NATURE OF THE DISORDERED PHASE OF LOW-DIMENSIONAL QUANTUM ANTIFERROMAGNETS

Subir Sachdev
Center for Theoretical Physics, P.O. Box 6666
Yale University, New Haven, CT 06511
U.S.A.

Lectures presented at the winter school on 'Electronic Correlation and Disorder Effects in Metals', Puri, India, January 1989, S. N. Behera ed., World Scientific, Singapore (1990)

ABSTRACT

I review work done in collaboration with N. Read on the nature of the non-Néel phase of quantum antiferromagnets on bipartite lattices in one and two dimensions. I examine in detail the global phase diagram of a class of nearest-neighbor $SU(N)$ antiferromagnets all of which have a two-sublattice Néel state as their classical ground state. The non-Néel or resonating valence bond state is shown to display spin-Peierls or valence-bond-solid order whose characteristics vary periodically with "spin"; the periodicity is the co-ordination number of the lattice. Consequences of our results for frustrated $SU(2)$ antiferromagnets on a square lattice are discussed.

1. INTRODUCTION

The recent discovery of high temperature superconductivity in a class of cuprate compounds [1] has led to a resurgence of interest in the properties of the $SU(2)$ Heisenberg antiferromagnet on a square lattice [2]. By a judicious choice of exchange constants, it may be possible for this model to have a ground state without long-range Néel order. A complete understanding of the nature of these possible spin-disordered states is lacking, and reliable results on closely related but tractable models will be useful. In these lectures I will present some new results on the global phase diagram of a $SU(N)$ generalization of the nearest-neighbor Heisenberg antiferromagnet on one-dimensional chains and the two-dimensional square lattice. I will then argue that in $d = 2$, with some *reasonable additional assumptions*, the results have strong consequences for frustrated $SU(2)$ antiferromagnets on bipartite lattices which have a two-sublattice Néel state as their classical ground state.

Most of the following is an elaboration of results already presented in recent papers with N. Read [3, 4, 5, 6]. Some of the work in Section 3.3 was done in collaboration with R. Shankar.

The ultimate objective of these lectures is to understand the different types of possible ground states of the following $SU(2)$ spin Hamiltonian on a square lattice:

$$H = \sum_{i,j} J_{ij} \mathbf{S}(i) \cdot \mathbf{S}(j), \tag{1}$$

where the sum extends over all pairs of sites on the square lattice and the $\mathbf{S}$ are $SU(2)$ operators satisfying the commutation relation

$$[S_a, S_b] = i\epsilon_{abc} S_c. \tag{2}$$

The exchange constants J_{ij} are chosen to be invariant under the full symmetry group of the square lattice and the states at each site transform under a representation of $SU(2)$ with spin S. Strictly speaking, the results of our $SU(N)$ calculations can only be applied to models whose ground state in the classical ($S \to \infty$) limit is the two-sublattice Néel state. This limits us to J_{ij} obtained by *weakly* frustrating the nearest-neighbor antiferromagnet. Having chosen a set of J_{ij} which satisfy this criterion, we argue in Section 7 that H has the following types of ground states:

1. Néel State

This state always occurs for large enough S and has the same long range order present in the classical ground state:

$$\langle \mathbf{S}(i) \rangle = N \varepsilon_i \mathbf{n}, \tag{3}$$

where $\varepsilon_i = 1$ for $i \in A$ sublattice and $\varepsilon_i = -1$ for $i \in B$ sublattice, N is a positive constant, and $\mathbf{n}$ is a fixed unit vector. The low-lying excitations in this state are spin-waves with two polarizations and energy-momentum relation $\omega = ck$.

2. Disordered State

For smaller values of S, H may undergo a transition to a state with exponentially decaying *two-spin* correlation functions

$$\langle \mathbf{S}(i) \cdot \mathbf{S}(j) \rangle \sim \varepsilon_i \varepsilon_j \exp\left(- \frac{|\mathbf{r}_i - \mathbf{r}_j|}{\xi} \right), \tag{4}$$

where ξ is the spin correlation length. The surprising new feature of the disordered state is the presence of *spin-Peierls or valence-bond-solid* order. To describe the nature of this order we introduce the field $\mathcal{Q}$ on every link of the square lattice

$$\mathcal{Q}_{i,i+\hat{\eta}} = -\mathbf{S}(i) \cdot \mathbf{S}(i+\hat{\eta}), \tag{5}$$

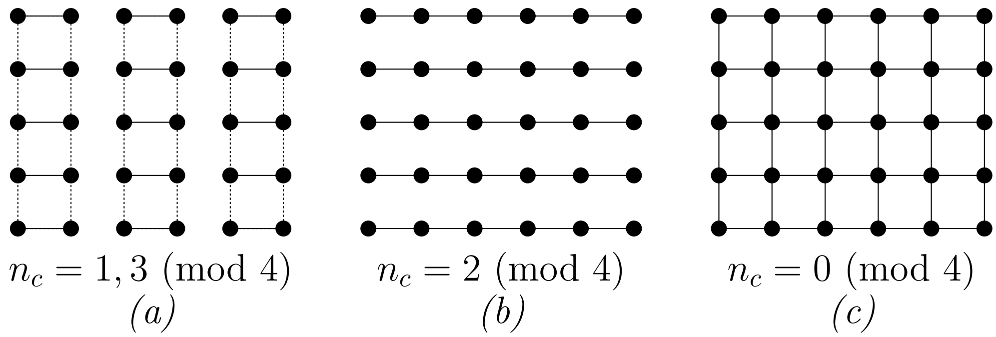


Figure 1: *Symmetry of non-Néel ground states of H as a function of $n_c \pmod 4$ with the minimum possible degeneracies of 4,2,1 respectively ($n_c = 2S$ for $SU(2)$).*

where $\hat{\eta}$ takes the values $\hat{x}$, $-\hat{x}$, $\hat{y}$ and $-\hat{y}$, and all hatted vectors are a lattice spacing in length. Except for the case $2S = 0 \pmod 4$, *the symmetry group of lattice rotations is spontaneously broken* and the values of $\langle \mathcal{Q}_{i,i+\hat{\eta}} \rangle$ depend upon the orientation and location of the link. States with the *minimum possible* ground state degeneracy are shown in Figs 1a-c. Solid lines denote larger values of $\langle \hat{S}(i) \cdot \hat{S}(i+1) \rangle$ for a link, dashed lines denote intermediate values, and no lines denote smaller values. Thus for $2S = 1, 3 \pmod 4$, the Z_4 lattice rotation symmetry is completely broken and there is a four-fold degeneracy in the ground state. For $2S = 2 \pmod 4$, the Z_4 symmetry is broken down to Z_2, and there is a two-fold ground state degeneracy. Only for $2S = 0 \pmod 4$ can the values of $\langle \mathcal{Q}_{i,i+\hat{\eta}} \rangle$ be independent of the link. The elementary excitations in this phase are spin-1/2 bose particles which are confined in pairs. There is also a spinless collective mode with a gap at all wavevectors. Ground states of H with breaking of *additional lattice symmetries are not ruled out.* We emphasize however, that for $2S \neq 0 \pmod 4$, our calculations do *not* display an intermediate phase with no broken lattice symmetry between the spin-Peierls and Néel states.

For many sets of J_{ij}, it is possible that the Néel state is the ground state for all values of S. It has in fact been shown [7] that for the case of a *nearest-neighbor* Heisenberg antiferromagnet on a d-dimensional hypercubic lattice, the ground state has Néel order for all S when $d \geq 3$ and for $S \geq 1$ when $d = 2$. The interesting case of $S = 1/2$ on a square lattice remains tantalizingly out of reach of these rigorous methods. Nevertheless, the consensus arising from many computer simulations [8] is that the ground state of this model has long-range Néel order. To obtain a non-Néel ground state for $S = 1/2$, it appears therefore that models with non-nearest neighbor interactions will have to be considered.

As noted earlier, most of this paper will present explicit calculations on a nearest neighbor $SU(N)$ antiferromagnet. The arguments necessary to apply the understanding gained from these calculations to non-nearest neighbor $SU(2)$ antiferromagnets will be presented in Section 7. We will begin in Section 2 by introducing the $SU(N)$ antiferromagnets and reviewing some elementary facts about the representations of $SU(N)$. An overview of the remaining calculations in this paper is best presented after the structure of the representations of $SU(N)$ is known; this summary is therefore postponed until Section 2.1.

2. DEFINITION OF THE MODELS

In this section we will introduce the $SU(N)$ antiferromagnet $\mathcal{H}$ (Eqn (27)), whose solution forms the bulk of these lectures. We will also review the theory of representations of $SU(N)$; in particular we will introduce a quantum number which generalizes the notion of "spin" to groups other than $SU(2)$. We begin by rewriting H in a way which makes the $SU(2)$ symmetry manifest. Introduce the 2×2 matrix of operators

$$\hat{S} = \frac{1}{2} \begin{pmatrix} S_z & S_x - iS_y \\ S_x + iS_y & -S_z \end{pmatrix}. \tag{6}$$

In terms of this operator, the Hamiltonian has an explicit $SU(2)$ invariance

$$H = \frac{1}{2} \sum_{(i,j)} \sum_{\alpha\beta} J_{ij} \hat{S}_\alpha^\beta(i) \hat{S}_\beta^\alpha(j), \tag{7}$$

where the $\hat{S}$ satisfy the commutation relation

$$[\hat{S}_\alpha^\beta, \hat{S}_\gamma^\delta] = \delta_\gamma^\beta \hat{S}_\alpha^\delta - \delta_\alpha^\delta \hat{S}_\beta^\gamma. \tag{8}$$

The generalization of these models to $SU(N)$ is now straightforward; the indices $\alpha\beta\ldots$ now run over $1 \ldots N$ and the commutation relations are otherwise unchanged. The reason for considering the $SU(N)$ models (as opposed to the $O(N)$ models for example) will become clear in the course of the lectures. We will find that all of the essential topological properties of the $SU(2)$ model are preserved by the $SU(N)$ models. In contrast the $O(N)$ models have some important topological properties which are special to the case $N = 3$.

We now show how the various representations of $SU(N)$ can be realized at each site by describing the states in terms of boson or fermion creation operators. Since all of the operations are carried out independently at each site, we will drop the site index for the remainder of this section.

We begin by reviewing the familiar case of $SU(2)$. The Schwinger boson method describes the $2S+1$ states of spin $S = n_c/2$ in terms of two bosons $b_\uparrow^\dagger$ and $b_\downarrow^\dagger$ with the restriction that there must be n_c bosons at each site. The operators

$$\hat{S}_\alpha^\beta = b_\alpha^\dagger b^\beta - \frac{n_c}{2} \delta_\alpha^\beta, \tag{9}$$

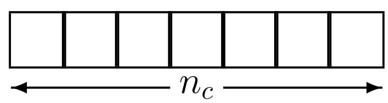
with α, β running over $\uparrow,\downarrow$ satisfy the $SU(2)$ commutation relations, and the states

$$|\alpha\beta\ldots\rangle = b^\dagger_\alpha b^\dagger_\beta \cdots |0\rangle, \tag{10}$$

where the dots denote n_c boson creation operators, span the Hilbert space of the representation with spin S. This method of generating representations has a simple generalization to $SU(N)$. We simply allow α, β to now extend over the indices $1 \ldots N$ and maintain the boson occupation number restriction

$$\sum_{\alpha=1}^{N} b^\dagger_\alpha b^\alpha = n_c. \tag{11}$$

The states $|\alpha\beta\ldots\rangle$ now transform according to the $SU(N)$ representation which is a totally symmetric product of n_c fundamental representations and is denoted by a Young tableau with one row of n_c boxes.



An important point is that the *same* representation can also be described in terms of fermionic operators. It becomes necessary, however, to introduce a secondary 'color' or 'orbital' index to generate representations with $n_c > 1$. The principle behind this method is quite familiar in the context of Hund's rule of atomic physics. An atom like Mn has 5 $3d$ electrons in a state of total spin $5/2$; the wavefunction achieves this by keeping the orbital wavefunction totally antisymmetric (and hence minimizing Coulomb repulsion). By the Pauli principle, the spin wavefunction then has to be totally symmetric. So we introduce fermion creation operators $c^\dagger_{\alpha a}$ where the spin index α runs through $1 \ldots N$ and the color (orbital) index runs from 1 to n_c $(2S)$. The spin operators now take the form

$$\hat{S}^\beta_\alpha = \sum_{a=1}^{n_c} c^\dagger_{\alpha a} c^{\beta a} - \frac{n_c}{2}\delta^\beta_\alpha. \tag{12}$$

The states which are totally antisymmetric in color space (color singlets), and therefore totally symmetric in spin space are

$$|\alpha\beta\cdots\rangle = \epsilon^{ab\cdots} c^\dagger_{\alpha a} c^\dagger_{\beta b} \cdots |0\rangle, \tag{13}$$

where each state has n_c fermions and $\epsilon^{ab\cdots}$ is the totally antisymmetric tensor with n_c indices. These states are completely equivalent to those introduced in Eqn (10). It is easy to check that the operator identity

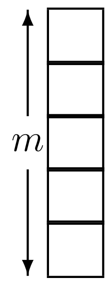
$$\sum_{\alpha=1}^{N} c^\dagger_{\alpha a} c^{\alpha b} = \delta^b_a \tag{14}$$

is satisfied when both sides of the equation act upon the states in Eqn (13).

The fermion operators also generate new representations of $SU(N)$ which exist only for $N > 2$. Consider the states

$$|\alpha\beta\ldots\rangle = c^\dagger_\alpha c^\dagger_\beta \cdots |0\rangle, \tag{15}$$

where $\alpha, \beta \ldots$ are $SU(N)$ indices and there are m fermion creation operators. These states transform according to a totally *antisymmetric* representation of $SU(N)$ which is represented by a Young tableau with 1 column of m boxes.



