

# Non-Fermi liquid behavior from two-dimensional antiferromagnetic fluctuations: a renormalization-group and large- $N$ analysis

Sergey Pankov,<sup>1</sup> Serge Florens,<sup>2</sup> Antoine Georges,<sup>2</sup> Gabriel Kotliar,<sup>1</sup> and Subir Sachdev<sup>3</sup>

<sup>1</sup>*Center for Materials Theory, Department of Physics and Astronomy,  
Rutgers University, Piscataway, New Jersey 08854, USA*

<sup>2</sup>*Laboratoire de Physique Théorique, Ecole Normale Supérieure,  
24 Rue Lhomond, 75231 Paris Cedex 05, France*

<sup>3</sup>*Department of Physics, Yale University,  
P.O. Box 208120, New Haven CT 06520-8120, USA*

(Dated: April 17, 2003)

We analyze the Hertz-Moriya-Millis theory of an antiferromagnetic quantum critical point, in the marginal case of two dimensions ( $d = 2, z = 2$ ). Up to next-to-leading order in the number of components ( $N$ ) of the field, we find that logarithmic corrections do not lead to an enhancement of the Landau damping. This is in agreement with a renormalization-group analysis, for arbitrary  $N$ . Hence, the logarithmic effects are unable to account for the behavior reportedly observed in inelastic neutron scattering experiments on  $\text{CeCu}_{6-x}\text{Au}_x$ . We also examine the extended dynamical mean-field treatment (local approximation) of this theory, and find that only subdominant corrections to the Landau damping are obtained within this approximation, in contrast to recent claims.

## I. INTRODUCTION

Materials in the vicinity of a quantum critical point (QCP) continue to be the subject of intensive investigations<sup>1</sup>. Particular attention has been paid to the heavy fermion system  $\text{CeCu}_{6-x}\text{Au}_x$  which undergoes a phase transition from a paramagnetic, heavy-fermion metal to an antiferromagnetic metal as a function of chemical composition and pressure<sup>2,3</sup>. Another remarkable case is the stoichiometric compound  $\text{YbRh}_2\text{Si}_2$  which is apparently located very close (on the magnetic side) of an antiferromagnetic QCP<sup>4</sup>. Hydrostatic pressure stabilizes the magnetic phase in that case. As the critical point is approached, these systems exhibit an enhancement of the specific heat coefficient  $C/T$ , magnetic susceptibility, resistivity  $\Delta\rho$  and thermoelectric power, which is not easily described within the conventional theory of a three dimensional quantum critical antiferromagnet.

Neutron scattering experiments<sup>3,5,6,7</sup> on  $\text{CeCu}_{6-x}\text{Au}_x$  have revealed that the spin fluctuation spectrum of this compound has a *two-dimensional* character in a wide range of temperatures. Indeed, maxima of the neutron scattering intensity are observed along rod-like structures in reciprocal space along which spin fluctuations are independent of one component of the wavevector. This motivated Rosch et al.<sup>5</sup> to propose that the observed anomalies are the results of the coupling of three-dimensional electrons to quasi two-dimensional critical spin fluctuations. In such a situation, conventional Hertz-Moriya-Millis theory<sup>8,9,10,11</sup> of quantum critical behavior leads to  $C/T \propto \ln(1/T)$ ,  $\Delta\rho \propto T$ , consistent with experimental findings. A similar observation applies to the thermoelectric power<sup>12</sup>. The microscopic reason for the two-dimensional nature of spin fluctuations is not clear however for  $\text{CeCu}_{6-x}\text{Au}_x$ , while magnetic frustration might provide a natural explanation<sup>4</sup> in the case of  $\text{YbRh}_2\text{Si}_2$  (for which inelastic neutron scattering data are not yet available).

The approach of Rosch et al.<sup>5</sup> requires fine tuning on a microscopic level (since a three-dimensional dispersion of the electrons will generically lead to three-dimensional critical fluctuations at the QCP) but it is internally consistent on a theoretical level. Indeed, the vertex corrections to the electron-spin fluctuation interaction is finite<sup>12</sup> in the case of a coupling to a three-dimensional fermion spectrum. (Note that the situation is different for two-dimensional fermions, which leads to a singular vertex<sup>13</sup>). Since the vertex is non-singular, the physics of long-wavelength magnetic fluctuations is described by a  $\phi^4$  field theory with ohmic damping. The quartic term is marginally irrelevant for  $d = 2$ , which is the upper critical dimension of this theory. Hence, the flow of the quartic term will lead to *logarithmic deviations* from mean-field critical behavior. These effects have not been investigated previously for the frequency-dependent response function in this context. Inelastic neutron scattering<sup>3,6,7</sup> on  $\text{CeCu}_{6-x}\text{Au}_x$  at the QCP ( $x_c \simeq 0.1$ ) have revealed an anomalous enhancement of the Landau damping. Whether logarithmic effects at the upper critical dimension could account for such an enhancement in the range of experimentally accessible frequencies is an outstanding open question which we address in this paper. We use the number of components ( $N$ ) of the field as a control parameter, and perform a calculation of the dynamical susceptibility to order  $1/N$ . We find that, to this order, the sign of the corrections to Landau damping *cannot explain the experimental data*.

An alternative theoretical viewpoint on the physics at the QCP is that the spin-fluctuation self-energy at the QCP is purely local (i.e momentum independent) and has an anomalous power-law dependence on frequency and temperature. This was first proposed<sup>3,6</sup> as a phenomenological fit to the inelastic neutron scattering data in the form:

$\text{Im}\chi_{q=Q}^{-1} = \omega^\alpha f(\omega/T)$  with  $\alpha \simeq 0.7 - 0.8$ . Si et al.<sup>14</sup> have further developed this point of view in the framework of the extended dynamical mean-field (EDMFT) theory of the Kondo lattice<sup>15,16,17</sup>. It has been argued in this context that the separation of electronic degrees of freedom and long-wavelength magnetic fluctuations is not legitimate<sup>14,18</sup>, because the (Kondo) coherence scale below which electron degrees of freedom can be eliminated might vanish at the QCP. Recently however, Grempel and Si showed<sup>19</sup> that *starting above the Kondo temperature* the EDMFT of the  $\phi^4$  theory for magnetic modes (see also the earlier work by Sengupta and one of us<sup>17</sup>). They further suggested that, in two dimensions, this theory would lead to an anomalous Landau damping, with the power-law form mentioned above ( $\alpha < 1$ ). We study this effective theory in the EDMFT approximation, using both a strict  $1/N$  expansion and a numerical solution using the large- $N$  expansion as an "impurity solver". We find that, contrary to the statements of Ref. 19, *no anomalous power-law is generated*. In fact, the EDMFT approximation to the  $\phi^4$  field-theory in the marginal case does not correctly reproduce the logarithmic corrections to the damping rate found in our direct calculation for  $d = 2$ , and leads only to subdominant  $O(\omega^2)$  corrections at low-energy. This extends to the marginal case the previous work of Motome and two of us<sup>20</sup> on the the EDMFT approximation applied to critical behavior, in which it was indeed found that the approximation is better in higher dimensions, and becomes unreliable below the upper critical dimensions.

Hence, the general conclusion of our paper is that logarithmic effects at the upper critical dimensions in the conventional theory of an antiferromagnetic QCP cannot explain the behavior observed experimentally for  $\text{CeCu}_{6-x}\text{Au}_x$ , at least at dominant order in the  $1/N$  expansion.

This paper is organized as follows: in section II, we perform a general renormalization-group analysis of the  $\phi^4$  theory with damping. In section III, we take a large- $N$  perspective, and present an explicit solution up to next to leading order in the  $1/N$  expansion. We also describe the local (EDMFT) approximation to this theory, up to order  $1/N$ . In section IV we discuss in more details the EDMFT approach, and solve numerically the EDMFT equations using a large- $N$  impurity solver. We conclude with a discussion of the implications of our work for the understanding of non-Fermi liquid behavior in heavy-fermion materials close to an antiferromagnetic QCP.

