obtained

$$\begin{aligned} \widetilde{I}_2^a(0) &= 4.17384 \times 10^{-6} \\ \widetilde{I}_2^b(0) &= -6.92307 \times 10^{-6} \\ \widetilde{I}_b(0) &= -1.06867 \times 10^{-7} \\ \widetilde{I}_c^{(1)}(0) &= -6.41276 \times 10^{-7} \\ \widetilde{I}_c^{(2)}(0) &= -2.93446 \times 10^{-7} \\ \mathcal{A} &= 2.28769 \times 10^{-5} \end{aligned} \tag{14}$$

We can combine (4), (11) and (12) to obtain the following corrections to the Aslamazov-Larkin result valid for $g \gg 1$:

$$\sigma(0) = \frac{e^{*2}\overline{\tau}}{8\pi\hbar g}\left[1 + \frac{1}{\pi g}\ln(13.3gT/T_c) + \frac{1}{\pi^2 g^2}\left(\ln^2(13.3gT/T_c) - \ln(13.3gT/T_c)\right) + \frac{1728\pi\mathcal{A}}{g^2} + \mathcal{O}(1/g^3)\right] \tag{15}$$

The overall pre-factor is the Aslamazov-Larkin result. The series is (15) one of the main new results of this paper. Using the value of g in (3) as input, it yields the T dependence of the fluctuation conductivity. Of course, it is more accurate to numerically solve (4) for $\mathcal{G}$ and to then use (12): such a procedure would account for all logarithms.

The results for the frequency dependence of the conductivity are much more cumbersome, and we quote only the terms to order $\mathcal{G}^2$:

$$\Upsilon(\mathcal{G}, y) = \frac{\mathcal{G}}{48\pi}\frac{2i(-y + (i+y)\ln(1-iy))}{y^2} + \mathcal{O}(\mathcal{G}^3). \tag{16}$$

This result is obtained by taking the $d \to 2$ limit of Eq. (3.7) in Ref. [9].

3.2 $g \approx 0$

The Kosterlitz-Thouless transition occurs at $\mathcal{G} = \mathcal{G}_c$, and the system continues to be described by (11) across the transition. The singularities of the transition appear as singularities in the function Υ as a function of the $\mathcal{G}$. The functional form of these singularities will similar to those in Ref. [2] as a function of T. Of course, there are no arbitrary scale factors in the function Υ near this singularity, but these can only be determination by suitable simulations.

3.3 $g \ll -1$

For $T < T_c$, it is useful to expand about a saddle point with $\Psi \neq 0$, and to organize the perturbation theory in powers of $\mathcal{G}_D$ defined in (6). We therefore rewrite (11) as

$$\sigma(\omega) = \frac{e^{*2}}{\hbar^2}\rho_s(T)\delta(\omega) + \frac{e^{*2}\overline{\tau}}{\hbar}\ \Upsilon_D\left(\mathcal{G}_D, \frac{\hbar\omega\mathcal{G}_D\overline{\tau}}{12k_BT}\right) \tag{17}$$

Again, both $\rho_s(T)$ and Υ_D can be expanded in a series involving only integer powers of $\mathcal{G}_D$. For $\rho_s(T)$, terms in this series were already computed in Ref. [3]:

$$\frac{\rho_s(T)}{k_BT} = \frac{3}{\mathcal{G}_D} - \frac{\mathcal{G}_D}{36} + \mathcal{O}(\mathcal{G}_D^2) \tag{18}$$

We computed Υ_D to one loop order by the methods of Ref. [9], and found

$$\Upsilon_D(\mathcal{G}_D, y) = -\frac{\mathcal{G}_D}{12\pi}\frac{\ln(1/2 - iy)}{(1 + 2iy)} + \mathcal{O}(\mathcal{G}_D^2). \tag{19}$$

4 Conclusions

This paper has obtained systematic corrections to the Aslamazov-Larkin fluctuation conductivity of thermally fluctuating superconductors in two dimensions. The main results are contained in (11) and (12), and take the form of an expansion in integer powers of a renormalized dimensionless coupling $\mathcal{G}$. When $\mathcal{G}$ is re-expressed in terms of bare microscopic parameters (such as the temperature), numerous logarithms appear. However, this microscopic relationship is known exactly, and consequently, so is the structure of the all the logarithms; this relationship involves non-perturbative information in the form of universal constants which were computed in recent numerical simulations [3,5,6]. The renormalized coupling $\mathcal{G}$ is related to a bare coupling g by (4), and an expansion for the conductivity in inverse powers of g is in (15). The coupling g is related to parameters in the Ginzburg-Landau free energy by (2). Alternatively, assuming a linear dependence of $a(T)$ on T, we can use the simpler expression (3), where $\rho_s(0) \equiv -\hbar^2 a(0)/(m^* b)$, is the helicity modulus of the Ginzburg-Landau free energy extrapolated to $T = 0$.

It is useful to mention here the expected values of the dimensionless parameter $\rho_s(0)/(k_B T_c)$ appearing in (3). In BCS theory, we can use standard expressions for the parameters in (1), obtained for electrons in a simple parabolic band [1]: keeping only the linear T dependence near T_c, and extrapolating to $T = 0$ we obtain

$$\left.\frac{\rho_s(0)}{k_B T_c}\right|_{BCS} = \frac{E_F}{\pi k_B T_c} \tag{20}$$

where E_F is the Fermi energy. The right-hand-side involves the very large ratio E_F/T_c, and this ensures a large regime where $g \gg 1$ above T_c. We can also compare to the numerical studies of the cuprate superconductors in Ref [10]. They defined a dimensionless parameter η, in terms of which we can write $\rho_s(0)/(k_B T_c) = (2/\eta)(T_c^{MF}/T_c)$; for their parameter values we have $\rho_s(0)/(k_B T_c) = 6.8$, which also ensures a reasonable large regime over which the $g \gg 1$ expansion is useful [7]. Finally, we note that near a superfluid-insulator quantum phase transition, $\rho_s(0)/(k_B T_c)$ is the ratio of two energy scales which vanish at the quantum critical point, and so is a universal number [3]: this universal number has been computed in a simple model for such a transition.

It would be interesting to extend the present results to other transport coefficients, such as the Nernst effect [10,11]. The present quantitative approach is likely to be most useful for observables [7] that do not diverge at T_c.

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