It is clear from the Pauli principle that $m \leq N$. For $SU(2)$, $m = 1$ is the usual spin-1/2 representation while $m = 2$ is a singlet with spin 0 and therefore not of any interest in lattice models. It should not come as any surprise that these totally antisymmetric representations can also be generated by boson creation operators after the introduction of a subsidiary color index. We use the operator representation

$$\hat{S}^\beta_\alpha = \sum_{a=1}^{m} b^\dagger_{\alpha a} b^{\beta a} - \frac{1}{2}\delta^\beta_\alpha, \tag{16}$$

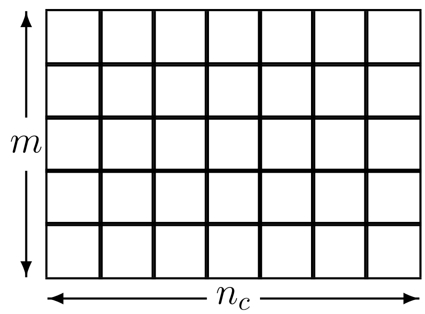
and the following states, which are color singlets,

$$|\alpha\beta\cdots\rangle = \epsilon^{ab\cdots} b^\dagger_{\alpha a} b^\dagger_{\beta b} \cdots |0\rangle, \tag{17}$$

with m bosons per site. These states therefore transform according to the representation with 1 column of m boxes. The restriction upon the Hilbert space can also be written in the form

$$\sum_{\alpha=1}^{N} b^\dagger_{\alpha a} b^{\alpha b} = \delta^b_a. \tag{18}$$

Finally we will also find it useful to consider representations with a mixed symmetry, denoted by a Young tableau with m rows of n_c boxes.



The arguments for generating these representations closely parallel the ones presented above and we simply state the results.

(i) Fermionic Operators

We use the operators

$$\hat{S}_{\alpha}^{\beta} = \sum_{a=1}^{n_c} c_{\alpha a}^{\dagger} c^{\beta a} - \frac{n_c}{2} \delta_{\alpha}^{\beta}, \tag{19}$$

where $\alpha\beta \in 1 \ldots N$, and $a \in 1 \ldots n_c$. We introduce the n_c fold *symmetric* tensor operators

$$T_{\alpha\beta\ldots} = \epsilon^{ab\cdots} c_{\alpha a}^{\dagger} c_{\beta b}^{\dagger} \cdots, \tag{20}$$

where there are n_c fermion creation operators on the RHS. The states at each site can then be written in the form

$$|\alpha\beta\cdots;\gamma\delta\cdots;\cdots\rangle = T_{\alpha\beta\ldots} T_{\gamma\delta\ldots} \cdots |0\rangle, \tag{21}$$

where there are m copies of the T operators. Equivalently the restriction on the Hilbert space is

$$\sum_{\alpha=1}^{N} c_{\alpha a}^{\dagger} c^{\alpha b} = m \delta_{a}^{b}. \tag{22}$$

(ii) Bosonic Operators

We use the operators

$$\hat{S}_{\alpha}^{\beta} = \sum_{a=1}^{m} b_{\alpha a}^{\dagger} b^{\beta a} - \frac{n_c}{2} \delta_{\alpha}^{\beta}, \tag{23}$$

where $\alpha\beta \in 1 \ldots N$, and $a \in 1 \ldots m$. We introduce the m fold *antisymmetric* tensor operators

$$A_{\alpha\beta\ldots} = \epsilon^{ab\cdots} b_{\alpha a}^{\dagger} b_{\beta b}^{\dagger} \cdots, \tag{24}$$

where there are m boson creation operators on the RHS. The states at each site can then be written in the form

$$|\alpha\beta\cdots;\gamma\delta\cdots;\cdots\rangle = A_{\alpha\beta\ldots} A_{\gamma\delta\ldots} \cdots |0\rangle, \tag{25}$$

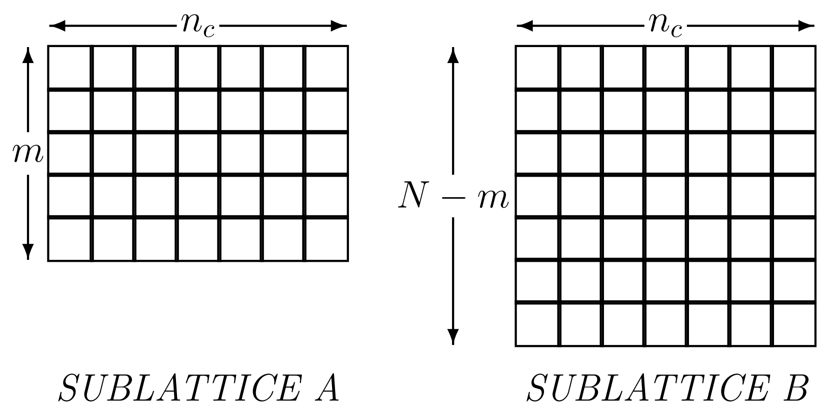


Figure 2: *Young tableau of the $SU(N)$ representations on sublattice A and B respectively*

where there are n_c copies of the A operators. Equivalently the restriction on the Hilbert space is

$$\sum_{\alpha=1}^{N} b^{\dagger}_{\alpha a} b^{\alpha b} = n_c \delta_a^b. \tag{26}$$

This completes our discussion of the classification of the states on a single site.

We are now finally in a position to present the Hamiltonian $\mathcal{H}$ which will considered in the remainder of these lectures:

$$\mathcal{H} = \frac{J}{N} \sum_{<ij>} \sum_{\alpha\beta} \hat{S}_\alpha^\beta(i) \hat{S}_\beta^\alpha(j). \tag{27}$$

The sum over i, j extends over all nearest-neighbor pairs on a chain in $d = 1$ and on the square lattice in $d = 2$. To obtain the correct generalization of the antiferromagnet to $SU(N)$, sites on the two sublattices have to transform as *conjugate* representations of $SU(N)$ *i.e.* we will take the "spins" on sites on sublattice A to transform as a Young tableau with m rows of n_c boxes and sites on sublattice B transform as the Young tableau with $N - m$ rows and n_c boxes (the representations with $m = N/2$ are clearly self-conjugate). The states on sublattice A can be realized by using either bosonic or fermionic operators as discussed above. It is most convenient to generate the states on sublattice B by a generalized particle-hole transformation, *i.e.* with fermionic operators we shall use

$$-\hat{S}_\alpha^\beta = \sum_{a=1}^{n_c} \bar{c}^{\beta a \dagger} \bar{c}_{\alpha a} - \frac{n_c}{2} \delta_\alpha^\beta, \tag{28}$$

with the constraints identical to those in Eqn (22) while with the bosonic operators we use

$$-\hat{S}_\alpha^\beta = \sum_{a=1}^{m} \bar{b}^{\beta a \dagger} \bar{b}_{\alpha a} - \frac{n_c}{2} \delta_\alpha^\beta, \tag{29}$$

with constraints identical to those in Eqn (26).

It is instructive at this juncture to consider the *two site* $SU(N)$ antiferromagnet with the states transforming under the Young tableau with m rows and n_c columns on the first site and the conjugate representation on the second site. The ground state is the unique singlet that can be written down by contracting lower and upper $SU(N)$ indices. We consider first the case $n_c = 1$. The ground state is simplest in terms of the fermionic operators:

$$|G(n_c = 1, m)\rangle = \left(\sum_{\alpha=1}^{N} c_{\alpha}^{\dagger} \bar{c}^{\alpha\dagger} \right)^{m} |0\rangle. \tag{30}$$

For the case $m = 1$ we use the bosonic operators

$$|G(n_c, m = 1)\rangle = \left(\sum_{\alpha=1}^{N} b_{\alpha}^{\dagger} \bar{b}^{\alpha\dagger} \right)^{n_c} |0\rangle. \tag{31}$$

Finally for general m,n_c we may use either

$$|G(n_c, m)\rangle = \left(\sum_{\alpha\beta\cdots=1}^{N} T_{\alpha\beta\ldots} \bar{T}^{\alpha\beta\cdots} \right)^{m} |0\rangle, \tag{32}$$

or the equivalent

$$|G(n_c, m)\rangle = \left(\sum_{\alpha\beta\cdots=1}^{N} A_{\alpha\beta\ldots} \bar{A}^{\alpha\beta\cdots} \right)^{n_c} |0\rangle. \tag{33}$$

The evaluation of the ground state energy is now straightforward though tedious; we find

$$E_{n_c,m} = -\frac{J n_c m}{N}(N - m + n_c) + \frac{J n_c^2}{4}\left(1 - \frac{4m}{N}\right). \tag{34}$$

It is gratifying to note that for $N = 2$, $m = 1$, $n_c = 2S$ this energy is $-JS(S + 1)$.

2.1 Overview of Remaining Sections

The bulk of the paper will now address the properties of the Hamiltonian $\mathcal{H}$ (Eqn (27)) as a function of n_c, m, and N using a variety of approximation schemes. We will find that the results are quite insensitive to the value of m, and can therefore be presented in the form of a phase diagram which is a function of n_c and N (Fig 3). We will begin in Section 3 by a discussion of the semiclassical limit ($n_c \to \infty$, with N, m, fixed). The ground state $\mathcal{H}$ in this limit is the Néel state. Semiclassical fluctuations about the classical state can be described by a $U(N)/(U(m) \times U(N - m))$ non-linear sigma model. This non-linear sigma model is shown to display a transition in a $d = 1 + \epsilon$ expansion to a disordered phase at a value of $n_c = \kappa N$; this determines the dashed line in Fig 3. A large-N analysis of the non-linear sigma model in the disordered phase will reveal an effective action consisting of N massive relativistic scalar

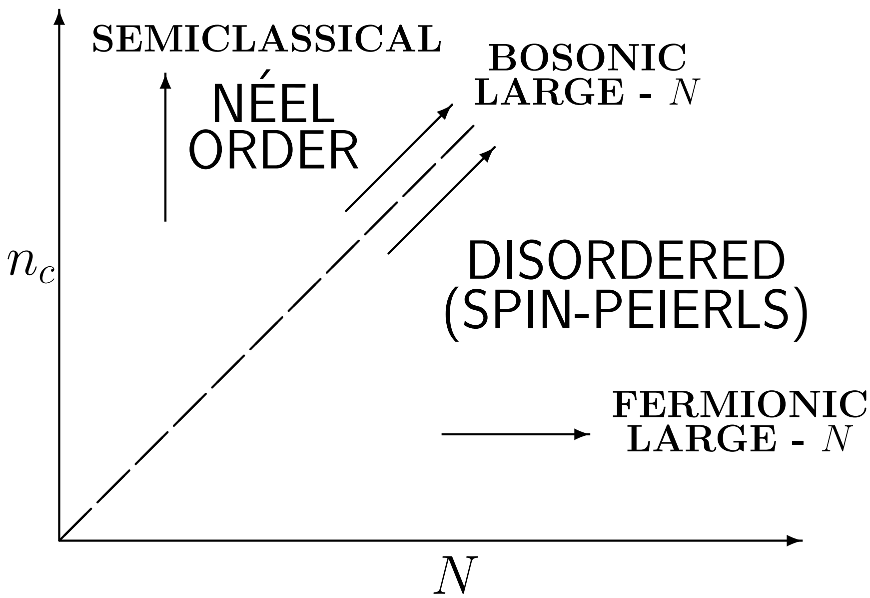


Figure 3: *Phase diagram of the square lattice $SU(N)$ antiferromagnet as a function of the "spin" n_c (= $2S$ for $SU(2)$).*

particles coupled to a $U(1)$ gauge field. We will also review in Section 3 the arguments of Haldane [9] and their generalization to $SU(N)$ [3], on the effect of 'hedgehog' tunneling events in the non-linear sigma model; these will be argued to predict a *minimum* degeneracy of the disordered phase of $1, 4, 2, 4$ for $n_c \pmod 4 = 0, 1, 2, 3$ respectively. Section 4 contains the most interesting results of these lectures. Using the bosonic representation of the $SU(N)$ operators, we take the limit $n_c \to \infty$, $N \to \infty$, n_c/N and m *fixed* at arbitrary values. We find a mean-field transition from a Néel state to a spin-disordered state, again determining a line $n_c = \kappa' N$ in the phase diagram (Fig 3). The $1/N$ fluctuations lead to an effective action involving N massive charged particles coupled to a *compact* $U(1)$ gauge field, very similar to that obtained in Section 3. The monopole-like instantons of the compact $U(1)$ gauge field are argued to be the remnants of the hedgehogs of the non-linear sigma model. The dynamics of these instantons is used to show that the nature of the disordered state is determined by $n_c \pmod 4$: for $n_c \pmod 4 = 0$ the ground state is a valence-bond-solid (Fig 1c) while for $n_c \pmod 4 \neq 0$, there is long range spin-Peierls order with a broken lattice rotational symmetry (Fig 1a,b). The degeneracies of these states are consistent with those obtained in Section 3. In Sections 5 and 6, we go deep in the disordered phase by considering the extreme quantum limit (n_c fixed, $N \to \infty$). In Section 5, we recapitulate results for the case $m = N/2$: the ground state is always a spin-Peierls state with the symmetry of Fig 1a, and the minimum degeneracy of the low-lying states is the same function of $n_c \pmod 4$ obtained in Section 3. In Section 6, we consider the case $m = 1$. The $1/N$ fluctuations are shown to be described by a quantum dimer model: numerical results on the quantum dimer model for $n_c = 1$ have shown that it has a spin-Peierls ground state of the type shown in Fig 1a. Finally, Section 7 presents a synthesis of the results obtained and discusses consequences for *weakly* frustrated $SU(2)$ antiferromagnets.

3. SEMICLASSICAL THEORY

In this section we shall examine the most familiar limit of the antiferromagnets: the semiclassical limit. This limit is obtained by sending $n_c \to \infty$ while keeping N and m fixed. We shall find that the quantum fluctuations about the classical ground state are described by a $U(N)/(U(m) \times U(N-m))$ non-linear sigma model [10]. We shall begin, in Section 3.1, with an exact path-integral representation of the partition function of $SU(2)$ quantum spins; the methods of this section are easily generalizable to arbitrary $SU(N)$ spins [3]. The semiclassical limit of this path integral will be used to derive the non-linear sigma model and associated Berry phase terms in Section 3.2. We will also discuss the relationship between the non-linear sigma models and the CP^{N-1} model. In Section 3.3, we will discuss the consequences of the Berry phases in $d = 1$ and $d = 2$ as can be deduced from the semiclassical analysis. Finally, in Section 3.4 we shall examine some features of the CP^{N-1} model which will be useful in the analysis of Section 4.

3.1 Path-Integral Representation of $SU(N)$ Spins

The standard method for deriving the path-integral representation of a quantum problem proceeds by deriving the coherent state representation of the Hilbert space [11]; we will follow this method here. For simplicity we will focus mostly on the case of $SU(2)$ spins and indicate the generalizations to $SU(N)$; further details of the general $SU(N)$ case can be found in Ref [3]. In this subsection we will make no explicit reference to the site index of the spin; all of the manipulations can be performed independently on each spin.