## II. RENORMALIZATION GROUP ANALYSIS

### A. Model and general renormalization-group framework

In this paper, we consider the following field theory<sup>8,10,11</sup> for an  $N$ -component field  $\phi_a$  ( $a = 1, \dots, N$ ):

$$S = \frac{1}{2} \sum_{q,\omega} D_0^{-1}(q, i\omega) \sum_a \phi_a^2(q, i\omega) + \frac{u_0}{4!} \int d^d x d\tau \left[ \sum_a \phi_a(x, \tau) \right]^2 \quad (1)$$

in which the free propagator reads, for imaginary frequencies:  $D_0^{-1}(q, i\omega) = r + |\omega| + q^2$ . This theory describes the vicinity of an antiferromagnetic quantum critical point, corresponding to the (bare) value  $z = 2$  of the dynamical critical exponent. In this context, the dissipative term is induced by the coupling to electronic degrees of freedom (Landau damping). The  $q$ -vector is the difference from the ordering wavevector, and  $r$  is a parameter which measures the distance to the critical point. We are particularly interested in this paper in the two-dimensional case, but we shall also briefly consider lower dimensions  $1 \leq d < 2$  (Sec. II C). In this section, we shall use the following normalized coupling constant:

$$g_0 \equiv S_d u_0 \quad \text{with} : \quad S_d = \frac{2}{\pi \Gamma(d/2) (4\pi)^{d/2}} \quad (2)$$

When considering the bare theory, an (ultra-violet) cutoff  $\Lambda$  on momenta is introduced. We consider the bare irreducible 2-point function  $\Gamma \equiv \chi^{-1}$ , which is the inverse of the correlation function (dynamical susceptibility  $\chi(q, i\omega) = \langle \phi(q, i\omega) \phi(q, -i\omega) \rangle$ ). The renormalization group (RG) specifies<sup>21</sup> how  $\Gamma$  changes upon a rescaling of the cutoff  $\Lambda \rightarrow \lambda\Lambda$ . The corresponding RG equation is obtained by imposing that the renormalized 2-point function  $\Gamma_R = Z\Gamma$  is independent of the original cutoff (with  $Z$  the field- or "wave function" renormalization). Focusing first, for simplicity, on the field-theory (1) in the critical (massless) case,  $\Gamma$  satisfies the following equation (all Green's functions in this and the next subsection are at  $T = 0$ ):

$$\Gamma[q, i\omega; g_0, \Lambda] = (\lambda\Lambda)^2 Z_\lambda^{-1} \Gamma \left[ \frac{q}{\lambda\Lambda}, \frac{Z_\lambda}{(\lambda\Lambda)^2} i\omega; g_\lambda, \Lambda = 1 \right] \quad (3)$$

In this expression,  $Z_\lambda$  and the running coupling constant  $g_\lambda$  are defined from the usual RG functions:

$$\lambda \frac{d}{d\lambda} g_\lambda = \beta(g_\lambda) \quad , \quad g_{\lambda=1} = g_0 \quad (4)$$

$$\lambda \frac{d}{d\lambda} \ln Z_\lambda = \eta(g_\lambda) \ , \ Z_{\lambda=1} = 1 \quad (5)$$

Let us emphasize a key aspect of Eq.(3), namely that the damping term does not require an independent renormalization so that the frequency dependence in the r.h.s of (3) involves only the wave-function renormalization  $Z_\lambda$ . This can be proven to hold to all orders, using a field theoretic RG<sup>22</sup>. For a use of this method in a different context see Ref.23

We have calculated the RG function  $\beta$  at one-loop order and the function  $\eta$  at two-loop order. The diagrams contributing to the 2-point function  $\Gamma$  up to this order are depicted in Fig. 1. We obtain (with  $\epsilon = d - 2$ ):

$$\begin{aligned} \beta(g) &= -\epsilon g + b_2 g^2 + O(g^3) \\ \eta(g) &= c_2 g^2 + O(g^3) \end{aligned}$$

where the coefficients  $b_2$  and  $c_2$  read, in the N-component theory:

$$b_2 = \frac{N+8}{6} \ , \ c_2 = \frac{(N+2)(12-\pi^2)}{144} \quad (6)$$

Details of calculation of the coefficient  $c_2$  are given in Appendix A.

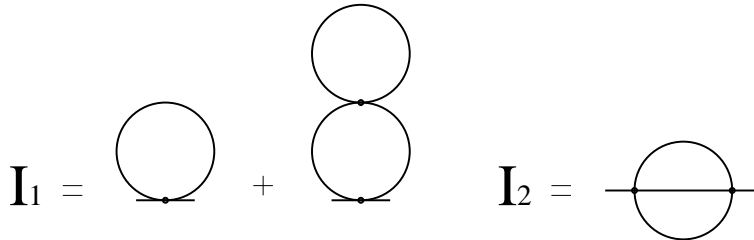


FIG. 1: Graphs contributing to the 2-point function at two-loop order.

### B. The marginal case $d = 2$ : logarithmic corrections to Landau damping

We focus here on the marginal case  $d = 2$  at the quantum-critical point, and integrate the above RG equations in order to obtain the corrections to the Landau damping at  $T = 0$ . Integrating the RG equations (4,5) using (6) yields, to this order:

$$\begin{aligned} g_\lambda &= \frac{g_0}{1 - b_2 g_0 \ln \lambda} \\ Z_\lambda &= \exp \left( \frac{c_2 g_0^2 \ln \lambda}{1 - b_2 g_0 \ln \lambda} \right) \end{aligned} \quad (7)$$

As expected in the marginal case,  $g_\lambda$  flows logarithmically to zero as  $\lambda \rightarrow 0$ , while  $Z_\lambda$  tends to some (non-universal) constant  $Z^*$ . To analyze the frequency dependence of  $\Gamma$  we use the general RG equation (3), setting  $q = 0$  and choosing  $\lambda = \lambda^*$  such that  $\omega Z_{\lambda^*} / (\lambda^* \Lambda)^2 = 1$ . This leads to:

$$\Gamma(q = 0, i\omega; g_0, \Lambda) = |\omega| \Psi(g_{\lambda^*}) \quad (8)$$

in which we use the notation  $\Psi(g) \equiv \Gamma(q = 0, i\omega = i; g, \Lambda = 1)$ . We then expand the r.h.s of (8) in powers of  $g_{\lambda^*}$ , with:  $g_{\lambda^*} \sim -1/(b_2 \ln \lambda^*) \sim -2/[b_2 \ln(Z^* |\omega| / \Lambda^2)]$ . This expansion actually starts at *second-order* in  $g_\lambda$ , since the 2-point function  $\Gamma$  is subtracted to insure that one sits at the critical point (in other words, the tadpoles  $I_1$  do not contribute to the frequency dependence). Noting that:  $I_2(q = 0, i\omega = 0; g_\lambda, \Lambda) - I_2(q = 0, i\omega, g_\lambda, \Lambda) = c g_\lambda^2 |\omega|$ , coefficient  $c$

computed in Appendix A, we finally obtain a correction to the frequency dependence of the 2-point function at the  $T = 0$  QCP of the form:

$$\Gamma(q = 0, i\omega) = |\omega| \left( 1 + \frac{[6\pi^2 \ln 2 - 11\zeta(3)](N+2)}{12(N+8)^2} \frac{1}{\ln^2(|\omega|Z^*/\Lambda^2)} \right) \quad (9)$$

Hence, only subdominant corrections to the Landau damping are generated in the marginal case  $d = 2$ . Furthermore, the positive coefficient in the above expression can be interpreted as an effective exponent  $\alpha = 2/z > 1$ . This is in agreement with the  $1/N$  expansion, which we consider later in this paper.

### C. $\epsilon = 2 - d$ expansion of critical exponents and corrections to scaling

We consider here the case  $d < 2$ , in which the quartic term is relevant, and the effective coupling  $g_\lambda$  tends to a non-trivial fixed point. The case  $d = 1$  is relevant, for example, when considering<sup>11</sup> an Ising chain in a transverse field ( $N = 1$ ) or a chain of quantum rotors ( $N > 1$ ) coupled to a dissipative environment.

We make use of the general RG equation (3) by choosing  $\lambda = \lambda^*$  such that  $q/\lambda^*\Lambda = 1$ . For  $\lambda \rightarrow 0$ , we now have:  $g_\lambda \rightarrow g^*$  and  $Z_\lambda \sim \lambda^\eta$  (with the critical exponent  $\eta \equiv \eta(g^*)$ ). Hence, we obtain (at  $T = 0$ ) for the dynamical susceptibility  $\chi = \Gamma^{-1}$  at low-frequency and small momentum:

$$\chi(q, \omega) = q^{-2+\eta} \phi(\omega/cq^{2-\eta}) \quad (10)$$

In this expression,  $\phi$  is a universal scaling function associated with the fixed point. From this expression, the dynamical critical exponent is identified as:

$$z = 2 - \eta \quad (11)$$

An expression which holds to all orders, and is the result of the existence of a unique RG function, as explained above. This scaling form can actually be generalized to finite temperature in the quantum critical regime in the form:

$$\chi(q, \omega) = \frac{1}{T} \Phi \left( \frac{\omega}{T}, \frac{c_1 q}{T^{1/z}} \right) \quad (12)$$

with  $\Phi$  a universal scaling function, and  $c_1$  a non-universal constant. In particular, at zero-momentum,  $T\chi(q = 0, \omega)$  is an entirely universal scaling function of  $\omega/T$ . At small  $\omega/T$ , we expect an analytic dependence on  $\omega/T$ , with  $\Gamma(0, \omega)/T = \mathcal{C}_1 - i\mathcal{C}_2\omega/T + \dots$ . In contrast, at large  $\omega/T$  we have  $\Gamma(0, \omega)/T = -i\mathcal{C}_3\omega/T$ , and the subdominant terms are not analytic, but are related to those in (19) below. Here  $\mathcal{C}_{1-3}$  are all non-trivial *universal* constants. The value of  $\mathcal{C}_3$  is determined in the  $\epsilon$  expansion by the computations in Appendix A ( $\mathcal{C}_3 = 1 + c(\epsilon/b_2)^2 + \dots$ ), and this universality is clearly related to the universal logarithmic correction in (9). The values of  $\mathcal{C}_{1,2}$  require a computation of the damping at  $T > 0$ , and accurate determination of these is likely to require a self-consistent treatment of the loop corrections, as discussed in Ref 24. In particular,  $\mathcal{C}_2 \neq \mathcal{C}_3$  and hence

$$\lim_{\omega \rightarrow 0} \lim_{T \rightarrow 0} \frac{\text{Im}\Gamma(q, \omega)}{\omega} \neq \lim_{T \rightarrow 0} \lim_{\omega \rightarrow 0} \frac{\text{Im}\Gamma(q, \omega)}{\omega}. \quad (13)$$

So the Landau damping co-efficient is sensitive to the order of limits of  $\omega \rightarrow 0$  and  $T \rightarrow 0$ ; we expect that similar considerations will also apply to the logarithmic corrections in  $d = 2$  noted in (9).