The Hilbert space of spin S is spanned by the $2S + 1$ eigenvectors of S_z: $|S, m\rangle$, $m = -S \ldots S$. The heighest "weight" state

$$|\Psi_0\rangle = |S, m = S\rangle \tag{35}$$

will be of particular use. The coherent states for spin S are defined as follows [12]

$$|\mathbf{n}\rangle = \exp\left(q\hat{S}_1^2 - q^*\hat{S}_2^1\right)|\Psi_0\rangle, \tag{36}$$

where $\mathbf{n}$ is a unit 3-vector pointing in the direction with spherical co-ordinates (θ, ϕ), $q = -(\theta/2)e^{-i\phi}$ is a complex number, and $\hat{S}_\alpha^\beta$ is the 2×2 matrix of operators introduced in Eqn

(6). The states $|\mathbf{n}\rangle$ are normalized to unity and obey the following important identity

$$\langle \mathbf{n}|\hat{S}_\alpha^\beta|\mathbf{n}\rangle = SQ_\alpha^\beta, \tag{37}$$

where the matrix Q is defined by

$$Q = U\Lambda U^\dagger \equiv \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}. \tag{38}$$

The unitary matrix U represents the action of the unitary transformation in Eqn (36) upon the fundamental representation and is given by

$$U = \exp\left[\begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}\right], \tag{39}$$

and the constant matrix Λ is

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{40}$$

The matrix Q therefore satisfies $Q^2 = 1$ and extends over the manifold $U(2)/(U(1) \times U(1))$ which is isomorphic to the surface of a sphere.

The standard method of coherent state quantization [11] may now be used to obtain the following representation for the partition function

$$Z = \int \mathcal{D}Q(\tau) \exp\left\{\int_0^\beta d\tau \left[\frac{\langle \mathbf{n}(\tau)|\mathbf{n}(\tau+\delta\tau)\rangle - 1}{\delta\tau} - H(Q(\tau))\right]\right\}, \tag{41}$$

where $H(Q)$ is obtained by replacing every occurrence of $\hat{S}_\alpha^\beta$ in the Hamiltonian by SQ_α^β, $Q(0) = Q(\tau)$, and $\mathcal{D}Q(\tau)$ is the invariant measure over the sphere. It now remains to evaluate the first term in the action, S_B, the Berry phase term. Using Eqn (36) and the following identity for the derivative of the exponential of any operator M [13]

$$\frac{d}{dx}e^M = \int_0^1 du e^{M(1-u)} \frac{dM}{dx} e^{Mu}, \tag{42}$$

we may easily show that

$$S_B = \int_0^\beta d\tau \int_0^1 du \langle \Psi_0| \exp\left(-u(q\hat{S}_1^2 - q^*\hat{S}_2^1)\right) \left(\frac{\partial q}{\partial \tau}\hat{S}_1^2 - \frac{\partial q^*}{\partial \tau}\hat{S}_2^1\right)$$
$$\exp\left(u(q\hat{S}_1^2 - q^*\hat{S}_2^1)\right)|\Psi_0\rangle. \tag{43}$$

Using the fundamental property of the coherent states in Eqn. (37), the above expression reduces immediately to

$$S_B = S\int_0^\beta d\tau \int_0^1 du \left[\frac{\partial q}{\partial \tau}Q_1^2(\tau,u) - \frac{\partial q^*}{\partial \tau}Q_2^1(\tau,u)\right], \tag{44}$$

where we have now introduced a u and τ dependent matrix Q which is defined as in Eqn (38) in terms of the unitary matrix $U(\tau, u)$,

$$U = \exp\left[u\begin{pmatrix} 0 & q(\tau) \\ -q^*(\tau) & 0 \end{pmatrix}\right]. \tag{45}$$

As a function of u , Q therefore satisfies $Q(\tau, 0) = \Lambda$ and $Q(\tau, 1) \equiv Q(\tau)$. We may now integrate Eqn (44) by parts and obtain the simple expression

$$S_B = -S\int_0^\beta d\tau \int_0^1 du \mathrm{Tr}\left[\begin{pmatrix} 0 & q(\tau) \\ -q^*(\tau) & 0 \end{pmatrix} \partial_\tau Q(\tau, u)\right]. \tag{46}$$

Using the easily established identity

$$\begin{pmatrix} 0 & q(\tau) \\ -q^*(\tau) & 0 \end{pmatrix} = -\frac{1}{2}Q(\tau, u)\frac{\partial Q(\tau, u)}{\partial u}, \tag{47}$$

we obtain our final result for the action, S_{act}, of the path integral

$$S_{act} = \int_0^\beta d\tau \int_0^1 du \left[\frac{S}{2}\mathrm{Tr}\left(Q(\tau, u)\frac{\partial Q(\tau, u)}{\partial u}\frac{\partial Q(\tau, u)}{\partial \tau}\right)\right] - \int_0^\beta d\tau H(Q(\tau)). \tag{48}$$

This expression can be generalized to $SU(N)$ spins transforming under a Young tableau with n_c columns and m rows: Q becomes a $N \times N$ matrix satisfying $Q^2 = 1$ and taking values over a $U(N)/((U(m) \times U(N-m))$ manifold *i.e.* Q satisfies Eqn (38) with U an arbitrary $N \times N$ unitary matrix and

$$\Lambda = \begin{pmatrix} 1_m & 0 \\ 0 & -1_{N-m} \end{pmatrix}, \tag{49}$$

where 1_r is a $r \times r$ unitary matrix. The constant S in Eqn (48) is replaced by $n_c/2$. A similar result has been quoted recently by Wiegmann [14].

Returning to the case of $SU(2)$, we may use the representation of Q in terms of $\mathbf{n}$ in Eqn (38) to obtain

$$S_{act} = iS\int_0^\beta d\tau \int_0^1 du \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial u} \times \frac{\partial \mathbf{n}}{\partial \tau}\right) - \int_0^\beta d\tau H(\mathbf{n}(\tau)). \tag{50}$$

The function $\mathbf{n}(\tau, u)$ satisfies $\mathbf{n}(\tau, u = 0) = (1, 0, 0)$, $\mathbf{n}(\tau, u = 1) = \mathbf{n}(\tau)$ and $\mathbf{n}(\tau = 0, u) = \mathbf{n}(\tau = \beta, u)$. We can thus visualize the u dependence of $\mathbf{n}(\tau, u)$ as a "string" connecting the physical spin direction $\mathbf{n}(\tau)$ at $u = 1$ to the North pole at $u = 0$. The term S_B is then easily seen to be the iS times the area swept out by this string on the surface of the sphere. This interpretation will be quite useful in visualizing the consequences of S_B for a lattice of spins.

3.2 Non-linear Sigma Model Representation

All of results in Section 3.1 were exact, and the functional integral over Eqn (50) in principle yields the exact partition function of the spins. In this section we shall make an approximate analysis of S_{act}. We note that the following discussions returns to general case of $SU(N)$ spins. In the large n_c (recall that $n_c = 2S$ for $SU(2)$) limit, the antiferromagnet behaves semiclassically, and we may decompose the Q field fluctuations into staggered and uniform components

$$Q(i) \approx \varepsilon_i \Omega_i \sqrt{1 - a^2 L_i^2} + a L_i, \tag{51}$$

where ε_i equals $+1$ on sublattice A and -1 on sublattice B, a is the lattice spacing, $\Omega_i^2 = 1$, and L_i is small and satisfies $L_i \Omega_i + \Omega_i L_i = 0$. Substituting this into Eqn (48), dropping total time derivatives and taking the continuum limit following Refs. [15] and [10], we obtain

$$S_{act} = S_B' + \frac{1}{2} \int_0^\beta d\tau \int d^d\mathbf{r} \ \mathrm{Tr} \left[\frac{J n_c^2}{4N} (\nabla_{\mathbf{r}} \Omega)^2 + \frac{d J n_c^2}{N} L^2 - \frac{n_c}{2a} L \Omega \partial_\tau \Omega \right], \tag{52}$$

where

$$S_B' = i n_c \sum_j \varepsilon_j \omega_j \tag{53}$$

is defined in terms of the spatial field ω_j

$$\omega_j = \frac{1}{4i} \int_0^\beta d\tau \int_0^1 du \mathrm{Tr}[\Omega_j \partial_u \Omega_j \partial_\tau \Omega_j]. \tag{54}$$

We may now integrate out the L fluctuations and obtain the action of a (d+1) non-linear sigma model with a residual Berry phase term

$$S_{act} = S_B' + \frac{1}{2} \int_0^\beta d\tau \int d^d\mathbf{r} \frac{\rho_s}{2} \ \mathrm{Tr} \left[(\nabla_{\mathbf{r}} \Omega)^2 + \frac{1}{c^2} (\partial_\tau \Omega)^2 \right], \tag{55}$$

where we have introduced the spin-wave stiffness $\rho_s = J n_c^2 / 2N$ and the spin-wave velocity $c = 2\sqrt{d} J n_c a / N$.

It useful at this point to specialize the following discussion to the case of $m = 1$; all of the following results hold for all values of m but the calculations are considerably more complicated and do not yield any additional insight [16]. In this case we can use the parametrization

$$(\Omega_j)_\alpha^\beta = -\delta_{\alpha\beta} + 2 z_\alpha^*(j) z^\beta(j), \tag{56}$$

where $z^\alpha(j)$ are N complex fields ($\alpha = 1, \cdots, N$) satisfying the constraint $\sum_\alpha |z^\alpha|^2 = 1$. Notice however the presence of a residual gauge-invariance: the transformation $z^\alpha(j) \to z^\alpha(j) e^{i\phi(j)}$ leaves the value of Ω_j unchanged. In terms of the z's, the action takes the form

$$S_{act} = S_B' + \frac{2}{g a^{d-1}} \int d^{d+1}x \left[|\partial_\mu z^\alpha|^2 - |z_\alpha^* \partial_\mu z^\alpha|^2 \right], \tag{57}$$

where x is the spacetime coordinate $(\mathbf{r}, \tilde{\tau})$, $\tilde{\tau} = c\tau$, $\tilde{\rho}_s = \rho_s/c$, μ extends over the $d+1$ coordinates $x, y \ldots \tilde{\tau}$, and $g = c/\rho_s$. Without the Berry phase term, S_{act} is the action for the well-known CP^{N-1} model. The Berry phase term S'_B still satisfies Eqn(53) but with

$$\omega_j = i \int_0^\beta d\tau \sum_\alpha z^*_\alpha(j) \frac{dz^\alpha(j)}{d\tau}. \tag{58}$$

For the subsequent analysis it is useful to make the gauge invariance of the CP^{N-1} model manifest:

$$Z = \int \mathcal{D}z^\alpha \mathcal{D}A_\mu \delta(|z^\alpha|^2 - 1) \exp(-S_z),$$
$$S_z = S'_B + \frac{2}{ga^{d-1}} \int d^{d+1}x |(\partial_\mu - iA_\mu) z^\alpha|^2, \tag{59}$$

where A_μ is an *independent* gauge field. The action is quadratic in A_μ and integrating out A_μ constrains

$$A_\mu = \frac{i}{2}(z^\alpha \partial_\mu z^*_\alpha - z^*_\alpha \partial_\mu z^\alpha), \tag{60}$$

and we regain the original CP^{N-1} action in Eqn (57). Under a gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \phi$; A_μ has therefore all the characteristics of a $U(1)$ gauge field.

Finally, to place everything in a familiar context, it is useful to rewrite these results for the special case of $SU(2)$. In this case we have the vector field $\mathbf{n}$

$$\mathbf{n}^i = z^*_\alpha \sigma^i_{\alpha\beta} z_\beta, \tag{61}$$

where σ^i are the Pauli matrices and $\mathbf{n}^2 = 1$; the field $\mathbf{n}$ is the usual Néel order parameter. The action now takes the form

$$S = S'_B + \frac{1}{2ga^{d-1}} \int d^{d+1}x (\partial_\mu \mathbf{n})^2 \tag{62}$$

with the Berry phase as in Eqn (53), but with

$$\omega_j = \frac{1}{2} \int_0^\beta d\tau \int_0^1 du \mathbf{n}_j \cdot \left(\frac{\partial \mathbf{n}_j}{\partial u} \times \frac{\partial \mathbf{n}_j}{\partial \tau} \right). \tag{63}$$

Before discussing topological effects of the residual Berry phase, we present results on the stability of the Néel phase. Following the analysis of Ref [17] and [18] we introduce the coupling constant $\tilde{g} = gk_M = 4\sqrt{2}k_M a/n_c$, where k_M is an upper cutoff in momentum space, and derive the following one-loop renormalization group equation in a $(d-1)$ expansion

$$\frac{d\tilde{g}}{dl} = -(d-1)\tilde{g} + \frac{N}{4} K_d \tilde{g}^2, \tag{64}$$

where e^l is the length rescaling factor, and $K_d = 1/(2^{d-1}\pi^{d/2}\Gamma(d/2))$ is a numerical constant.

In $d = 1$ the coupling $\tilde{g}$ always flows to strong coupling. This implies that the Néel phase is never stable. As we shall discuss in section 3.3.1, the nature of the disordered phase is strongly influenced by the Berry phase terms.

In $d = 2$, the RG equation predicts that the Néel phase will be stable provided $g < 4/(NK_d)$ or

$$n_c > (K_d \sqrt{2}\pi k_M a)N. \tag{65}$$

This determines a line of second-order transitions in the (N, n_c) plane across which the Néel phases transforms into a phase with exponentially decaying spin correlations. This transition is represented by the dashed line in Fig 3. Note that at one-loop order the position of the line is independent of m. As an alternative to the $(d-1)$ expansion we may examine the non-linear sigma model field theory in the large N limit. Taking the limit with m and Ng fixed (implying $n_c \sim N$), the renormalization group equation (64) is in fact exact. For sufficiently large N, therefore, the statements of this paragraph have a validity beyond a $(d-1)$ expansion.

3.3 Topological Berry Phases

We now turn to a discussion of the effects of the Berry phase term S'_B. We discuss the cases $d = 1, 2$ in turn.

3.3.1 One dimension

Assuming a slowly varying $z^\alpha(j)$, S'_B can be easily shown to be

$$S'_B = \frac{\Theta}{2\pi} \int dx \int d\tau \epsilon_{\mu\nu} \partial_\mu z^*_\alpha \partial_\nu z^\alpha. \tag{66}$$

The integrand is the well-known Θ term [15], with $\Theta = \pi n_c$. With periodic boundary conditions the integrand can be shown to quantized in multiples of $2\pi i$ [19]. This quantization is perhaps more familiar in the $O(3)$ language whence

$$S'_B = \frac{iS}{2} \int dx \int d\tau \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial \tau} \times \frac{\partial \mathbf{n}}{\partial x} \right). \tag{67}$$

S'_B now measures the number of times the spacetime order-parameter configuration is wrapped around the sphere.

Very little is known directly about the non-linear sigma model with the Θ term. Assuming that the integrand is quantized, we may conclude that for *even* n_c, the topological term has no effect. For odd-values of n_c, we will show in Section 4.1 that for large N, the topological term implies that the ground state has *spin-Peierls* order. This spin-Peierls order is expected to persist down to $N = 3$ [3, 20]. For $N = 2$, a massless phase is expected to appear corresponding to the Bethe ansatz solution of the $S = 1/2$ chain [21].

The Θ term as written makes explicit reference to the field z_α (or $\mathbf{n}$); as we will see in Section 4, the gauge field A_μ becomes the more useful dynamical variable in the phase without long range order. In terms of A_μ,

$$S'_B = \frac{i\Theta}{2\pi} \int dx \int d\tau F_{x\tau}, \tag{68}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field. See the very clear discussion of Witten [22] on the differences between the two ways of writing down the Θ term.

3.3.2 Two dimensions

The spatial summation in S'_B has been shown to vanish [23] for any order parameter configuration which is smooth on the scale of the lattice spacing. Following Ref [9], it is therefore necessary to consider space-time singularities in the order parameter. These are the 'hedgehogs' characterized by the integer charge m

$$\begin{aligned} m &= \frac{i}{2\pi} \int_\Sigma dS_{\mu\nu} (\partial_\mu z^*_\alpha \partial_\nu z^\alpha - \partial_\nu z^*_\alpha \partial_\mu z^\alpha) \\ &= \frac{1}{2\pi} \int_\Sigma dS_{\mu\nu} F_{\mu\nu} \\ &= \frac{1}{4\pi} \int_\Sigma dS_{\mu\nu} \mathbf{n} \cdot (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}) \;\; \text{for } SU(2), \end{aligned} \tag{69}$$

where Σ is any surface surrounding the point $(\mathbf{R}, \tau)$ at which the hedgehog is centered: $\mathbf{R}$ is located at the center of a plaquette of the lattice of spins. The hedgehogs can be interpreted as instantons in tunneling events involving a change in the total 'skyrmion' number, Q, of the instantaneous spin configuration[24]:

$$\begin{aligned} Q &= \frac{i}{2\pi} \int dxdy (\partial_x z^*_\alpha \partial_y z^\alpha - \partial_y z^*_\alpha \partial_x z^\alpha) \\ &= \frac{1}{2\pi} \int dxdy F_{xy} \\ &= \frac{1}{4\pi} \int dxdy \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial x} \times \frac{\partial \mathbf{n}}{\partial y} \right) \qquad \text{for } SU(2). \end{aligned} \tag{70}$$

Therefore, for $SU(2)$ the skyrmion number is simply the number of times the spatial order-parameter configuration wraps around the sphere. In the language of the $U(1)$ gauge field, the skyrmion number is linked to the total 'magnetic flux' piercing the lattice and an instanton of charge m changes this flux by $2\pi m$. The Berry phase for such tunneling events has been evaluated in the ordered phase of the CP^{N-1} model using Eqn (58) [9, 3]; for the case of the

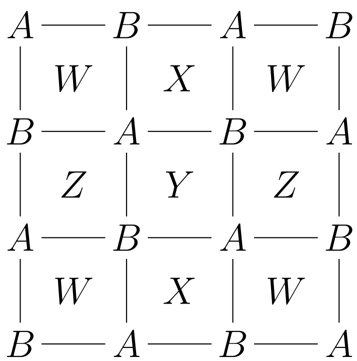


Figure 4: *The A, B sublattices of the lattice of spins and the sublattices W, X, Y, Z of the dual lattice.*

square lattice we find

$$S_B = \sum_s i \frac{n_c \pi}{2} \zeta_s m_s. \tag{71}$$

The hedgehogs are represented by integer charges m_s located at spacetime co-ordinates $(\mathbf{R}_s, \tau_s)$ and the integer $\zeta_s = 0, 1, 2, 3$ for $\mathbf{R}_s$ on four dual sublattices W, X, Y, Z (Fig 4). We note there is a gauge choice involved in specifying the values of ζ_s: a gauge transformation can rotate the values of ζ_s among the four dual sublattices. We emphasize that no symmetry has been broken by this gauge choice: the four sublattices are still completely equivalent.

We thus find the remarkable results that a tunneling event which changes the skyrmion number by m is associated with a phase factor

$$(\xi_s)^{n_c m}. \tag{72}$$

Here ξ_s is $+1$, -1, $+i$, or $-i$ depending upon whether the core of the hedgehog $\mathbf{R}_s$ is on sublattice $W, X, Y,$ or Z. For $n_c (\mathrm{mod}\, 4) = 1$ or 3, the phase factor ξ_j will lead to destructive interference between tunneling events except those involving a change in the skyrmion number m by a multiple of 4. This suggests that the Hilbert space of the system splits up into 4 separate sectors characterized by the number of skyrmions modulo 4, with vanishing tunneling matrix elements between the sectors. In the Néel phase there is a finite energy gap towards the creation of skyrmions, and therefore this argument only affects some high-lying states. In the disordered phase however, the skyrmions proliferate, and this argument suggests a *minimum* degeneracy of *all* low-lying states of 4. In a similar manner we can argue that, in the spin-disordered phase for $n_c (\mathrm{mod}\, 4) = 2$, all low-lying states have a degeneracy of at least 2. The degeneracy can be arbitrarily small for $n_c (\mathrm{mod}\, 4) = 0$.

3.4 Gauge Theory of the Disordered Phase of the CP^{N-1} Model

This subsection presents some results on the physics of the CP^{N-1} models which will be useful in subsequent section. D'Adda *et al.* [25] and Witten [22] have presented an analysis of the disordered phase of the CP^{N-1} model in $d = 1$ in which they emphasized the importance of the fluctuations of the $U(1)$ gauge field, A_μ. For completeness, we recall features of their results. We will also present the straightforward generalization of their results to $d = 2$ [26]. All the results in this subsection can be interpreted as the lowest non-trivial order in a $1/N$ expansion, provided n_c is of order N (using the expressions for ρ_s and c at the beginning of this section this implies $g = c/\rho_s$ is of order $1/N$).

We begin by expressing the constraint in Eqn(57) by a Lagrange multiplier field λ

$$Z = \int \mathcal{D}z^\alpha \mathcal{D}A_\mu \mathcal{D}\lambda \exp\left[-\frac{2}{ga^{d-1}} \int d^{d+1}x \left(|(\partial_\mu - iA_\mu)z^\alpha|^2 + i\lambda(|z^\alpha|^2 - 1) \right) \right. \\ \left. + \; S_B(A_\mu) \right]. \tag{73}$$

We have also indicated that the S_B term is to be evaluated using the A_μ dependent expressions in Section 3.3.2. Integrating out the z^α we find

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\lambda \exp\left[-NS_{eff}(A_\mu, \lambda) \; + \; S_B(A_\mu) \right], \tag{74}$$

where the effective action S_{eff} will be preceded by a factor N provided $g \sim 1/n_c \sim 1/N$. In the large N limit we may evaluate the partition function by expanding about a minimum of S_{eff}. We search for a minimum with $A_\mu = 0$ and $i\lambda = \Delta^2/c^2$ where Δ is a positive constant. Such a minimum always exists in $d = 1$ and for $g > g_c$ in $d = 2$ (g_c is a number of order $1/N$); for $g < g_c$ in $d = 2$, the CP^{N-1} model is in the Néel phase.

In the disordered phase, and at length scales larger than c/Δ, we may perform a gradient expansion of S_{eff} and obtain

$$Z = \int \mathcal{D}z^\alpha \mathcal{D}A_\mu \exp\left[-\int d^{d+1}x \left(\frac{2}{ga^{d-1}} \left\{ |(\partial_\mu - iA_\mu)z^\alpha|^2 + \frac{\Delta^2}{c^2}|z^\alpha|^2 \right\} \right.\right. \\ \left.\left. + \frac{N}{4e^2}F_{\mu\nu}^2 \right) \; + \; S_B(A_\mu) \right]. \tag{75}$$

We have reintroduced the z^α bosons to display their coupling to A_μ, neglected the massive λ fluctuations, and introduced the coupling constant $e^2 \sim (\Delta/c)^{3-d}$. Notice the dynamical generation of the A_μ kinetic energy and the absence of an explicit constraint on the magnitude of z^α. We postpone further analysis of this partition function to Section 4 where we will obtain a very similar result by performing a large-N (with n_c of order N) calculation *directly* on the spin Hamiltonian $\mathcal{H}$. The advantage of the latter procedure is that it retains the coupling to lattice spin-Peierls order parameter which has been lost in the continuum limit of this section.

We conclude this section with a reiteration of the main results established in $d = 2$. We have shown that in the semiclassical (large n_c) limit, the $SU(N)$ spin model is described by a non-linear sigma model. At a value of $n_c = \kappa N$ (where κ is a constant of order 1) the Néel phase undergoes a second order phase transition to a state with exponentially decaying spin correlations. We also showed the connection between the non-linear sigma models and the CP^{N-1} models; in the disordered phase, the CP^{N-1} models were shown to be equivalent at large-N to a theory of massive charged relativistic bosons coupled to a $U(1)$ gauge field. We evaluated the Berry phases of 'hedgehog' tunneling events; in the semiclassical theory, the Berry phases were argued to lead to a degeneracy of all low-lying states in the massive phase of at least 1,4,2, or 4, as $n_c (\bmod 4)$ took the values 0,1,2, or 3. The discussion of the effect of skyrmions is somewhat unsatisfactory to the extent that it uses the language of the ordered phase to describe a phase with short-range spin correlations. We will show explicitly in Section 4 that the topological effects *survive* the destruction of Néel order and are properly described in terms of the gauge field A_μ introduced earlier. We will also find that the ground state degeneracy is realized by a *broken lattice symmetry.*

4. BOSONIC LARGE-N ($n_c \propto N$)

In Section 3 we found in a semiclassical $d-1$ expansion that the boundary of stability of the Néel phase was given by the relation $n_c = \kappa N$. In this section we shall develop a large-N theory which investigates this transition in greater detail. We shall therefore examine the properties of $\mathcal{H}$ (Eqn (27)) by fixing n_c proportional to N and then taking the large N limit. This is most conveniently done by using the *bosonic* representation of the $SU(N)$ operators introduced in Section 2. For simplicity, we shall restrict our discussion in this section to $m = 1$; all of the calculations in this section can be generalized to arbitrary m and there is no essential change in the form of the results.

We recall the bosonic representation of the generators for $m = 1$

$$\hat{S}_\alpha^\beta(i) = b_\alpha^\dagger(i) b^\beta(i) - \frac{n_c}{2}\delta_\alpha^\beta, \qquad i \in A \text{ sublattice}, \tag{76}$$

and,

$$\hat{S}_\alpha^\beta(j) = -\bar{b}^{\beta\dagger}(j)\bar{b}_\alpha(j) + \frac{n_c}{2}\delta_\alpha^\beta, \qquad j \in B \text{ sublattice}, \tag{77}$$

where the $\bar{b}$ bosons are implied by the placement of indices to transform as the conjugate representation to b, which are in the fundamental representation of $SU(N)$. The constraints

$$b_\alpha^\dagger b^\alpha = n_c \qquad \bar{b}^{\alpha\dagger}\bar{b}_\alpha = n_c \tag{78}$$

are imposed at all sites. This representation has been used previously by Arovas and Auerbach [27] to obtain a $1/N$ expansion with $n_c \propto N$ in order to study mainly the Néel ordered phase.

We may represent the partition function of $\mathcal{H}$ by

$$Z = \int \mathcal{D}Q\mathcal{D}b\mathcal{D}\bar{b}\mathcal{D}\lambda \exp\left(-\int_0^\beta \mathcal{L}d\tau\right), \tag{79}$$

where

$$\mathcal{L} = \sum_{i\in A}\left[b_\alpha^\dagger(i)\left(\frac{d}{d\tau} + i\lambda(i)\right)b^\alpha(i) - i\lambda(i)n_c\right] + \sum_{j\in B}\left[\bar{b}^{\alpha\dagger}(j)\left(\frac{d}{d\tau} + i\lambda(j)\right)\bar{b}_\alpha(j)\right.$$
$$\left. - i\lambda(j)n_c\right] + \sum_{i\in A,\hat{\eta}}\left[\frac{N}{J}|Q_{i,i+\hat{\eta}}|^2 - Q_{i,i+\hat{\eta}}^* b^\alpha(i)\bar{b}_\alpha(i+\hat{\eta}) + h.c.\right].$$

Here the $\lambda(i)$ fix the boson number of n_c at each site; τ-dependence of all fields is implicit; Q was introduced by a Hubbard-Stratonovich decoupling of H, and $\hat{\eta}$ runs over nearest neighbor vectors and has length a. An important feature of the lagrangian $\mathcal{L}$ is its $U(1)$ gauge invariance under which

$$\begin{aligned}
b_\alpha^\dagger(i) &\rightarrow b_\alpha^\dagger(i)\exp\left(i\phi(i,\tau)\right)\\
\bar{b}^{\alpha\dagger}(j) &\rightarrow b^{\alpha\dagger}(j)\exp\left(i\phi(j,\tau)\right)\\
Q_{i,i+\hat{\eta}} &\rightarrow Q_{i,i+\hat{\eta}}\exp\left(-i\phi(i,\tau) - i\phi(i+\hat{\eta},\tau)\right)\\
\lambda(i) &\rightarrow \lambda(i) + \frac{\partial\phi}{\partial\tau}(i,\tau).
\end{aligned}$$

The functional integral over $\mathcal{L}$ faithfully represents the partition function as long as we fix a gauge, *e.g.* by the condition $d\lambda/d\tau = 0$ at all sites.