From the RG expressions in Section II A, the  $\epsilon$ -expansion of the critical exponent  $\eta = 2 - z$  reads, to order  $\epsilon^2$ :

$$\eta = \frac{(N+2)(12-\pi^2)}{4(N+8)^2} \epsilon^2 + O(\epsilon^3) \quad (14)$$

We have also obtained the correlation length exponent at order  $\epsilon$ :

$$\nu = \frac{1}{2} + \frac{(N+2)}{4(N+8)} \epsilon + O(\epsilon^2) \quad (15)$$

Finally, we comment briefly on the next-to-leading corrections at  $T = 0$  to the Landau damping at the QCP. These are obtained from corrections to scaling, i.e. depend on the manner in which  $g_\lambda$  flows to the fixed point:

$$g_\lambda = g^* + (g_0 - g^*)\lambda^\Omega + \dots \quad (16)$$

with:

$$\Omega = \beta'(g^*) = \epsilon + O(\epsilon^2) \quad (17)$$

The RG equation then yields the following form for the 2-point function, including corrections to scaling:

$$\Gamma(q, \omega) = q^{2-\eta} [\gamma_0(\omega q^{-z}) + q^{2\Omega}\gamma_2(\omega q^{-z}) + \dots] \quad (18)$$

Taking the limit  $q \rightarrow 0$ , this yields the following form of the Landau damping term:

$$\Gamma(q = 0, i\omega) = C_0|\omega|^{\frac{2-\eta}{z}} + C_2|\omega|^{\frac{2-\eta+\Omega}{z}} + \dots \quad (19)$$

In the present case, the exact identity  $z = 2 - \eta$  implies that the dominant term is simply  $\propto |\omega|$ , while a correction of the form  $|\omega|^{1+\frac{\Omega}{2-\eta}}$  is found. The logarithmic correction found above can be seen as the limiting behavior of this correction in the marginal case  $d = 2$  ( $\epsilon = 0$ ).

### III. LARGE-N ANALYSIS OF THE $\phi^4$ THEORY OF AN ANTIFERROMAGNETIC QUANTUM CRITICAL POINT IN TWO DIMENSIONS ( $z = 2$ )

In this section, we treat the field-theory (1) within the large-N expansion. For this purpose, we scale the coupling constant in (1) by  $1/N$  and set:

$$u_0 = \frac{6u}{N} \quad (20)$$

We denote by  $\Sigma_0$  and  $\Sigma_1$  the contributions to the self-energy of order  $1/N^0$  and  $1/N$ , respectively ( $\langle\phi\phi\rangle^{-1} = D_0^{-1} - \Sigma$ ). The (skeleton) diagrams contributing to  $\Sigma_0$  and  $\Sigma_1$  are depicted in Fig. 2. The corresponding expressions read:

$$\Sigma_0(q, i\omega) = -uT \sum_{\nu_n} \int \frac{d^2k}{(2\pi)^2} D(k, i\nu_n) \quad (21)$$

$$\Sigma_1(q, i\omega) = -2\frac{u}{N}T \sum_{\nu_n} \int \frac{d^2k}{(2\pi)^2} \frac{D(k+q, i\nu_n + i\omega_n)}{1 + u\Pi_0(k, i\nu_n)} \quad (22)$$

where:

$$\Pi_0(q, i\omega) = T \sum_{\nu_n} \int \frac{d^2p}{(2\pi)^2} D(p, i\nu_n) D(p+q, i\nu_n + i\omega_n) \quad (23)$$

In this expression,  $D$  denotes the full propagator to order  $1/N^0$  (i.e including the self-energy  $\Sigma_0$ ). Hence, Eq. (21) should be viewed as a self-consistent equation for  $\Sigma_0$ . We first consider this equation at  $\omega = q = 0$ , defining a "gap"

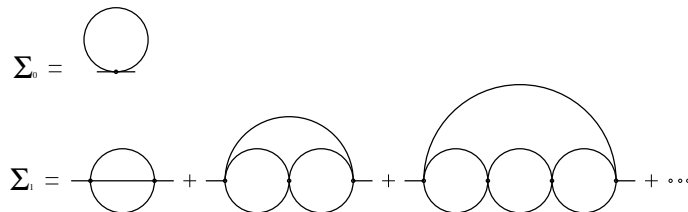


FIG. 2: Self-energy graphs in the  $1/N$  expansion.

$\Delta(T) = r - \Sigma(q = 0, i\omega = 0)$  related to the correlation length  $\xi$  by  $\Delta = \xi^{-2}$ .  $\Delta$  vanishes ( $\xi$  diverges) at the  $T = 0$

QCP. At the saddle point ( $N = \infty$ ) level  $\Delta(T) = \Sigma_0(T=0) - \Sigma_0(T)$ . We show in Appendix B that, at this order, the self-consistent equation for the gap reads (see also Ref. 25):

$$\frac{(2\pi)^2}{u} \tilde{\Delta} = \frac{1}{2} \ln \frac{\tilde{\Delta}}{\tilde{\Delta} + \tilde{\Lambda}_{q^2}} + \ln \frac{\Gamma(\tilde{\Delta})}{\Gamma(\tilde{\Delta} + \tilde{\Lambda}_{q^2})} + \tilde{\Lambda}_{q^2} \ln \tilde{\Lambda}_{q^2} - \tilde{\Lambda}_{q^2} \quad (24)$$

where  $\Lambda_{q^2}$  is a cutoff for  $q^2$ ,  $\tilde{\Lambda}_{q^2} = \Lambda_{q^2}/(2\pi T)$  and  $\tilde{\Delta} = \Delta/(2\pi T)$ . In the limit  $\tilde{\Delta} \ll 1$  and  $\tilde{\Delta} \ll \tilde{\Lambda}_{q^2}$  Eq.(24) reduces to:

$$\frac{(2\pi)^2}{u} \tilde{\Delta} = -\frac{1}{2} \ln \tilde{\Delta} - \tilde{\Delta} \ln \tilde{\Lambda} \quad (25)$$

We dropped the index in  $\Lambda_{q^2}$  as it is the only scale we use. In the limit  $u \ln \tilde{\Lambda} \gg 1$  the approximate solution of Eq.(25) is<sup>25</sup>:

$$\Delta = \pi T \frac{\ln(\ln \Lambda/2\pi T)}{\ln \Lambda/2\pi T} \quad (26)$$

Hence,  $\Delta$  vanishes faster than  $T$  as  $T \rightarrow 0$ . The contributions of the marginally irrelevant coupling are smaller than temperature (or energy) itself. As we shall now see, this also applies to the  $T = 0$  contributions to the Landau damping at the critical point, which is the main quantity of interest in this paper.

We perform a calculation of the damping rate at the  $T = 0$  QCP up to order  $1/N$ . At the saddle-point level ( $1/N^0$ ), the propagator at the QCP reads:  $D(q, i\omega) = (|\omega| + q^2)^{-1}$ . The polarization bubble  $\Pi_0(q, i\omega)$  defined by (23) can then be calculated exactly. Details are given in Appendix C. Here we present the result:

$$\begin{aligned} 8\pi^2 \Pi_0(q, i\omega) = & \frac{\pi^2}{4} - \frac{|\omega| + q^2}{q^2} \left( \ln \frac{|\omega| + \Lambda}{\Lambda} + 2 \ln \frac{|\omega| + q^2}{|\omega|} \right) + 2 \ln \frac{|\omega| + \Lambda}{|\omega|} \\ & + \frac{s}{q^2} \ln \frac{|\omega| + q^2 + 2\Lambda + s}{|\omega| + q^2 + 2\Lambda - s} - \text{Li}_2 \left( \frac{q^2}{2|\omega| + q^2} \right) + \text{Li}_2 \left( -\frac{q^2}{2|\omega| + q^2} \right) \\ & + \text{Li}_2 \left( \frac{q^2}{|\omega| + s} \right) - \text{Li}_2 \left( -\frac{q^2}{|\omega| + s} \right) + \text{Li}_2 \left( \frac{q^2}{|\omega| - s} \right) - \text{Li}_2 \left( -\frac{q^2}{|\omega| - s} \right) \end{aligned} \quad (27)$$

where  $s = \sqrt{(|\omega| + q^2)^2 + 4q^2\Lambda}$  and  $\text{Li}_2(x) = -\int_0^x dy \ln(1-y)/y$ . The above expression has the following asymptotics at  $\omega/\Lambda \rightarrow 0$ ,  $q^2/\Lambda \rightarrow 0$ :

$$\begin{aligned} & 4 - 2 \ln \frac{q^2}{\Lambda}, & \frac{|\omega|}{q^2} \rightarrow 0 \\ & 2 + \frac{\pi^2}{4} - 2 \ln \frac{|\omega|}{\Lambda}, & \frac{q^2}{|\omega|} \rightarrow 0, & \frac{\omega^2}{q^2\Lambda} \rightarrow 0 \\ & -2 + \frac{\pi^2}{4} - 2 \ln \frac{|\omega|}{\Lambda}, & \frac{q^2}{|\omega|} \rightarrow 0, & \frac{q^2\Lambda}{\omega^2} \rightarrow 0 \end{aligned} \quad (28)$$

We use the explicit expression (27) into (22) and perform the frequency and momentum integrals numerically. The resulting self-energy  $\Sigma_1(q, i\omega)$  is plotted in Fig.3. It is seen that the  $1/N$  corrections to the damping rate are *less singular* than  $\omega$  at low-frequency. Alternatively, the corrections can be put in the form of an effective, scale-dependent exponent  $\alpha(\omega)$ , defined by:  $\alpha(\omega) = \omega \partial(\ln \chi^{-1}(i\omega))/\partial\omega$  with  $\chi^{-1}(i\omega) = |\omega| - \Sigma_1(q, i\omega = 0) + \Sigma_1(q = 0, i\omega = 0)$  the  $q = 0$  dynamical susceptibility at the QCP. This effective exponent is plotted on Fig. 4: it is seen to be only weakly dependent on frequency in the range where it is displayed. Hence, the logarithmic corrections in the marginal case can be mimicked by a power-law, but correspond to an effective exponent  $\alpha > 1$ , in contrast to the experimental observation  $\alpha < 1$  for the two compounds mentioned above.

Incidentally, we would like to comment on the data analysis made in Ref. 6 in order to support  $\omega/T$  scaling of the inelastic neutron scattering data in  $\text{CeCu}_{5.9}\text{Au}_{0.1}$ . To this aim, we perform a similar analysis on our analytical result, by plotting  $T^\alpha \chi(\omega, T)$  versus  $\omega/T$  (Fig. 5). In this plot, the frequency-dependent susceptibility at finite temperature and  $q = 0$  is approximated by:  $\chi(\omega, T)^{-1} \simeq -i\omega + \Delta(T) - \Sigma_1(q = 0, \omega) + \Sigma_1(q = 0, \omega = 0)$  (this approximation does not include the temperature dependence of the Bose functions in the calculations which would be required to obtain the correct damping coefficient, as is required to obtain (13)). Such a scaling plot is attempted for a varying range of  $\alpha$ , and the optimal value of the exponent is chosen such as to yield the best collapse<sup>6</sup>, as measured by the standard deviation plotted in Fig. 6. It is seen that an excellent collapse is obtained with  $\alpha \simeq 1.09$ . We emphasize

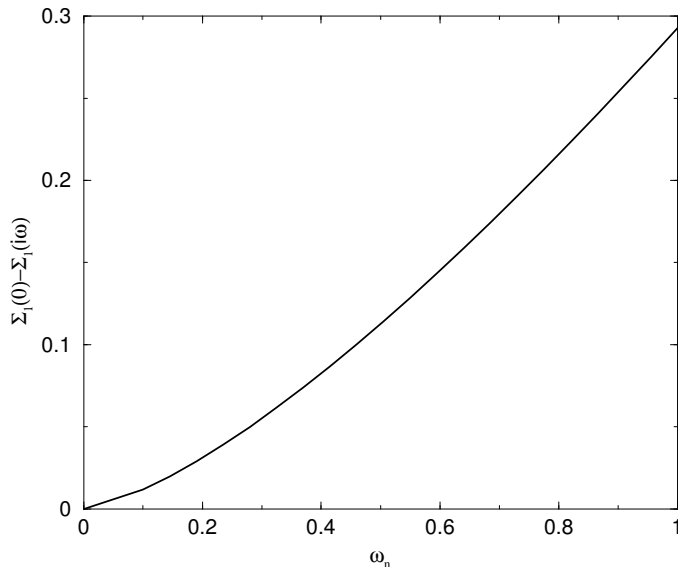


FIG. 3: Self-energy at the  $d = 2, z = 2$  QCP, up to order  $1/N$ , as a function of (imaginary) frequency  $\omega$ . The calculation is for  $u/(2\pi)^3 = 5$ , and  $\omega$  is measured in units of the cutoff  $\Lambda$ .

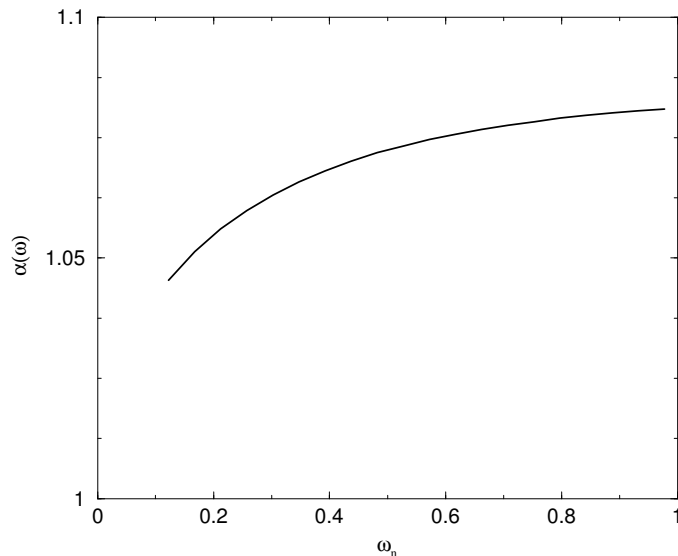


FIG. 4: Effective exponent  $\alpha(\omega)$ , as defined in the text, as a function of (imaginary) frequency  $\omega$ . The parameters are as in Fig.3

that  $\omega/T$  scaling, in a strict asymptotic sense, does apply here, but with a trivial Gaussian exponent and scaling function:  $T\chi(\omega, T) = i/\tilde{\omega}$  ( $\tilde{\omega} = \omega/T$ ). The logarithmic corrections characteristic of the marginal case apparently mimic non-trivial scaling properties over quite an extended range of  $\omega/T$ . This should serve as a warning for the interpretation of the experimental results.

We end this section by considering the *local approximation* to the damped  $\phi^4$  theory in the marginal case, up to order  $1/N$ . In a previous paper, Motome and two of us<sup>20</sup> have investigated the EDMFT approximation to the critical behavior of various models, and shown that this local approximation is quite satisfactory above the upper critical

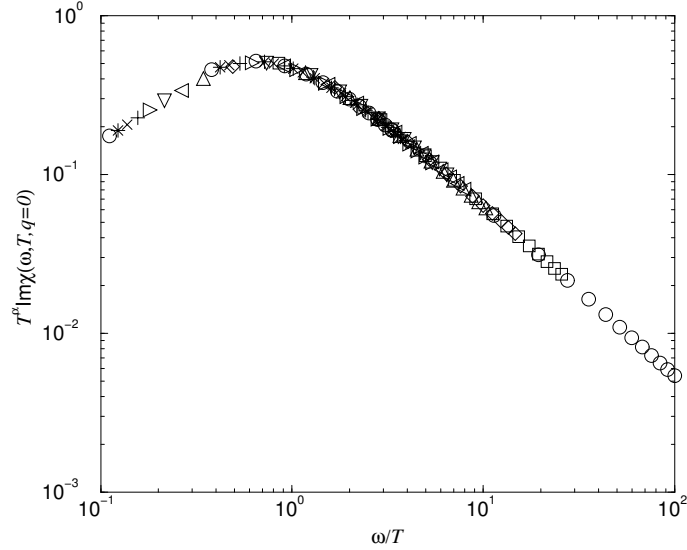


FIG. 5: Scaling plot for  $T^\alpha \text{Im}\chi(\omega, T, q=0)$  versus  $\omega/T$ . The temperatures displayed are:  $T = .03 - .9\Lambda$ . The choice  $\alpha = 1.09$  provides a good collapse of the data over this frequency range. Different symbols correspond to different values of  $T$ .