The $1/N$ expansion of the free energy can be obtained by integrating out of $\mathcal{L}$ the N-component b,$\bar{b}$ fields to leave an effective action for Q, λ having co-efficient N (since $n_c \propto N$). Thus the $N \rightarrow \infty$ limit is given by minimizing the effective action with respect to "mean-field" values of Q,λ. This is in turn equivalent to solving the mean-field Hamiltonian

$$H_{MF} = \sum_{i\in A,\hat{\eta}}\left(\frac{N|\bar{Q}|^2}{J} - \bar{Q}b^\alpha(i)\bar{b}_\alpha(i+\hat{\eta}) + h.c.\right) + \bar{\lambda}\sum_{i\in A}(b_\alpha^\dagger(i)b^\alpha(i) - n_c)$$
$$+ \bar{\lambda}\sum_{j\in B}(\bar{b}^{\alpha\dagger}(i)\bar{b}_\alpha(i) - n_c). \tag{80}$$

In writing H_{MF} we used the fact that $i\lambda(i) = \bar{\lambda}$ and $Q_{i,i+\hat{\eta}}$ are found to be uniform and independent of $\hat{\eta}$ at the saddle point. The constant $\bar{\lambda}$ is found to be real, and $\bar{Q}$ can be taken real, positive by a gauge transformation. The Hamiltonian H_{MF} can be diagonalized by Bogoluibov's method and we find two modes for each wavevector in the (reduced) Brillouin zone, of energy

$$\omega_{\mathbf{k}} = (\bar{\lambda}^2 - 4d^2\bar{Q}^2\gamma_{\mathbf{k}}^2)^{1/2}, \tag{81}$$

where

$$\begin{aligned} \gamma_{\mathbf{k}} &= (1/2d)\sum_{\hat{\eta}} e^{i\mathbf{k}\cdot\hat{\eta}} \\ &= \frac{1}{2}\left(\cos(k_x a) + \cos(k_y a)\right) \text{ in } d = 2 , \end{aligned} \tag{82}$$

and $\bar{\lambda} \sim \bar{Q} \sim J$. At $\mathbf{k} = 0$, $\omega_{\mathbf{k}} = \Delta = (\bar{\lambda}^2 - 4d^2\bar{Q}^2)^{1/2} \geq 0$ is the energy gap; a non-zero Δ implies the absence of long-range Néel order. In $d = 1$, $\Delta \to 0$ as $n_c/N \to \infty$; thus, in agreement with the results of Section 3.2, we find that there is no Néel ordered phase for finite n_c. In $d = 2$, $\Delta \to 0$ as temperature $T \to 0$ for all $n_c/N \geq 0.19$ while for $n_c/N < 0.19$, the gap Δ remains non-zero at $T = 0$. Thus this mean-field analysis determines a line in the $n_c - N$ plane (shown in Fig 3), with slope 0.19 for large N, above which there is long-range Néel order. This conclusion is again in agreement with the non-linear sigma model analysis of Section 3.2 For $d > 2$, Δ vanishes above some critical value of n_c/N for all $T < T_{N\acute{e}el}(n_c/N)$, the Néel ordering temperature. The vanishing of Δ may be physically identified with the presence of long-range Néel order by noting that $\Delta = 0$ requires $< b >, < \bar{b} >$ to be non-zero due to condensation into the zero-energy states [28]. In the remainder of this section, we shall focus exclusively on the properties of the *disordered* state at $T = 0$ in $d = 1, 2$ ($n_c/N < 0.19$ for $d = 2$) where $SU(N)$ symmetry is *unbroken*.

The subsequent analysis is simplest close to the transition line in Fig 3 where $\Delta \ll J$; the bosonic spectrum has the relativistic form

$$\omega_{\mathbf{k}} = \sqrt{\Delta^2 + c^2\mathbf{k}^2}, \tag{83}$$

where the speed of "light" (spin-wave velocity) $c = \bar{\lambda}a/d^{1/2}$. We can also define a spin correlation length ξ

$$\xi = \frac{c}{\Delta} \gg a, \tag{84}$$

which is much greater than the lattice spacing. This length diverges as one approaches the transition to the Néel phase (Fig 3): we therefore expect to obtain a simple continuum description of the system in this limit. The ground state of H_{MF} has the form for $\Delta > 0$

$$|\Omega\rangle \propto \exp\left(\sum_{\mathbf{k}} f_{\mathbf{k}} b_{\mathbf{k}\alpha}^{\dagger} \bar{b}_{-\mathbf{k}}^{\alpha\dagger}\right)|0\rangle, \tag{85}$$

which represents a condensate of singlet pairs of bosons ("valence bonds"); the bonds have ends on opposite sublattices and their characteristic size is ξ. When projected onto n_c bosons per site, $|\Omega\rangle$ is an $SU(N)$ generalization of the short-range resonating-valence-bond states of Sutherland and Liang *et. al.* [29], which are thus *exact* in the present large-N limit provided the distribution of bond-lengths is chosen correctly. The eigenmodes of H_{MF} are clearly *bosons* in agreement with recent calculations [30]. The mean-field results for the disordered phase are thus close in spirit to the resonating-valence-bond scenario of Kivelson *et.al.* [31], and consist of a featureless fluid of singlet pairs of bosons. We will now show that topological effects in the fluctuations about the mean-field dramatically alter the nature of the disordered phase. We will begin in this section by examining the structure of a straightforward $1/N$ expansion. The consequences topological fluctuations will be examined in the subsequent subsections.

We begin by considering the fate of the $U(1)$ gauge invariance of $\mathcal{L}$ in the mean-field theory of the disordered state. It is useful to examine first *global* (site and τ independent) transformations; since our system has two sites per unit cell, there are two such invariances: *(i)* uniform: $b \to e^{i\phi} b$, $\bar{b} \to e^{i\phi}\bar{b}$, and, *(ii)* staggered: $b \to e^{i\phi} b$, $\bar{b} \to e^{-i\phi}\bar{b}$. Clearly the "uniform" symmetry is broken by non-zero value of $\bar{Q} \sim < b^\alpha \bar{b}_\alpha >$ while the "staggered" symmetry is not. Considering the full group of *local* gauge transformations we see that it splits into two parts: the uniform part which is broken, and the staggered part which is not. Fluctuations of Q and λ can be written in the form (for each unit cell labeled by $i \in A$):

$$\begin{aligned} Q_{i,i+\hat{\eta}} &= \left(\bar{Q} + q_{\hat{\eta}}(i + \tfrac{1}{2}\hat{\eta})\right) \exp\left(i\theta_{\hat{\eta}}(i + \tfrac{1}{2}\hat{\eta})\right) \\ i\lambda(i) &= \bar{\lambda} + i\lambda_1(i) \qquad i\lambda(i+\hat{x}) = \bar{\lambda} + i\lambda_2(i+\hat{x}). \end{aligned} \tag{86}$$

We now separate out the "uniform" and "staggered" gauge field components of the phase θ and the Lagrange multipliers λ by defining in momentum space

$$\begin{aligned} aA_{\hat{\eta}}(\mathbf{k}) &= \tfrac{1}{2}(\theta_{\hat{\eta}}(\mathbf{k}) - \theta_{-\hat{\eta}}(\mathbf{k})) = -aA_{-\hat{\eta}}(\mathbf{k}) \\ A_\tau(\mathbf{k}) &= \tfrac{1}{2}(\lambda_1(\mathbf{k}) - \lambda_2(\mathbf{k})) \\ M_{\hat{\eta}}(\mathbf{k}) &= \tfrac{1}{2}(\theta_{\hat{\eta}}(\mathbf{k}) + \theta_{-\hat{\eta}}(\mathbf{k})) = M_{-\hat{\eta}}(\mathbf{k}) \\ M_\tau(\mathbf{k}) &= \tfrac{1}{2}(\lambda_1(\mathbf{k}) + \lambda_2(\mathbf{k})). \end{aligned} \tag{87}$$

The notation has been chosen in a suggestive manner so that with $\hat{\eta}$ in a positive axis direction, the $A_{\hat{\eta}}$,A_τ are the components ($\mathbf{A}$,A_τ) of the gauge field for the unbroken staggered $U(1)$ symmetry, while the M's are related to the broken uniform symmetry. We now insert these definitions into $\mathcal{L}$, perform a gradient expansion, and obtain

$$\mathcal{L} \approx \int \frac{d^d\mathbf{r}}{a^d} \left[b^\dagger_\alpha \left(\frac{d}{d\tau} + iA_\tau\right) b^\alpha + \bar{b}^{\alpha\dagger} \left(\frac{d}{d\tau} - iA_\tau\right) \bar{b}_\alpha \right.$$

$$+\bar{\lambda}\left(|b^\alpha|^2+|\bar{b}_\alpha|^2\right)-4\bar{Q}\left(b^\alpha\bar{b}_\alpha+b_\alpha^\dagger\bar{b}^{\alpha\dagger}\right)\Big]$$

$$+\int\frac{d^d\mathbf{r}}{a^{d-2}}\bar{Q}\left[\left(\nabla_{\mathbf{r}}+i\mathbf{A}\right)b^\alpha\left(\nabla_{\mathbf{r}}-i\mathbf{A}\right)\bar{b}_\alpha+\left(\nabla_{\mathbf{r}}-i\mathbf{A}\right)b_\alpha^\dagger\left(\nabla_{\mathbf{r}}+i\mathbf{A}\right)\bar{b}^{\alpha\dagger}\right]. \tag{88}$$

plus additional terms involving M and $q_{\hat{\eta}}$. Note that the two modes of H_{MF} have charges ± 1 with respect to the staggered symmetry, *i.e.* they are particle and anti-particle. We now find it convenient to introduce the boson fields

$$\begin{aligned} z^\alpha &= (b^\alpha+\bar{b}^{\alpha\dagger})/2 \\ \pi^\alpha &= (b^\alpha-\bar{b}^{\alpha\dagger})/2. \end{aligned}$$

From Eqn (88), it is clear that the the π fields turn out to have mass $\bar{\lambda}+4\bar{Q}$, while the z fields have a mass $\bar{\lambda}-4\bar{Q}$ which *vanishes* at the transition to the Néel phase. The π fields can therefore be safely integrated out, and we obtain finally the effective action

$$S'_{eff}=\int d^d r\int_0^{c\beta}d\tilde{\tau}\frac{a^{1-d}}{2\sqrt{d}}\left\{|(\partial_\mu-iA_\mu)z^\alpha|^2+\frac{\Delta^2}{c^2}|z^\alpha|^2\right\}, \tag{89}$$

plus additional terms coupling the M and $q_{\hat{\eta}}$ fields to the z^α fields. Here $\tilde{\tau}=c\tau$, $A_{\tilde{\tau}}=A_\tau/c$ and μ runs over $x,y\ldots\tilde{\tau}$. Remarkably, this action is identical in form to the CP^{N-1} action (75) obtained in the *semiclassical* limit in Section 3.4. However there are several advantages to the present derivation:

1. We have a microscopic interpretation of the spatial components of the gauge-field $\mathbf{A}$ as the phase of a bond variable.

2. The *compact* nature of the gauge-field fluctuations are apparent from the definition in Eqn (86) - this fact will be crucial in understanding topological effects below.

3. As shown in Eqn (90) below, the present large-N limit is useful in describing the lattice-scale coupling to the spin-Peierls order parameter.

To complete the large-N calculation we must now integrate out the N-component z^α boson. This, at one-loop order, will lead to a quadratic action in the variables A, M and q with co-efficient N. If we restrict our attention to phenomena at distances larger than ξ, we obtain the simple effective action

$$S_{eff}=N\int d^d r\int_0^{c\beta}d\tilde{\tau}\left[\frac{1}{4e^2}F_{\mu\nu}^2+i\gamma\sum_{\hat{\eta}>0}(q_{\hat{\eta}}-q_{-\hat{\eta}})F_{\hat{\eta}\tilde{\tau}}\right], \tag{90}$$

and additional terms involving q and M; $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the electromagnetic field. The term proportional to γ arises from the coupling between $q_{\hat{\eta}}$ and z^α which was not displayed in Eqn (89). For $d < 3$, $e^2 \sim (\Delta/c)^{3-d}$ can be calculated in the continuum limit, but γ (needed for the spin-Peierls calculation below) has to be calculated using the underlying lattice regularization, giving $\gamma \sim a^{1-d}/\bar{\lambda}$.

It is easy to show that the terms of S_{eff} considered so far do not change any essential features of the mean-field analysis. One expects that, in all orders in a $1/N$ expansion, the fluctuation corrections will merely renormalize the mean-field parameters and be insensitive to the value of n_c (mod 4) . Such a sensitivity however arises when we consider the effects of topologically non-trivial gauge field configurations. Such configurations are expected to give rise to Berry phase factors in the functional integral. These phases should be obtainable by integrating out $b, \bar{b}$ in the presence of a non-trivial background gauge field. However, it is much simpler to perform the equivalent procedure of calculating the phase due to adiabatic evolution of the ground state (85) in such a gauge-field background. We will find by explicit calculations in the disordered phase that the Berry phases are closely related to the semiclassical Berry phases calculated in the ordered phase in Section 3.3. We will now discuss the details of this calculation in $d = 1, 2$ in turn.

4.1 Berry Phases in $d = 1$

The calculations in Section 3.3.1 showed that the topological term in the CP^{N-1} model could be written in the form

$$\frac{i\Theta}{2\pi}\int dx d\tilde{\tau} F_{x\tilde{\tau}}, \tag{91}$$

with $\Theta = \pi n_c$. We will prove the existence of such a term by a direct calculation in the disordered phase.