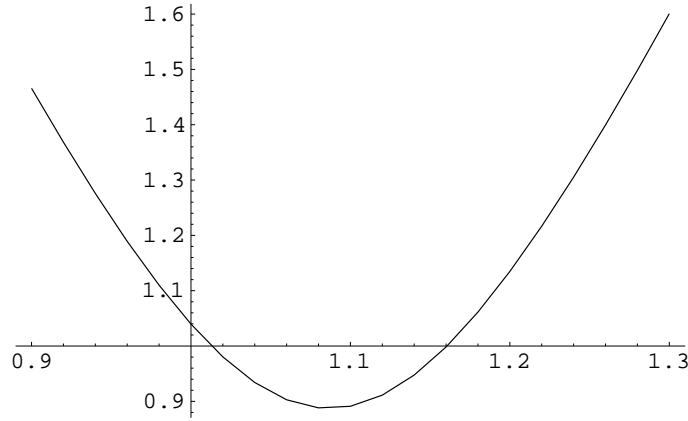


FIG. 6: Standard deviation  $\sigma_{\log(T^\alpha \chi)}$ , measuring the quality of the  $\omega/T$  collapse<sup>6</sup>, versus  $\alpha$ .

dimension (i.e here for  $d + z > 4$ ), when the quartic coupling is irrelevant. In this approach, all skeleton graphs are taken to be local. For arbitrary  $z$  and  $d = 2$ , the local propagator at the  $T = 0$  QCP reads:

$$D(i\omega) = \int \frac{qdq}{2\pi} \frac{1}{|\omega|^{\frac{z}{2}} + q^2} = \frac{1}{4\pi} \ln \left( \frac{\Lambda + |\omega|^{\frac{z}{2}}}{|\omega|^{\frac{z}{2}}} \right) \quad (29)$$

Performing the Fourier transform, this yields the asymptotic behavior at long (imaginary) time:

$$D(\tau) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{1}{4\pi} \ln \left( \frac{\Lambda + |\omega|^{\frac{z}{2}}}{|\omega|^{\frac{z}{2}}} \right) e^{-i\omega\tau} \approx \frac{1}{(2\pi)^2 |\tau|} \int_0^{+\infty} dx \ln x^{-\frac{z}{2}} \cos x = \frac{1}{4\pi z |\tau|} \quad (30)$$

The  $1/N$  correction to the spin-fluctuation self-energy is:

$$\Sigma_1(\tau) = \frac{2u^2}{N} D(\tau)^3 \approx \frac{1}{N} \frac{2u^2}{(4\pi)^3 z^3 |\tau|^3} \quad (31)$$



which corresponds to the following low-frequency behavior (with  $\Lambda_1$  a cutoff of order 1):

$$\Sigma_1(i\omega) \approx \frac{1}{N} \frac{2u^2}{(4\pi)^3 z^3} \left(-\frac{3}{2} + \gamma - \ln \Lambda_1 + \ln |\omega|\right) \omega^2 \quad (32)$$

For real frequencies (31) corresponds to corrections to the Landau damping of the form  $\text{Im}\Sigma_1(\omega + i0^+) \propto \omega^2 \text{Sign}(\omega)$ . This is much smaller (by a factor of order  $\omega$ , up to logarithms) than the corrections obtained from the direct calculation in  $d = 2$  detailed above (Fig. 3). Hence, we conclude that the local (EDMFT) approximation is not very reliable at the upper critical dimension (marginal case). As we shall see in the following section, a numerical solution of the EDMFT equations for the Kondo lattice using an approximate "impurity solver" based on the large- $N$  expansion leads to a similar conclusion: only subdominant corrections to the Landau damping rate are generated instead of the anomalous (power-law like) enhancement observed experimentally.

#### IV. "LOCALLY" CRITICAL POINT: SELF-CONSISTENT LARGE- $N$ SOLUTION OF THE EDMFT EQUATIONS

##### A. EDMFT of the Kondo lattice and mapping onto a local spin-fluctuation model

It has been recently argued<sup>14,18</sup> that the understanding of non-Fermi liquid behavior in heavy-fermion compounds close to a QCP requires a formalism in which electronic degrees of freedom and spin fluctuations can be treated on the same footing, at least as a starting point. This is a non-trivial task, which is however made easier in the context of the extended dynamical mean-field theory (EDMFT)<sup>15,16,17</sup> of the Kondo lattice model, which we consider here.

Let us consider the Kondo lattice Hamiltonian with an explicit exchange coupling between localized spins, chosen to be Ising-like:

$$H = \sum_{k\sigma} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma} + J_K \sum_{i\sigma\sigma'} \vec{S}_i \cdot c_{i\sigma}^\dagger \vec{\tau}_{\sigma\sigma'} c_{i\sigma'} + \sum_{i<j} I_{ij} S_i^z S_j^z \quad (33)$$

In this expression,  $J_K$  is the (antiferromagnetic) Kondo coupling, and  $\vec{\tau}$  stands for the Pauli matrices. EDMFT maps this model onto a local impurity model with effective action:

$$S = - \int_0^\beta d\tau \int_0^\beta d\tau' \sum_\sigma [c_\sigma^\dagger(\tau) \mathcal{G}_0^{-1}(\tau - \tau') c_\sigma(\tau') + S^z(\tau) \chi_0^{-1}(\tau - \tau') S^z(\tau')] \\ + \int_0^\beta d\tau J_K \sum_{\sigma\sigma'} \vec{S} \cdot c_\sigma^\dagger \vec{\tau}_{\sigma\sigma'} c_{\sigma'} \quad (34)$$

Both  $\mathcal{G}_0$  and  $\chi_0$  are effective "Weiss fields" which must be determined in such a way that the following self-consistency conditions hold:

$$G(i\omega_n) \equiv \langle c_\sigma^\dagger(i\omega_n) c_\sigma(i\omega_n) \rangle_S = \int d\epsilon \frac{\rho_0(\epsilon)}{i\omega_n - \Sigma_c(i\omega_n) - \epsilon} \quad (35)$$

$$\chi(i\nu_n) \equiv \langle S^z(i\nu_n) S^z(-i\nu_n) \rangle_S = \int d\epsilon \frac{\rho_I(\epsilon)}{M(i\nu_n) - \epsilon} \quad (36)$$

In this expression,  $\Sigma_c(i\omega_n)$  and  $M(i\nu_n)$  are respectively the fermionic and local spin self-energies calculated within the impurity problem (34), *i.e.*:

$$G^{-1}(i\omega_n) = \mathcal{G}_0^{-1}(i\omega_n) - \Sigma_c(i\omega_n) \quad (37)$$

$$\chi^{-1}(i\nu_n) = -\chi_0^{-1}(i\nu_n) + M(i\nu_n) \quad (38)$$

The densities of states  $\rho_0$  and  $\rho_I$  associated respectively with the conduction band and to the spin interactions read:  $\rho_0(\epsilon) = \sum_k \delta(\epsilon - \epsilon_k)$ ,  $\rho_I(\epsilon) = \sum_q \delta(\epsilon - I_q)$ . The self-consistency conditions (35,36) simply express that the impurity model Green's functions and self-energies should coincide with their lattice counterpart (assuming momentum-independence of the self-energies). They can be recast into a more compact form, relating directly the local correlation functions to the two Weiss fields appearing in the effective action (34):

$$G(i\omega_n) = \int d\epsilon \frac{\rho_0(\epsilon)}{i\omega_n + G^{-1}(i\omega_n) - \mathcal{G}_0^{-1}(i\omega_n) - \epsilon} \quad (39)$$

$$\chi(i\nu_n) = \int d\epsilon \frac{\rho_I(\epsilon)}{\chi^{-1}(i\nu_n) + \chi_0^{-1}(i\nu_n) - \epsilon} \quad (40)$$

The key question at this stage is whether the electronic degrees of freedom can be integrated out, particularly near the QCP. The local EDMFT framework allows to handle this issue in a more controlled manner. Indeed, electronic degrees of freedom can be integrated out<sup>17</sup> from the impurity model (34) using an expansion in the spin-flip part of the Kondo coupling à la Anderson-Yuval-Hammann<sup>26,27</sup>. This can be conveniently performed, for example, using bosonization methods<sup>19,28,29</sup>. This leads to the following action for the spin degrees of freedom:

$$S = \int_0^\beta d\tau \int_0^\beta d\tau' S^z(\tau) [-\chi_0^{-1}(\tau - \tau') + K(\tau - \tau')] S^z(\tau') + \int_0^\beta d\tau \Gamma S^x \quad (41)$$

In this expression,  $K(\tau - \tau') \sim 1/(\tau - \tau')^2$  is a long-range interaction induced by the electronic modes. It corresponds to the bare Landau damping ( $K(i\nu_n) = \kappa|\nu_n|$  at low-frequency). The coupling in front of the spin-flip term is proportional to the Kondo coupling:  $\Gamma \propto \tau_0 J_K$  (with  $\tau_0$  a short-time cutoff). Hence, the problem has been mapped onto an quantum Ising model in a transverse field with long range interactions<sup>17,19,29</sup>.

## B. Numerical solution based on the large- $N$ approximation

This mapping has been used in order to perform analytical<sup>17</sup> and numerical<sup>29</sup> studies of the behavior at the QCP in mean-field models with random exchange couplings  $I_{ij}$ . It was demonstrated that this case is formally similar<sup>30</sup> to the Hertz-Moriya-Millis theory of an antiferromagnetic QCP in  $d = 3$ . Recently, Grepel and Si have analyzed the EDMFT theory of the QCP in the *two-dimensional* case using this mapping<sup>19</sup>. Their claim is that anomalous Landau damping ( $\propto \omega^\alpha$ ,  $\alpha < 1$ ) and  $\omega/T$  scaling can be demonstrated in this case. Here, we present a solution of this problem in which the local impurity problem is solved using a large- $N$  method. In contrast to the claims of Ref. 19, we find that only subdominant corrections to the Landau damping are generated.

Following Ref. 17, we deal with the local action(41) for the transverse field Ising model by extending it to an  $N$ -component rotor model  $\vec{n}$ :

$$S = \frac{1}{2} \int_0^\beta d\tau \left[ \frac{(\partial_\tau \vec{n})^2}{g} + \lambda(\tau)(\vec{n}^2 - N) \right] + \frac{1}{2} \int_0^\beta d\tau \int_0^\beta d\tau' \vec{n}(\tau) [-\chi_0^{-1}(\tau - \tau') + K(\tau - \tau')] \vec{n}(\tau') \quad (42)$$

In this expression,  $g$  plays a role to similar to that of the transverse field in the original Ising model (disordering term). We define a self-energy correction for the local spin susceptibility  $\chi(\tau) \equiv \langle S^z(0)S^z(\tau) \rangle$ :

$$\chi(i\nu_n) = \frac{1}{\nu_n^2/g + K(i\nu_n) - \chi_0^{-1}(i\nu_n) - \Sigma(i\nu_n)} \quad (43)$$

Let us first consider the problem at the saddle-point level ( $N = \infty$ ), for a given Weiss function  $\chi_0$ . The self-energy is then frequency- independent (as noted by Grepel and Si<sup>19</sup>) and is given by the saddle-point value of the Lagrange multiplier:

$$\Sigma_0(i\nu_n) = -\lambda \quad (44)$$

such that:

$$\chi_\infty(\tau = 0) = 1 \quad (45)$$

In this expression,  $\chi_\infty$  is the local susceptibility at the saddle-point ( $N = \infty$ ) level:

$$\chi_\infty(i\nu_n) = \frac{1}{\nu_n^2/g + K(i\nu_n) - \chi_0^{-1}(i\nu_n) + \lambda} \quad (46)$$

Both  $\lambda$  and  $\chi_\infty$  depend on the Weiss function  $\chi_0$ . We approximate the self-energy by its first two terms in the  $1/N$  expansion:

$$\Sigma(i\nu_n) \simeq \Sigma_0(i\nu_n) + \Sigma_1(i\nu_n) - \Sigma_1(i0) \quad (47)$$

$$\Sigma_1(\tau) = -\frac{2}{N}\Gamma(\tau)\chi_\infty(\tau) \quad (48)$$

$$\text{with } \Gamma(i\nu_n) = \frac{1}{\Pi(i\nu_n)} \text{ and } \Pi(\tau) = [\chi_\infty(\tau)]^2 \quad (49)$$

The EDMFT equations are thus solved by iterating (numerically) the following procedure. For a given  $\chi_0(i\nu_n)$ , the saddle-point quantities  $\lambda = -\Sigma_0$  and  $\chi_\infty$  are computed by solving Eqs. (45,46). Then, the self-energy is computed from (47,48,49). This is inserted into the EDMFT self-consistency condition in order to obtain the local susceptibility:

$$\chi(i\nu_n) = \int d\epsilon \frac{\rho_I(\epsilon)}{\nu_n^2/g + K(i\nu_n) - \Sigma(i\nu_n) - \epsilon} \quad (50)$$

From this, an updated value of the Weiss function is obtained as:  $\chi_0^{-1}(i\nu_n) = \nu_n^2/g + K(i\nu_n) - \Sigma(i\nu_n) - \chi^{-1}(i\nu_n)$ , and the procedure is iterated until convergence is reached.

The result of this numerical calculation for the (local) spin self-energy is displayed in Fig. 7. The behavior of the self-energy at low imaginary frequency is well fitted by  $\Sigma_1(i\omega) \sim \omega^2 \ln|\omega|$  (inset) when one tunes the parameters to sit at criticality (Fig. 8). This corresponds to subdominant corrections to the Landau damping  $\text{Im}\Sigma(\omega + i0^+) \propto \omega^2 \text{sign}(\omega)$ . These findings are in agreement with the expansion of the EDMFT equations for the  $\phi^4$  model at order  $1/N$  discussed at the end of the previous section, and provide an independent check of this result. The method followed in the present section has been to use the large- $N$  method as an approximate "impurity solver" for the local problem. It can be easily checked that if, instead, the equations used here are all expanded up to order  $1/N$ , the equations of the local approximation presented in the previous section are recovered.

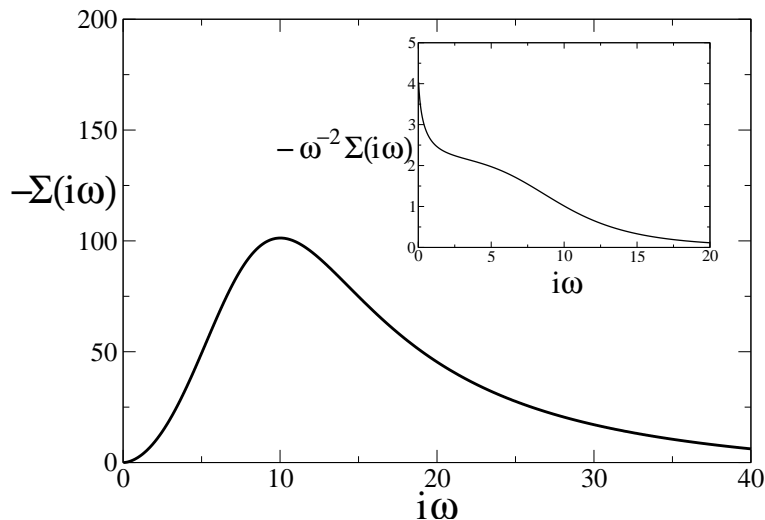


FIG. 7: Self-energy  $\Sigma(i\omega)$  close to the critical point (at inverse temperature  $\beta = 500$  and  $g = g_c = 0.223$ ), as a function of imaginary frequency. For this calculation, we have set  $N = 2$ , and used a flat density of states  $\rho_I(\epsilon)$  of half-width equal to unity. Inset: plot of  $\omega^{-2}\Sigma(i\omega)$ , showing the logarithmic behavior of this quantity at small frequencies.

## V. CONCLUSION

We analyzed two of the approaches to a critical theory of the antiferromagnetic metal to paramagnetic metal in the  $\text{CeCu}_{6-x}\text{Au}_x$  and  $\text{YbRh}_2\text{Si}_2$  materials from the perspective of the antiferromagnetic spin fluctuations.

In the context of the model of Rosch et. al.<sup>5</sup> we analyzed the corrections to mean field theory due to the marginally irrelevant fourth order coupling. We find that while there are logarithmic corrections to mean field theory which vanish as  $1/\text{Log}^2|\omega|$  the coefficient of this corrections is such that it can only be mimicked by a decreased Landau damping, in clear disagreement with experimental observations. In our view, this rules out this model, in spite of the fact that it can account nicely for many of the electronic properties of these systems.

We also analyze the "local" version of this model using the large  $N$  expansion. Again in this case, we find that it is not possible to account for the anomalous frequency dependence of the damping coefficient. This is in disagreement with reference<sup>19</sup> but in agreement with a recent EDMFT study of the Anderson lattice model.<sup>31</sup>

The anomalous damping has been recently observed in NMR measurements in both<sup>32,33</sup>  $\text{YbRh}_2\text{Si}_2$  and  $\text{CeCu}_{6-x}\text{Au}_x$ . While it has different temperature dependencies in each system, it is clearly enhanced above the

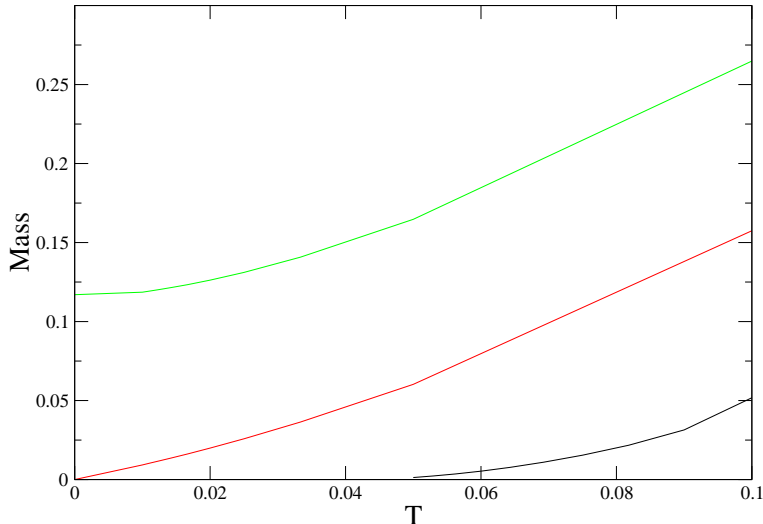


FIG. 8: Plot of the mass  $\Sigma(i0)$  as a function of temperature, demonstrating the existence of a continuous transition at  $T = 0$  within our approximate solution of the EDMFT equations. From bottom to top:  $g = 0.1$  (ordered state),  $g = 0.223$  (critical at  $T = 0$ ),  $g = 0.3$  (disordered case).

fermi liquid temperature dependence. Possible resolution of this puzzle may require a better treatment of the effects of disorder or better treatments of models which retain the fermionic nature of the spin fluctuations as proposed by Coleman et. al.<sup>18</sup> and Si et. al.<sup>14</sup>

### Acknowledgments

We thank D. Gempel and Q. Si for useful discussions and correspondence. The authors acknowledge the hospitality of the Kavli Institute for Theoretical Physics (UCSB, Santa Barbara), where part of this work was performed and supported under grant NSF-PHY99-07949 and NSF grant No. DMR-0096462. We also acknowledge the support of an international collaborative grant from CNRS (PICS 1062).