Consider a spin chain with N_s sites (N_s even) and periodic boundary conditions. Choosing the configuration in the phase of Q

$$\theta_{\hat{\eta}}(i + \tfrac{1}{2}\hat{\eta}, \tau) = \mathrm{sgn}(\hat{\eta})\phi(\tau), \tag{92}$$

where $\phi(\tau)$ increases slowly from 0 at $\tau = 0$ to the gauge equivalent value $2\pi\ell/N_s$ at $\tau = \beta$ (ℓ integer). Evaluating the value of A_μ from Eqn (87), this configuration yields a value

$$\int dx d\tilde{\tau} F_{x\tilde{\tau}} = 2\pi\ell. \tag{93}$$

We now adiabatically evolve the ground state wavefunction in this time-dependent background electric field. At $\tau = 0$ we have the wavefunction $|\Omega\rangle$ in Eqn (85) with f_k real, and the sum is over $k = 2\pi n/aN_s$, $n = 1 \dots N_s$. For $\tau > 0$ we choose the phase of the wavefunction such that $\langle\Omega|d/d\tau|\Omega\rangle = 0$; as a result

$$|\Omega(\tau = \beta)\rangle \propto \exp\left(\sum_k f_{k-2\pi\ell/aN_s} b^\dagger_{k\alpha}\bar{b}^{\alpha\dagger}_{-k}\right)|0\rangle. \tag{94}$$

Figure 5: *Symmetry of the ground states of the $d = 1$ chain for $N > 2$*

The gauge-invariant Berry phase is now just the change in the phase of the wave-function. A simple calculation shows that

$$P_{n_c}|\Omega(\tau = \beta)\rangle = (-1)^{n_c \ell} P_{n_c}|\Omega(\tau = 0)\rangle, \tag{95}$$

where P_{n_c} projects onto n_c bosons per site. For the action S_{eff} in Eqn (90) to reproduce this phase, it is clear that we have to add a topological term as in Eqn (91) with $\Theta = p\pi$, where the integer p is restricted to be even (odd) if n_c is even (odd). Each choice of Θ corresponds to a different metastable state of the spin chain with a mean static electric field [22] $iF_{x\tilde{\tau}} = e^2 p/N$ and energy per site $\sim ce^2p^2a/N$. Using the coupling between the electric field and the $q_{\hat{\eta}}$ fields in Eqn (90), we deduce the presence of a spin-Peierls order parameter

$$\langle \hat{S}(i) \cdot \hat{S}(i+1) - \hat{S}(i) \cdot \hat{S}(i-1) \rangle \sim N\bar{Q}\langle q_{\hat{x}} - q_{-\hat{x}} \rangle / J \sim a\gamma e^2 cp. \tag{96}$$

The ground state for n_c even is therefore obtained with the choice $p = 0$ and is non-degenerate; the linear Coulomb force confines the z^α bosons (spinons) in pairs. For n_c odd the ground state corresponds to $p = \pm 1$, and is *two-fold degenerate with a non-zero spin-Peierls order parameter*; the spinons are domain walls interpolating between the two ground states. A schematic of the two ground states is shown in Fig 5a,b. The spin-Peierls order for n_c odd was anticipated by Affleck [32] though not shown directly for $n_c \sim N$. This picture is now expected to be correct for *all* $N > 2$ [3, 20]. Only for $N = 2$ and odd values of n_c does a massless Bethe ansatz type of ground state appear.

4.2 Berry Phases in $d = 2$

From the close similarity in $d = 1$, between the semiclassical results of Section 3.3.1 and the calculations in Section 4.1, it is clear that we have to look for the remnant of the 'hedgehogs' in the disordered phase. One characterization of the hedgehog of charge m is that (see Eqn (69)):

$$m = \frac{i}{2\pi} \int dS_{\mu\nu} \partial_\mu z^\alpha \partial_\nu z^*_\alpha, \tag{97}$$

where the integral is over the surface of a sphere surrounding the hedgehog. By analogy with $d = 1$ we look for excitations in the gauge field A_μ with $\int F_{\mu\nu} dS_{\mu\nu} = 2\pi m$ [14]. These instantons are the "monopole" excitations of the 2+1 dimensional compact QED considered by Polyakov [33]. The field configuration is given by

$$H_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda}F_{\nu\lambda}$$
$$\nabla \times \mathbf{H} = 0 \qquad ; \qquad \nabla \cdot \mathbf{H} = 2\pi m \delta(r - r')$$

for an instanton at r'. Our picture of the instanton defect is as follows: at its core (of linear dimension ξ), the defect is better described using the language of the ordered side as a hedgehog; the instantons are bound by a linear r potential at distances smaller than ξ. At distances larger that ξ however, the gauge-field description is superior. Notice however that the structure of the topological singularity is preserved even with the loss of long-range Néel order. The Berry phase of the instantons can be calculated in a manner very similar to that employed for $d = 1$; details of the calculation may be found in Ref [5]. We find the following results for the total Berry phase for an instanton configuration with integer charges m_s located at spatial co-ordinates $\mathbf{R}_s$ at the centers of the plaquettes and imaginary time co-ordinates τ_s:

$$S_B = \sum_s i\frac{n_c\pi}{2}\zeta_s m_s, \tag{98}$$

where the integer $\zeta_s = 0, 1, 2, 3$ for $\mathbf{R}_s$ on four dual sublattices W, X, Y, Z (Fig 4). Remarkably, this result is *identical* to the hedgehog Berry phase calculated by Haldane [9] and its extension to $SU(N)$ [3] (see Eqn (71)).

The subsequent analysis follows closely Polyakov's solution [33] of 2+1 dimensional compact QED. Neglecting all fields except A at distances $> c/\Delta$, the action is evaluated for each instanton configuration. The field equations for the instanton charges are identical to those of ordinary electrostatics: we therefore obtain a Coulomb interaction between the instantons in the partition function

$$Z = \sum_{K,\{m_s\}} \frac{1}{K!}\prod_{s=1}^{K}\left(\sum_{\mathbf{R}_a}\int_0^{c\beta}\frac{d\tilde{\tau}_s}{\rho a}\right)\exp(-S_m(\{m_s\}))$$

$$S_m(\{m_s\}) = \frac{N\pi}{2e^2}\sum_{s\neq t}\frac{m_s m_t}{\left((\mathbf{R}_s - \mathbf{R}_t)^2 + (\tilde{\tau}_s - \tilde{\tau}_t)^2\right)^{1/2}} + \sum_s\left(NE_c m_s^2 + i\frac{n_c\pi}{2}\zeta_s m_s\right), \tag{99}$$

where ρ is a dimensionless constant of order unity, and NE_c, the instanton core-action is determined by physics at length scales shorter than c/Δ; assuming that the instanton is better described as a hedgehog at these length scales, we expect $E_c \sim \bar{\lambda}/\Delta$.

We now use the well-known equivalence between the d-dimensional Coulomb gas and the sine-Gordon model [33]. Introduce the sine-Gordon field $\chi(\mathbf{R}, \tilde{\tau})$ by the transformation

$$e^{-S_m} = \int \mathcal{D}\chi \exp \Bigg[-g \int_0^{c\beta} d\tilde{\tau} \left\{ \sum_{<a,b>} (\chi_a - \chi_b)^2 + \sum_a a^2 \left(\frac{\partial \chi_a}{\partial \tilde{\tau}} \right)^2 \right\} - \sum_{\{m_a\}} \left\{ N E_c m_a^2 + i \left(\frac{n_c \pi}{2} \zeta_a + \chi(\mathbf{R}_a, \tilde{\tau}_a) \right) m_a \right\} \Bigg], \tag{100}$$

where we have used the notation $\chi_a \equiv \chi(\mathbf{R}_a, \tilde{\tau})$ and $g = e^2/(4N\pi^2)$. Summing over the instanton charges (neglecting the effect of charges > 1 because of the small instanton fugacity $\sim \exp(-NE_c)$), we show finally that the properties of Z are equivalent to those of $\int \mathcal{D}\chi e^{-S_{sg}}$ with

$$S_{sg} = \frac{g}{2} \int_0^{c\beta} d\tilde{\tau} \left\{ \sum_{<s,t>} (\chi_s - \chi_t)^2 + \sum_s \left(a^2 \left(\frac{\partial \chi_s}{\partial \tilde{\tau}} \right)^2 - M^2 \cos \left[\chi_s - (n_c \pi/2)\zeta_s \right] \right) \right\}. \tag{101}$$

Here $M^2 = (2/g\rho a) \exp(-NE_c)$ is the *exponentially* small instanton fugacity.

In the absence of any non-trivial Berry phases (the case $n_c = 0 \pmod 4$), we note that there is in fact an *exact* mapping between the compact $U(1)$ gauge theory and the sine-Gordon model. As was first shown by Polyakov [33], for the case of a *periodic Gaussian* form for the gauge theory action

$$Z = \int dA_\mu \sum_{m_{\mu\nu}} \exp \left(- \sum_{\square} (\Delta_\mu A_\nu - \Delta_\nu A_\mu - 2\pi m_{\mu\nu})^2 \right), \tag{102}$$

($m_{\mu\nu}$ are integers at the centers of the plaquettes of a three-dimensional lattice, and Δ_μ indicates a lattice derivative) there exists an exact duality transformation to a discrete-Gaussian model and the sine-Gordon model [5]. We will show in Section 6 that there exist exact duality transformations which produce S_{sg} for the cases $n_c \neq 0 \pmod 4$.

If $n_c = 0 \pmod 4$, S_{sg} is the usual sine-Gordon model. For small M, it is solved by expanding perturbatively around a minimum [33]. This gives a "screening length" in the instanton plasma $\sim aM^{-1}$ and confinement of z quanta (spinons) into pairs of size $\sim aM^{-1}$. The fluctuations in F give a collective mode of gap $\sim cM/a$. This closely resembles the properties of the valence-bond-solid states recently introduced for $n_c = 2S = 4$ in an $SU(2)$ model [34], and gives the full lattice symmetry (Fig 1c).

For $n_c \neq 0 \pmod 4$, the uniform state $\chi_s =$ constant is *unstable*. The rotation symmetry between the four sublattices W, X, Y, Z is therefore *spontaneously broken*. For $n_c = 1 \pmod 4$ one stable minimum of S_{sg} is given to order M^2 by

$$\chi_W = \chi_X = -\frac{\pi}{4} - \frac{M^2}{4\sqrt{2}} \quad : \quad \chi_Y = \chi_Z = -\frac{\pi}{4} + \frac{M^2}{4\sqrt{2}} \tag{103}$$

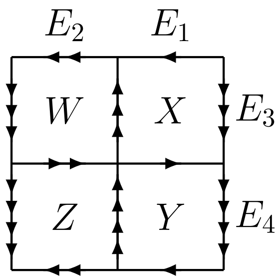


Figure 6: *Definition of the electric fields on the links of the square lattice produced by the condensation of the instantons*

(there are three other similar minima near $\pi/4$, $3\pi/4$, and $-3\pi/4$). If we calculate the instanton charge density ($\propto \Delta^2\chi$, where Δ^2 is the lattice-laplacian) associated with this minimum, we find different *non-zero mean instanton charge densities* on the W, X, Y, Z sublattices. This condensation of the instantons leads to a static electric field $iE_3 = iE_4 = 0$, $iE_1 = iE_2 = \pi g M^2/\sqrt{2}a$ shown in Fig 6. The coupling between the electric field and the $q_{\hat{\eta}}$ field in S_{eff} (Eqn (90)) now implies an exponentially small (in N) but *non-zero spin-Peierls order* of the type shown in Fig 1a. with $\langle q_{\hat{x}} - q_{-\hat{x}}\rangle \sim (\gamma\bar{\lambda}a)\pi g c M^2/\sqrt{2}$. A very similar analysis can be performed for $n_c = 3 \pmod 4$. For $n_c = 2 \pmod 4$, the minima of S_{sg} lead to the static electric fields $iE_2 = iE_3 = -iE_1 = -iE_4 = gM^2/4a$ and spin-Peierls order of the type shown in Fig 1b. These states with broken lattice symmetry also give confinement of spinons and a massive spinless collective mode but with gap (inverse confinement scale) $\sim cM^2/a$ and cM^4/a for $n_c = 2$ and $1, 3 \pmod 4$ respectively.

The results in the previous paragraph have been obtained in the large-N, small M^2 limit. It is also of interest to examine S_{sg} as a phenomenological description of the non-Néel phase of Heisenberg antiferromagnet in its own right, with the unknown parameters g and M^2 determined by microscopic details of the physics. The physics may be visualized by interpreting χ as the height of a three-dimensional solid-on-solid interface [35]. Such a point of view was taken in a recent paper [6] where it is shown in a spatial dimensionality $d = 1 + \epsilon$ expansion that under a renormalization group transformation, the coupling M^2 always flows to infinity. This instability is related to the fact that three-dimensional solid-on-solid interfaces are always smooth [36]. The structure of the *large* M^2 limit was found to bear a remarkable similarity to the fermionic large N theory to be discussed in Section 6. It is also shown in Ref [6] that this large M^2 limit displays spin-Peierls order of exactly the same type as discussed above for small M^2.

We have thus established the most interesting results of these lectures. We have shown that the non-Néel state is *generically unstable to the formation of spin-Peierls order* for

$n_c \pmod 4 \neq 0$. The instability arose as a consequence of careful consideration of the long-wavelength interference between topological instanton tunneling events.

5. FERMIONIC LARGE-N I ($N \to \infty$, $m = N/2$)

We will obtain further support for the presence of spin-Peierls order by performing a calculation deep in the disordered phase (Fig 3) using the fermionic representation of the $SU(N)$ generators. We shall consider two choices for the parameter m. In this section we will focus on $m = N/2$ while in Section 6 we shall consider $m = 1$. The methods of this section are more generally applicable to any m of order N while those Section 6 can be generalized to any m of order 1. In both these sections we will use the representation Eqn (19) for the generators of $SU(N)$. The case $m = N/2$ is self-conjugate (there is a particle-hole symmetry); the two sublattices are equivalent and considerable simplification in the following calculation results.

Generalizing the procedure of Ref [37] to n_c colors, the fermion interaction in $\mathcal{H}$ can be decoupled by introducing a $n_c \times n_c$ *matrix* field $\chi^a_b(ij)$ on every link (i,j); the a, b are color indices. It is easy to show that the partition function for Hamiltonian $\mathcal{H}$ in Eqn (27) can be expressed as follows (after subtracting a constant energy of $-Jn_c^2/2$ per site):

$$Z/Z_o = \int_{-\pi/\beta}^{\pi/\beta} \frac{d\lambda^a_b(i)}{2\pi} \int \mathcal{D}c\mathcal{D}c^\dagger\mathcal{D}\chi \exp\left(-\int_0^\beta d\tau \mathcal{L}(\tau)\right), \tag{104}$$

where the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = c^\dagger_{\alpha a}(i)\frac{\partial c^{\alpha a}(i)}{\partial\tau} + i\lambda^a_b(i)\left[c^\dagger_{\alpha a}(i)c^{\alpha b}(i) - \delta^b_a N/2\right] + \frac{N}{J}|\chi^a_b(ij)|^2$$
$$+ c^\dagger_{\alpha a}(i)c^{\alpha b}(j)\chi^a_b(ij) + \chi^b_a(ji)c^\dagger_{\alpha b}(j)c^{\alpha a}(i) \tag{105}$$

with all repeated indices summed over. The normalization Z_o is given by

$$Z_o = Z_F \int \mathcal{D}\chi \exp(-\int_0^\beta d\tau |\chi^a_b(ij)|^2), \tag{106}$$

where Z_F is a free fermion determinant.