### APPENDIX A: COMPUTATION OF RG FLOW

The computations leading to the RG flow (6) are standard. Here we only provide some further details of the computation of the number  $c_2$ , and of the constant  $c$  required to obtain the universal logarithmic correction in (9). These computations involve the  $|\omega|$  term in the propagator in an essential and novel manner.

Repeating the standard field-theoretic derivation for the two-loop graph in Fig 1 we have:

$$I_2(q, i\omega = 0) = \frac{u_0^2(N+2)}{18} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{(q_1^2 + |\omega_1|)(q_2^2 + |\omega_2|)} \times \frac{1}{((\mathbf{q} + \mathbf{q}_1 + \mathbf{q}_2)^2 + |\omega_1 + \omega_2|)} = g^2 q^{2d-2} \left( -\frac{c_2}{2\epsilon} + \mathcal{O}(\epsilon^0) \right) \quad (\text{A1})$$

$I_2$  is to be evaluated for  $d < 1$ , where it is convergent, and the result analytically continued to near  $d = 2$ , where it is expected have the small  $\epsilon$  expansion shown on the r.h.s. The coefficient of the pole fixes the value of the constant  $c_2$ .

Note that we are evaluating  $I_2$  in an external momentum  $\mathbf{q}$ , and in  $d = 2$  the pole on the r.h.s. corresponds to the appearance of  $q^2 \ln q$  term. In contrast, evaluation of  $I_2(0, i\omega)$  (presented below) shows that no pole appears, and hence there is no  $|\omega| \ln |\omega|$  term in  $d = 2$ . This absence of such a pole is, of course, the reason for the absence of an independent renormalization constant for the damping term noted below (5), and for the  $\omega$  dependence of  $I_2$  noted above (9). It is also responsible for the exponent identity  $z = 2 - \eta$  for  $d < 2$ .

We now turn to the evaluation of  $I_2$  in (A1) in an external momentum  $\mathbf{q}$ . First, we evaluate the integral over  $q_2$  by the standard Feynman parameter method

$$I_2(q, 0) = \frac{u_0^2(N+2)\Gamma(2-d/2)}{18(4\pi)^{d/2}} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \int \frac{d^d q_1}{(2\pi)^d} \int_0^1 dx \frac{1}{(q_1^2 + |\omega_1|)} \times \frac{1}{[(\mathbf{q} + \mathbf{q}_1)^2 x(1-x) + |\omega_2|(1-x) + |\omega_1 + \omega_2|x]^{2-d/2}}. \quad (\text{A2})$$

Similarly, performing the integral over  $q_1$  with a Feynman parameter  $y$  we obtain

$$I_2(q, 0) = \frac{u_0^2(N+2)\Gamma(3-d)}{18(4\pi)^d} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \int_0^1 dx [x(1-x)]^{\epsilon/2} \int_0^1 dy y^{\epsilon/2} \times \frac{1}{[q^2 x(1-x)y(1-y) + |\omega_1 + \omega_2|x(1-x)(1-y) + |\omega_1|y(1-x) + |\omega_2|xy]^{1+\epsilon}} \quad (\text{A3})$$

Now we perform the integral over  $\omega_1$  and  $\omega_2$  by using the useful formula

$$\int_{-\infty}^{\infty} d\omega_1 d\omega_2 \frac{1}{[A + B|\omega_1| + C|\omega_2| + D|\omega_1 + \omega_2|]^\sigma} = \frac{4A^{2-\sigma}(B+C+D)\Gamma(\sigma-2)}{(B+C)(C+D)(B+D)\Gamma(\sigma)}. \quad (\text{A4})$$

The formula (A4) is derived by explicitly performing the integrals over  $\omega_{1,2}$  over different regions in the  $\omega_{1,2}$  plane delineated by changes in signs of  $\omega_1$ ,  $\omega_2$ , and  $\omega_1 + \omega_2$ . Note that (A4) is to be evaluated at  $\sigma = 1 + \epsilon$ . Consequently, we obtain from (A4) the factor  $\Gamma(\sigma - 2) = \Gamma(-1 + \epsilon) = -1/\epsilon + \mathcal{O}(\epsilon^0)$ , which gives us the requisite pole in  $\epsilon$  appearing on the r.h.s. of (A1). So we may safely set  $d = 2$  in the remaining terms in (A3). In this manner, combining (A1), (A3) and (A4) we obtain

$$c_2 = \frac{(N+2)}{36} \int_0^1 dx \int_0^1 dy \frac{(1-y)[x(1-x)(1-y) + y(1-x) + xy]}{[x(1-y) + y][(1-x)(1-y) + y]} = \frac{(N+2)(12 - \pi^2)}{144}. \quad (\text{A5})$$

Let us now present a few details of the evaluation of  $I_2(0, i\omega)$ . In this case, (A3) is replaced by

$$I_2(0, i\omega) = \frac{u_0^2(N+2)\Gamma(3-d)}{18(4\pi)^d} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \int_0^1 dx [x(1-x)]^{\epsilon/2} \int_0^1 dy y^{\epsilon/2} \times \frac{1}{[|\omega + \omega_1 + \omega_2|x(1-x)(1-y) + |\omega_1|y(1-x) + |\omega_2|xy]^{1+\epsilon}}. \quad (\text{A6})$$

Now, instead of (A4), we need the following integral

$$\int_{-\infty}^{\infty} d\omega_1 d\omega_2 \frac{1}{[A|\omega_1| + B|\omega_2| + C|\omega + \omega_1 + \omega_2|]^\sigma} = \frac{4|\omega|^{2-\sigma} ABC \Gamma(\sigma-2)}{\Gamma(\sigma)} \times \left[ \frac{A^{1-\sigma}}{(A^2 - B^2)(A^2 - C^2)} + \frac{B^{1-\sigma}}{(B^2 - A^2)(B^2 - C^2)} + \frac{C^{1-\sigma}}{(C^2 - A^2)(C^2 - B^2)} \right]. \quad (\text{A7})$$

As below (A4), we need to evaluate (A7) at  $\sigma = 1 + \epsilon$  and pick out a possible pole in  $\epsilon$ . Indeed, a possible pole does appear to be present in the  $\Gamma(\sigma - 2)$  pre-factor in (A7). However, careful evaluation shows that the residue of such a pole vanishes, and (A7) in fact has a smooth  $\sigma \rightarrow 1$  limit:

$$\lim_{\sigma \rightarrow 1} (\text{A7}) = 4ABC|\omega| \left[ \frac{\ln A}{(A^2 - B^2)(A^2 - C^2)} + \frac{\ln B}{(B^2 - A^2)(B^2 - C^2)} + \frac{\ln C}{(C^2 - A^2)(C^2 - B^2)} \right] \quad (\text{A8})$$

The  $\epsilon \rightarrow 0$  limit of the remaining terms in (A6) is straightforward, and this establishes the claimed absence of a  $|\omega| \ln |\omega|$  term in  $I_2(0, i\omega)$  in  $d = 2$ . Evaluating the constant  $c$  in  $I_2(0, i\omega) = I_2(0, 0) - cg^2|\omega|$  we obtain:

$$c = \frac{(N+2)}{36} \mathcal{P} \int_0^1 dx \int_0^1 dy \frac{x^2(1-y)}{(1-2x)[y^2 - x^2(1-y)^2]} \ln \left( x \frac{1-y}{y} \right) = \frac{(N+2)(6\pi^2 \ln 2 - 11\zeta(3))}{1728} \quad (\text{A9})$$

We performed integrations in Eq.(A9) by changing variables to  $x' = x$ ,  $y' = x(1-y)/y$  and integrating first over  $x'$  and then over  $y'$ .