The structure of the theory is straightforward in the large N limit. After integrating out the fermions, the effective action for the χ and λ fields acquires a factor of N in front. The functional integral can therefore be approximated by the stationary phase point of the action. By particle-hole symmetry, the expectation value of the λ fields is zero at the stationary phase point. In addition, it can easily be shown the fluctuations of the λ fields make no contribution to the ground state energy; the λ field will therefore be omitted in the subsequent discussion. The effective action for the χ fields after integrating out the fermions is given by

$$\frac{S_{eff}}{N} = \int_0^\beta d\tau \left[-\text{Tr Ln} \left(\frac{\partial}{\partial \tau} \delta^a_b \delta_{ij} + \chi^a_b(ij) \right) + \frac{1}{J} \sum_{<ij>} \sum_{ab} |\chi^a_b(ij)|^2 \right], \tag{107}$$

with $\chi^a_b(ij) = \left(\chi^b_a(ji)\right)^*$.

A class of time-independent stationary phase solutions of S_{eff} are those with χ color-diagonal. The colors then decouple from each other and the subsequent minimization becomes identical to the one color calculation carried out by Affleck and Marston. The global minima found by Affleck and Marston correspond to the 'bond' solutions: for each color, a, the field $\chi^a_a(ij)$ has a mean field value of either $\bar{\chi}$ or 0 on every link ($\bar{\chi} = J/2$ at this order); every site has exactly one link with $\chi^a_a = J/2$ (a 'bond') for all values of the color a. The relative positions of bonds with different colors is however arbitrary. The ground state energy E_G is given by

$$E_G = -\frac{N_s N n_c J}{8}, \tag{108}$$

where N_s is the number of sites on the square lattice. (There is an additional continuous family of solutions to the mean field equations which is degenerate with the ones considered; however they are higher in energy at order $1/N$ and will not be considered further [3].)

Going beyond the assumption of color-diagonal solutions, we have carried out an extensive computer search for additional time-independent χ configurations which minimize S_{eff} with up to three colors. We examined all periodic configurations with a unit cell of four sites. No additional color-gauge inequivalent solutions with a lower energy were found for any n_c studied, and we believe that none exist for any n_c.

5.1 $1/N$ Corrections

To break the degeneracy in the ground state it is necessary to consider the $1/N$ corrections to the ground state energy. The procedure for carrying this out is, in principle, straightforward. We evaluate the quadratic action for the χ fields after integrating out the fermion fields. The resulting functional determinant can be evaluated in closed form and the $1/N$ correction to the ground state energy for any bond configuration evaluated. We will not present the details here: they have been discussed at length in Ref [3].

The results will be discussed for $d = 1, 2$ in turn.

(a) One Dimension

The ground state bond configuration turns out to have the form of Fig 5a for $n_c \pmod 2 = 1$ and the form of Fig 5b for $n_c \pmod 2 = 0$. This is completely consistent with the results from the bosonic representation in Section 4 and the field theoretical arguments of

Affleck [20]. Note however the magnitude of the spin-Peierls order parameter is now of order N while near the transition to the Néel phase it was calculated to be of order 1.

(b) Two Dimensions

The global ground state for *all* values of n_c turns out to have the form Fig 1c: *i.e.* the bonds of all the colors are superposed into the same set of columns. The line state (for $n_c \pmod 4 = 2$) and the valence-bond-solid state (for $n_c \pmod 4 = 0$) now turn out to be *metastable*. However the important point is that *none* of the low-lying states obtained violate the lower bound on the ground state degeneracy of 1,4,2,4 for $n_c(\bmod 4) = 0,1,2,3$ respectively, which was suggested by the semiclassical non-linear sigma model of Section 3.3.2. Moreover the magnitude of the spin-Peierls order in this extreme quantum limit is of order N unlike the bosonic calculation in which it was exponentially small in N. The different types of spin-Peierls order for the two calculations (for n_c even) suggests the existence of a first-order transition somewhere in the disordered phase.

6. FERMIONIC LARGE-N II ($N \to \infty$, $m = 1$)

In this section we will consider properties of the Hamiltonian $\mathcal{H}$ in Eqn (27) for the case $m = 1$. We will show that for $m = 1$, at order $1/N$, the antiferromagnet is exactly equivalent to a generalized quantum dimer Hamiltonian [38]. The $m = 1$ antiferromagnet has in fact already been considered by Affleck [32] in one dimension, where the dimer Hamiltonian is trivially solvable. While the overall phase diagram obtained by Affleck is correct, there are some minor errors in the structure of his perturbation theory; it will be important to correct these errors to understand the physics in two dimensions.

We begin by describing the model with $n_c = 2$, but we shall use a general formalism which will allow a straightforward generalization to arbitrary n_c. We will work with the symmetric tensor operators $T_{\alpha\beta}$ discussed in Eqn (20) to represent the states on sublattice A. On the B sublattice, we will use the $\tilde{T}^{\alpha\beta}(j)$ operators defined as in Eqn (20), with the $\tilde{c}^\dagger$ operators replacing the $c^\dagger$ operators. The Hilbert space is spanned by the action of the T operators upon the vacuum state on every site of the square lattice.

In the $N \to \infty$ limit the ground states of H can be easily deduced from the arguments of Affleck [32]. The ground state has a degeneracy of order $\exp(cN)$ for some constant c. Each ground state can be described as follows: contract the $SU(N)$ indices of the $T_{\alpha\beta}$ on neighboring sites in an arbitrary manner until there are no free indices left. All of the states so obtained are manifestly $SU(N)$ singlets and can be labeled by a set of non-negative integers, $\{n_\ell\}$, where $0 \leq n_\ell \leq n_c$ is the number of contractions ('bonds') on the link ℓ of the square lattice. For example, the state

$$|\cdots n_{(i,j)}\cdots n_{(p,q)}, n_{(q,r)}, n_{(r,s)}, n_{(s,p)}\cdots\rangle =$$

$$C'\left(\sum_{\alpha\beta\gamma\delta\nu\sigma}\cdots T_{\alpha\beta}(i)\tilde{T}^{\alpha\beta}(j)\cdots T_{\gamma\delta}(p)\tilde{T}^{\delta\nu}(q)T_{\nu\sigma}(r)\tilde{T}^{\sigma\gamma}(s)\cdots\right)|0\rangle, \qquad (109)$$

where C is a normalization constant, (i,j), (p,q), (q,r), (r,s) and (s,p) are links of the square lattice, has $n_{(i,j)} = 2$, and $n_{(p,q)} = n_{(q,r)} = n_{(r,s)} = n_{(s,p)} = 1$. The set of integers $\{n_\ell\}$ must also satisfy the constraint

$$n_{(i,i+\hat{x})} + n_{(i,i-\hat{x})} + n_{(i,i+\hat{y})} + n_{(i,i-\hat{y})} = n_c \qquad (110)$$

because there are n_c $SU(N)$ indices emerging from each site (the sum in the equation above extends over the four links ending at the site i).

A word is in order here about our phase convention for the states. We will always write the $T_{\alpha\beta\ldots}(i)$ operators by using a *fixed*, but arbitrary, ordering of the sites i of the square lattice. This convention now uniquely determines the state $|\{n_\ell\}\rangle$ once the values of the link variables n_ℓ are known.

6.1 Order $1/N$

We now show that at order $1/N$, there are matrix elements which mix the states in the ground state manifold; this mixing can be described by an effective Hamiltonian which is a generalization of the quantum dimer Hamiltonians considered by Rokhsar and Kivelson [38].

The results follow from the repeated use of the following commutation relation

$$[\hat{S}_\alpha^\beta, T_{\gamma\delta}] = T_{\gamma\alpha}\delta_\delta^\beta + T_{\delta\alpha}\delta_\gamma^\beta \qquad (111)$$

and its obvious generalization to arbitrary n_c. A similar result holds on sublattice B. The site index has been supressed in the above two equations. To evaluate the energy of the bonds between the sites i and j we will need the following commutator

$$\left[\hat{S}_\mu^\nu(i)\hat{\tilde{S}}_\nu^\mu(j)\ ,\ T_{\alpha\beta}(i)T^{\gamma\delta}(j)\right] = \delta_\alpha^\gamma T_{\mu\beta}(i)\tilde{T}^{\mu\delta}(j) + \delta_\alpha^\delta T_{\mu\beta}(i)\tilde{T}^{\mu\gamma}(j)$$

$$+\delta_\beta^\gamma T_{\alpha\mu}(i)\tilde{T}^{\mu\delta}(j) + \delta_\beta^\delta T_{\alpha\mu}(i)\tilde{T}^{\gamma\mu}(j) + \left\{T_{\mu\beta}(i)\tilde{T}^{\gamma\delta}(j)\hat{\tilde{S}}_\alpha^\mu(j)\right.$$

$$\left. +T_{\alpha\mu}(i)\tilde{T}^{\gamma\delta}(j)\hat{\tilde{S}}_\beta^\mu(j) + T_{\alpha\beta}(i)\tilde{T}^{\mu\delta}(j)\hat{S}_\mu^\gamma(i) + T_{\alpha\beta}(i)\tilde{T}^{\gamma\mu}(j)\hat{S}_\mu^\delta(i)\right\}. \qquad (112)$$

This result can be used to commute the Hamiltonian $\mathcal{H}$ in Eqn (27) through the T operators, and so determine the diagonal energies and the off-diagonal mixing terms between the states

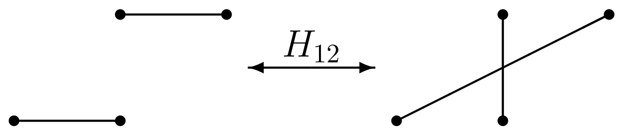


Figure 7: *Fluctuations caused by the action of* H_{12} *upon a candidate ground state for* $n_c = 1$. *This fluctuation changes the energy at order* $1/N^2$

$|\{n_\ell\}\rangle$. The terms in the curly brackets in Eqn (112) annihilate the ground state and can ignored in the subsequent discussion. The remaining terms in Eqn (112) have three different types of effects:

(i) They contribute a diagonal term to the energy of the state $|\{n_\ell\}\rangle$. The contribution of a n_ℓ-fold bond on a link, E_{n_ℓ}, to the ground state energy can easily be computed to be

$$E_{n_\ell} = -\frac{J}{N} n_\ell (N - 1 + 2n_c - n_\ell). \tag{113}$$

In Eqn (113), and in the remainder of this section, we have omitted a constant energy per site of $(Jn_c^2/2)(1 - 4m/N)$.

(ii) They generate states which have a bond between the sites which are *not* nearest neighbors. An example of this process is illustrated in Fig 7 for the case $n_c = 1$. It is easy to show that such processes contribute to the energy of states $|\{n_\ell\}\rangle$ in second order perturbation theory. They shift the energy of the states by an amount proportional to

$$J \frac{m^2 (N-m)^2}{N^2 (N-1)^2}. \tag{114}$$

For $m = 1$, this contribution is proportional to $1/N^2$, and can therefore be neglected. For m of order N however, these terms are of the *same* order as the first order perturbation theory result. This makes clear why the calculations of this section do not define a consistent perturbation theory for $m = N/2$, and the functional integral method of Section 5 is the appropriate way to proceed.

(iii) They lead to mixing between the states $|\{n_\ell\}\rangle$, as shown in Fig 8. The action of the Hamiltonian leads to a local rearrangement in the values of the n_ℓ field around a plaquette. A plaquette with n_1, $n_2 - 1$, n_3, $n_4 - 1$ bonds on the four links around it can be transformed to a configuration with $n_1 - 1$, n_2, $n_3 - 1$, n_4 bonds on the links. The matrix element for the process requires determination of the normalization constant C', and use of Eqn (112); it can be shown to equal

$$\Big\langle \cdots n_1, n_2 - 1, n_3, n_4 - 1 \cdots \Big| H \Big| \cdots n_1 - 1, n_2, n_3 - 1, n_4 \cdots \Big\rangle = -(2J/N)\sqrt{n_1 n_2 n_3 n_4}. \tag{115}$$

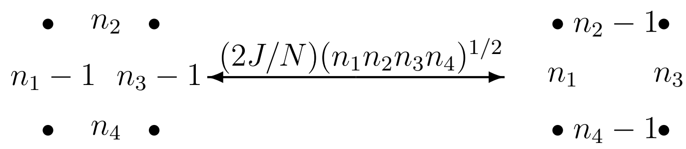


Figure 8: *The off-diagonal term in H_{eff} which changes the local n_ℓ values on the links around a plaquette.*

We are now in a position to express the $1/N$ corrections in terms of a effective Hamiltonian H_{eff} acting upon the Hilbert space of the $|\{n_\ell\}\rangle$ states. The non-orthogonality of the different $|\{n_\ell\}\rangle$ states can be shown to affect H_{eff} only at order $1/N^2$. Using the results in Eqn (110), (113), and (115) we can show

$$\frac{H_{eff}}{J/N} = \sum_{\{n_\ell\}} \Big|\{n_\ell\}\Big\rangle \left(\sum_\ell n_\ell^2\right) \Big\langle\{n_\ell\}\Big| -$$

$$\sum_{\ell_1,\ell_2,\ell_3,\ell_4 \in \Box} \Big|\cdot\cdot\, n_{\ell_1}, n_{\ell_2}-1, n_{\ell_3}, n_{\ell_4}-1, \cdot\cdot\Big\rangle 2\sqrt{n_{\ell_1} n_{\ell_2} n_{\ell_3} n_{\ell_4}} \Big\langle \cdot\cdot\, n_{\ell_1}-1, n_{\ell_2}, n_{\ell_3}-1, n_{\ell_4}, \cdot\cdot\Big|, \quad (116)$$

where the second sum extends over all plaquettes on the lattice. Equation (116) defines the effective Hamiltonian of the generalized dimer model. Notice that while the full Hamiltonian was not invariant under translation by one lattice spacing, the "reduced" dimer Hamiltonian (116) does have this property. For $n_c = 1$, it reduces to a special case of the quantum dimer model of Ref [38].

In one dimension, the off-diagonal term vanishes and H_{eff} is trivially soluble. The ground states, as noted by Affleck [32], for even n_c are non degenerate and have $n_\ell = n_c/2$ on every link of the chain. (For $N = 2$ these states are identical to the exact ground states of the models introduced in Ref [34]) For odd n_c, the ground states have a two-fold degeneracy and we have $n_\ell = (n_c + 1)/2$ on even links and $n_\ell = (n_c - 1)/2$ on odd links or vice versa. H_{eff} differentiates the candidate states obtained from the $N \to \infty$ limit at order $1/N$, whereas Affleck incorrectly found a difference only at order $1/N^2$.

The physics of H_{eff} is not so transparent in two dimensions. We find it useful to introduce *bond* bosons a_ℓ such that $n_\ell = a_\ell^\dagger a_\ell$ is now interpreted as a bond-boson *number operator* which satisfies the constraint in Eqn (110). The effective Hamiltonian now becomes

$$\frac{H_{eff}}{J/N} = \sum_\ell n_\ell^2 - \sum_{\ell_1,\ell_2,\ell_3,\ell_4 \in \Box} a_{\ell_1}^\dagger a_{\ell_3}^\dagger a_{\ell_2} a_{\ell_4}. \quad (117)$$

It has recently been shown [39, 5] that in this form H_{eff} is related by an exact duality transformation to a discrete-Gaussian model and the sine-Gordon model S_{sg} (Eqn (101)) discussed in section 4.2. Following the arguments in that section, we therefore conclude that H_{eff} displays spin-Peierls or valence-bond-solid order whose characteristics are, as before, determined by the value of $n_c \pmod 4$.

To obtain an independent verification of this result, I have performed some numerical calculations on H_{eff} for the case $n_c = 1$ [40]. Using the Lanczos method, the exact ground state of H_{eff} was determined on lattice sizes upto 6×6. A finite size scaling analysis shows clearly that the dimers crystallize in columns and H_{eff} therefore displays spin-Peierls order.

7. CONCLUSIONS

We conclude this paper by recalling the large number of models considered and the relationships between them (See Fig 9). The bulk of the paper obtained new results on the non-Néel phase of a nearest neighbor $SU(N)$ antiferromagnet on a square lattice as a function of n_c, m, and N. The main conclusion was that the non-Néel phase displays spin-Peierls or valence-bond-solid order of the types shown in Fig 1; the minimum degeneracy of all low-lying states in this phase was found to be $1, 4, 2, 4$ for $n_c \pmod 4 \; = 0, 1, 2, 3$ respectively. We now review the numerous methods used to obtain this result as shown in Fig 9.

- For large n_c (but with N, m arbitrary) and on length scales l much larger than the lattice spacing, the antiferromagnet was shown to be described by a non-linear sigma model (Section 3.2).

- A large N analysis on the non-linear sigma model (for $m = 1$) in the disordered phase gave a continuum theory consisting of N relativistic charged scalar particles coupled to a $U(1)$ gauge field; this is also the large-N limit of a CP^{N-1} model in its disordered phase (Section 3.4).

- A Schwinger boson representation of the $SU(N)$ spins [27] was used to take the large-N limit with n_c proportional to N; in the disordered phase the long-distance theory of the bosons is very similar to the CP^{N-1} model (Section 4). This mapping also shows that the $U(1)$ gauge field is in fact compact.

- At distances much larger than the spin-correlation length, the bosons can be integrated out and one obtains a pure compact $U(1)$ gauge theory in 2+1 dimensions.

- The gauge theory is related by a duality transformation (which is exact for a Villain-type model) to a 3D Coulomb gas of instantons (Section 4.2). Lattice scale physics imposes a complex fugacity on the instantons: these Berry phases can be calculated

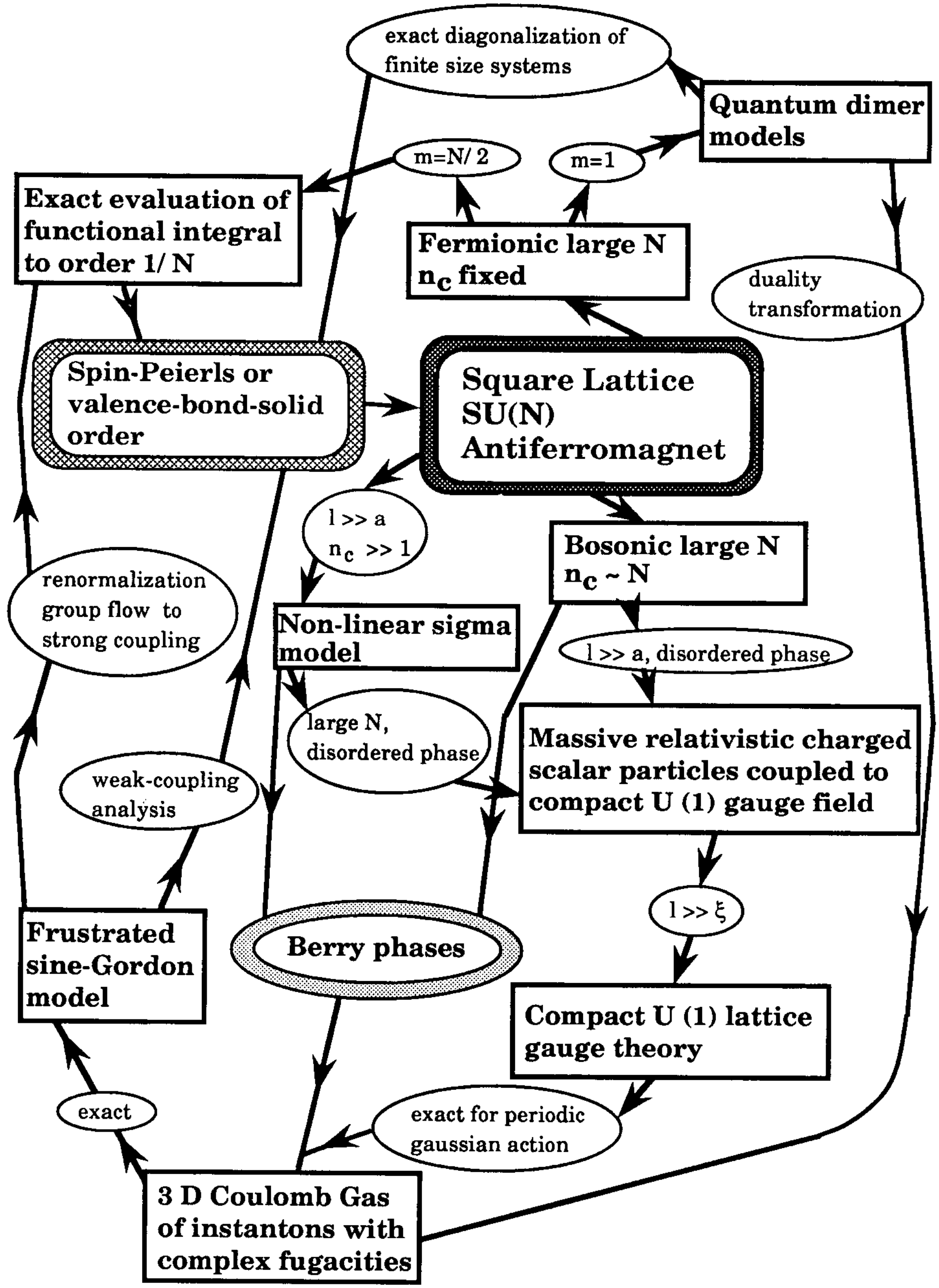


Figure 9: *Flow-chart summarizing the logical relationships between the various models considered in this paper. See Section 7 for further details.*

either from the non-linear sigma model (Section 3.3.2) or the Bosonic large-N (Section 4.2).

- The Coulomb gas is related by another duality transformation to a frustrated sine-Gordon model (Section 4.2).

- A weak-coupling analysis on the sine-Gordon model is shown to imply the presence of spin-Peierls or valence-bond-solid order for the $SU(N)$ antiferromagnet.

- Under a $d = 1 + \epsilon$ renormalization group transformation, the sine-Gordon theory flows to a strong-coupling limit which bears a remarkable resemblance to the fermionic large-N theories [6].

- A large N analysis for fixed n_c can be performed using fermionic operators. For $m = 1$, the $1/N$ fluctuations yield an effective quantum dimer model Hamiltonion (Section 6).

- This dimer model is related by a duality transformation to the 3D Coulomb gas with the *same* Berry phases obtained earlier [39, 5].

- Numerical results on the quantum dimer model for $n_c = 1$ indicate the presence of spin-Peierls order [40].

- For $m = N/2$ the fermionic large-N leads at order $1/N$ to an exactly solvable theory which displays spin-Peierls order of the type shown in Fig 1a for *all* n_c.

Finally we address the all-important question of what the consequences of our results are for weakly frustrated spin-1/2 $SU(2)$ antiferromagnets on a square lattice. This is related to the question of how large the 'universality class' of the transition denoted in Fig 3 by the dashed line is. There are two essential ingredients in the description of this transition

1. The long-distance fluctuations of the spins on the ordered side are described by a non-linear sigma model.

2. The hedgehogs are centered predominantly at the centers of the plaquettes and have the required Berry phases.

The first requirement is expected to by satisfied by any antiferromagnet which has a two-sublattice Néel state as its classical ground state. The existence of the exact duality transformations between *(i)* the compact $U(1)$ gauge theory and the Coulomb gas of instantons, and, *(ii)* the quantum dimer model and the gas of instantons with complex fugacities lends considerably solidity to the second assumption: in both exact transformations, the hedgehogs do indeed lie at the centers of the plaquettes. Thus the central conclusion from the large number of arguments reviewed in this paper is that $SU(2)$ antiferromagnets with a classical ($S = \infty$) Néel ground state are expected to display two types of ground states for finite S: *(i)* the quantum Néel state and *(ii)* a spin-Peierls or valence-bond-solid ordered state. An intermediate spin-disordered state with no broken lattice symmetry (for $n_c \neq 0 \pmod 4$) has been ruled out. Numerical studies of weakly frustrated antiferromagnets will therefore be of great interest; a finite-size scaling analysis with the order-parameters introduced in Ref [40] should yield a clear test of the presence of spin-Peierls order.

Acknowledgements

As noted earlier, the bulk of this work was done in collaboration with N. Read. I thank R. Shankar for many useful conversations and his contributions at the early stages of this project, Heinz Schulz for useful discussions and critical comments on the manuscript, and U. Pasi for comments on the manuscript. Finally, I am indebted to Prof. S.N. Behera and all the participants for a very useful and enjoyable workshop. This research was supported in part by a Presidential Young Investigators Award from the National Science Foundation and by a fellowship from the Alfred P. Sloan Foundation.

References

[1] J.G. Bednorz and K.A. Muller, Z. Phys. **B64**, 188 (1986); M.K. Wu *et. al.*, Phys. Rev. Lett. 58, 908 (1987).

[2] P.W. Anderson, Science **235**, 1196 (1987).

[3] N. Read and S. Sachdev, Nucl. Phys. B., to be published.

[4] N. Read and S. Sachdev, Phys. Rev. Lett., **62**, 1694 (1989).

[5] N. Read and S. Sachdev, to be published.

[6] W. Zheng and S. Sachdev, to be published.

[7] F. Dyson, E.H. Lieb, and B. Simon, J. Stat. Phys. **18**, 335 (1978); E. Jordao Neves and J. Fernando Perez, Phys. Lett. **114A**, 331 (1986); T. Kennedy, E.H. Lieb, and B.S. Shastry, J. Stat. Phys., to be published.

[8] D.A. Huse and V. Elser, Phys. Rev. Lett., **60**, 2531 (1988); J.D. Reger and A.P. Young, Phys. Rev. **B37**, 5978 (1988).

[9] F.D.M. Haldane, Phys. Rev. Lett. **61**, 1029 (1988).

[10] I. Affleck, Nucl. Phys. **B257**, 397 (1985); Nucl. Phys. **B265**, 409 (1985).

[11] *Coherent States* by J.R. Klauder and B. Skagerstam, World Scientific, Singapore (1985).

[12] *Generalized Coherent States and Their Applications*, by A. Perelomov, Springer-Verlag, New York (1986).

[13] R. P. Feynman, Phys. Rev. **84**, 108 (1951).

[14] P.B. Wiegmann, Phys. Rev. Lett. **60**, 821 (1988).

[15] F.D.M. Haldane, Phys. Lett. **93A**, 464 (1983); Phys. Rev. Lett. **50**, 1153 (1983); and J. Appl. Phys. **57**, 3359 (1985).

[16] A.J. MacFarlane, Physics Letters **82B**, 239 (1979).

[17] S. Chakravarty, B.I. Halperin, and D.R. Nelson, Phys. Rev. Lett., **60**, 1057 (1988).

[18] E. Brezin, S. Hikami, and J. Zinn-Justin, Nucl. Phys. **B165**, 528 (1980).

[19] See *e.g. Solitons and Instantons*, by R. Rajaraman, North-Holland, Amsterdam (1982).

[20] I. Affleck, preprint.

[21] I. Affleck and F.D.M. Haldane, Phys. Rev **B36**, 5291 (1987)

[22] E. Witten, Nucl. Phys. **B149**, 285 (1979); S. Coleman, Ann. of Phys. **101**, 239 (1976).

[23] X.G. Wen and A. Zee, Phys. Rev. Lett., **61**, 1025 (1988); E. Fradkin and M. Stone, Phys. Rev. **B38**, 7215 (1988); T. Dombre and N. Read, Phys. Rev. **B38**, 7181 (1988); R. Shankar and S. Sachdev, unpublished.

[24] A. Belavin and A. Polyakov, JETP Lett., **22**, 245 (1975).

[25] A. D'Adda, P. Di Vecchia, and M. Luscher, Nucl. Phys. **B146**, 63 (1978)

[26] X.G. Wen preprint

[27] D.P. Arovas and A. Auerbach, Phys. Rev. **B38**, 316 (1988); Phys. Rev. Lett. **61**, 617 (1988).

[28] D. Yoshioka, preprint.

[29] B. Sutherland Phys. Rev. **B37**, 3786 (1988); S. Liang, B. Doucot, and P.W. Anderson, Phys. Rev. Lett. **61**, 365 (1988).

[30] N. Read and B. Chakraborty, preprint; H. Levine and F.D.M. Haldane, preprint.

[31] S. Kivelson, D.S. Rokhsar and J.P. Sethna, Phys. Rev. **B35**, 8865 (1987).

[32] I. Affleck, Phys. Rev. Lett. **54**, 966 (1985).

[33] A.M. Polyakov, Nucl. Phys. **B120**, 429 (1977); *Gauge Fields and Strings*, Harwood, New York (1987).

[34] I. Affleck, T. Kennedy, E.H. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987); D.Arovas, A. Auerbach and F.D.M. Haldane, Phys. Rev. Lett. **60**, 531 (1988).

[35] S.-T. Chui and J.D. Weeks, Phys. Rev. **B14**, 4978 (1976).

[36] J.M. Kosterlitz, J. Phys. **C10**, 3753 (1977).

[37] I. Affleck and J. Brad Marston, Phys. Rev. **B37**, 3774 (1988).

[38] D. Rokhsar and S. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988).

[39] E. Fradkin and S. Kivelson, private communication.

[40] S. Sachdev, submitted to Phys. Rev. B