## APPENDIX B: GAP EQUATION

In this appendix we derive the gap equation Eq.(24). We need to evaluate:

$$\Sigma_0(T) = -uT \sum_{\omega_n} \int \frac{qdq}{2\pi} D(q, i\omega) = \frac{1}{4\pi} uT \sum_{\omega_n} \ln \frac{|\omega_n| + \Delta}{|\omega_n| + \Delta + \Lambda_{q^2}} \quad (\text{B1})$$

We consider a  $q^2$  cutoff  $\Lambda_{q^2}^2$ , which from now on we denote as  $\Lambda$ , and a frequency cutoff  $\Lambda_\omega \in 2\pi nT$  which is to be taken to infinity. For this choice of the sharp cutoff  $\Lambda_\omega$  one can show:

$$S_1(T) = \sum_{-\Lambda_\omega < \omega_n < \Lambda_\omega} \ln \frac{|\omega_n| + \Delta}{|\omega_n| + \Delta + \Lambda_{q^2}} = \ln \frac{\tilde{\Delta} + \tilde{\Lambda}}{\tilde{\Delta}} + 2 \ln \frac{\Gamma(\tilde{\Delta} + \tilde{\Lambda})\Gamma(\tilde{\Delta} + \tilde{\Lambda}_\omega)}{\Gamma(\tilde{\Delta})\Gamma(\tilde{\Delta} + \tilde{\Lambda} + \tilde{\Lambda}_\omega)} \quad (\text{B2})$$

where  $\tilde{\Lambda} = \Lambda/(2\pi T)$ ,  $\tilde{\Lambda}_\omega = \Lambda_\omega/(2\pi T)$  and  $\tilde{\Delta} = \Delta/(2\pi T)$ . Taking the limit  $\tilde{\Lambda}_\omega \rightarrow \infty$  we have:

$$S_1(T) = \ln \frac{\tilde{\Delta} + \tilde{\Lambda}}{\tilde{\Delta}} + 2 \ln \frac{\Gamma(\tilde{\Delta} + \tilde{\Lambda})}{\Gamma(\tilde{\Delta})} - 2\tilde{\Lambda} \ln \tilde{\Lambda}_\omega \quad (\text{B3})$$

The finite  $T$  self energy is  $\Sigma_0(T) = \frac{1}{4\pi} uT S_1(T)$ , and the zero temperature  $\Sigma_0(T=0)$  is:

$$\Sigma_0(T=0) = \frac{1}{4\pi} uT S_1(T) \Big|_{T \rightarrow 0} = \frac{u\Lambda}{(2\pi)^2} \left( \ln \frac{\Lambda}{\Lambda_\omega} - 1 \right) \quad (\text{B4})$$

Noticing that  $\Delta(T) = \Sigma_0(T=0) - \Sigma_0(T)$  we write Eq.(24).

## APPENDIX C: POLARIZATION BUBBLE $\Pi_0$

Here we show in detail how we compute  $\Pi_0(q, i\omega)$  in  $d=2$  with  $z=2$ . We use a finite cutoff  $\Lambda_{q^2} \equiv \Lambda$  for  $q^2$  in a momentum integration, and we use  $\Lambda_\omega \rightarrow \infty$  as a frequency cutoff. We use the Feynman parameterization to find:

$$\begin{aligned} \Pi_0(q, i\omega) &= \int \frac{d^2p}{(2\pi)^2} \frac{d\nu}{2\pi} D(p, i\nu) D(p+q, i\nu+i\omega) \\ &= \frac{1}{4\pi} \int_0^1 dx \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} \left( l(x, q, \nu, i\omega)^{-1} - [\Lambda + l(x, q, \nu, i\omega)]^{-1} \right) \end{aligned} \quad (\text{C1})$$

where  $l(x, q, \nu, i\omega) = x(1-x)q^2 + x|\nu| + (1-x)|\nu + \omega|$  and  $D^{-1} = |\omega| + q^2$  at the QCP. Integrating out frequency we get:

$$\begin{aligned} \Pi_0(q, i\omega) &= \frac{1}{8\pi^2} \mathcal{P} \int_0^1 dx \frac{4x}{1-2x} \left\{ \ln \left[ (1-x)(q^2x + |\omega|) \right] \right. \\ &\quad \left. - \ln \left[ \left( \frac{1}{2q^2}(q^2 - |\omega| + s) - x \right) \left( q^2x + \frac{1}{2}(|\omega| - q^2 + s) \right) \right] \right\} \end{aligned} \quad (\text{C2})$$

where  $s = \sqrt{(|\omega| + q^2)^2 + 4q^2\Lambda}$ . It is easy to show:

$$\mathcal{P} \int_0^1 dx \frac{4x}{1-2x} \ln(ax+b) = 2 - 2\frac{a+b}{a} \ln \frac{a+b}{b} - 2 \ln b - \text{Li}_2\left(\frac{a}{a+2b}\right) + \text{Li}_2\left(-\frac{a}{a+2b}\right) \quad (\text{C3})$$

for  $a+b > 0$  and  $b > 0$ . From Eq.(C2) and Eq.(C3), after simple algebra, Eq.(27) follows.

<sup>1</sup> For a recent review, see: G. R. Stewart, *Rev. Mod. Phys.* **73**, 797-855 (2001).

<sup>2</sup> H. von Löhneysen et al. *Phys. Rev. Lett.* **72**, 1994, 3262.

<sup>3</sup> A. Schroder et al., *Nature*, **407**, 351-355 (2000).

- <sup>4</sup> O. Tjorveit et al., *Phys. Rev. Lett.* **85**, 626 (2000).
- <sup>5</sup> A. Rosch, A. Schröder, O. Stockert, and H. v. Löhneysen, *Phys. Rev. Lett.* **79**, 159-162 (1997).
- <sup>6</sup> A. Schröder, G. Aeppli, E. Bucher, R. Ramazashvili and P. Coleman *Phys. Rev. Lett.* **80**, 5623 (1998).
- <sup>7</sup> O. Stockert, H. von Löhneysen, A. Rosch, N. Pyka and M. Loewenhaupt *Phys. Rev. Lett.* **80**, 5627 (1998).
- <sup>8</sup> J. Hertz. *Phys. Rev. B* **14** 1165 (1976).
- <sup>9</sup> T. Moriya and J. Kawabata *J. Phys. Soc. Japan* **34**, 639 (1973); **35**,669 (1973).
- <sup>10</sup> A.J. Millis, *Phys. Rev. B* **48**, 7183 (1993).
- <sup>11</sup> For a review, see: S. Sachdev "Quantum Phase Transitions" (Cambridge University Press, 1999).
- <sup>12</sup> I. Paul and G. Kotliar, *Phys. Rev. B* **64**, 184414 (2001).
- <sup>13</sup> A. Abanov and A. Chubukov, *Phys. Rev. Lett.* **84**, 5608 (2000).
- <sup>14</sup> Q. Si, S. Rabello, K. Ingersent, and L. Smith, *Nature* **413**, 804 (2001).
- <sup>15</sup> Q. Si and J. L. Smith, *Phys. Rev. Lett* **77**, 3391 (1996); *Phys. Rev. B* **61**, 5184 (2000)
- <sup>16</sup> H. Kajueter and G. Kotliar, *unpublished* (1995); H. Kajueter, *PhD thesis, Rutgers University*
- <sup>17</sup> A.M. Sengupta and A. Georges *Phys. Rev. B* **52**, 10295 (1995).
- <sup>18</sup> P. Coleman and C. Pépin, *cond-mat/0301001*
- <sup>19</sup> D.R. Grempel and Q. Si, *cond-mat/0207493*
- <sup>20</sup> S. Pankov, G. Kotliar, and Y. Motome, *Phys. Rev. B* **66**, 045117 (2002).
- <sup>21</sup> See e.g J. Zinn-Justin *Quantum Field and Critical Phenomena*, Clarendon Press (Oxford), 1989 (in particular Chap. 22).
- <sup>22</sup> A. Aharony, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1976), Vol. 6.
- <sup>23</sup> D. Boyanovsky and J. L. Cardy, *Phys. Rev. B* **27**, 5557 (1983).
- <sup>24</sup> S. Sachdev, *Phys.Rev.B* **55**, 142 (1997).
- <sup>25</sup> S. Sachdev, A.V. Chubukov and A. Sokol *Phys.Rev.B* **51**, 14874 (1995)
- <sup>26</sup> P.W Anderson, G. Yuval and D. R. Hamann, *Phys. Rev. B* **1**, 4464 (1970).
- <sup>27</sup> S. Chakravarty and J. Rudnick, *Phys. Rev. Lett.* **75** 501 (1995).
- <sup>28</sup> G. Kotliar and Q. Si, *Phys. Rev. B* **53**, 12373 (1996).
- <sup>29</sup> D.R. Grempel and M. J. Rozenberg, *Phys. Rev. B* **60** 4702 (1999).
- <sup>30</sup> It is not yet clear at this stage whether the EDMFT approach to the antiferromagnetic Kondo lattice in the non-random case follows the conventional Hertz-Millis behaviour, or undergoes a first-order transition from the Fermi liquid to the ordered phase. A Large-N study for an  $SU(N)$ -symmetric exchange coupling (in contrast to the Ising case studied here) provides indications in favor of the latter possibility<sup>34</sup>.
- <sup>31</sup> P. Sun and G. Kotliar, *cond-mat/0303539*.
- <sup>32</sup> K. Ishida et al. *Phys. Rev. Lett.* **89**, 107202 (2002).
- <sup>33</sup> R. E. Walstedt et al. *Phys. Rev. Lett.* **90**, 067601 (2003).
- <sup>34</sup> S. Burdin, M. Grilli and D. Grempel, *cond-mat/0206174*.