Low energy theory of a single vortex and electronic quasiparticles in a d-wave superconductor

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Abstract

We highlight the properties of a simple model (contained in our recent work) of the quantum dynamics of a single point vortex interacting with the nodal fermionic quasiparticles of a d-wave superconductor. We describe the renormalization of the vortex motion by the quasiparticles: at $T = 0$, the quasiparticles renormalize the vortex mass and introduce only a weak sub-Ohmic damping. Ohmic (or 'Bardeen-Stephen' damping) appears at $T > 0$, with the damping co-efficient vanishing $\sim T^2$ with a universal prefactor. Conversely, quantum fluctuations of the vortex renormalize the quasiparticle spectrum. A point vortex oscillating in a harmonic pinning potential has no zero-bias peak in the electronic local density of states (LDOS), but has small satellite features at an energy determined by the pinning potential. These are proposed as the origin of sub-gap LDOS peaks observed in scanning tunneling microscopic studies of the LDOS near a vortex.

Key words: vortex, cuprates, Dirac quasiparticles

1 The Model

A more complete discussion of the quantum dynamics of a vortex and nodal quasiparticles in a two-dimensional d-wave superconductor appears in our recent papers [1,2,3] which also describe the connections to microscopic theory. Here, we focus on a simple low-energy model, which captures all the essential features. Our model has no zero-bias peak in the electronic LDOS at the vortex center (see also [4]), in contrast to computations in the traditional BCS framework [5,6,7]. The zero-point quantum motion of the vortex in the pinning

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potential leads to sub-gap satellite peaks in the LDOS. Finally, as discussed in earlier work [8] (and not reviewed here) a proper accounting of Aharanov-Bohm-like phases in the vortex motion leads to periodic spatial modulations in the LDOS (see also [9]). These features have the potential to explain the key characteristics of scanning tunneling microscopic (STM) observations of vortices in the cuprate superconductors [10,11,12].

The degrees of freedom of our model are (*i*) the low energy $S = 1/2$ fermionic quasiparticles in the vicinity of the nodes of a *d*-wave superconductor, described by the second-quantized Nambu spinor $\Psi(\boldsymbol{r})$, and (*ii*) a (point) vortex, described by its first-quantized position $\boldsymbol{r}_\mathrm{v}$ and canonically conjugate momentum $\boldsymbol{p}_\mathrm{v}$. In the absence of vortex pinning (considered later), the Hamiltonian has the very simple form ($\hbar = 1$):

$$H = \sum_{\mathrm{nodes}} \int \mathrm{d}^2 r \Psi^\dagger(\boldsymbol{r}) H_D(\boldsymbol{r}, \boldsymbol{r}_\mathrm{v}) \Psi(\boldsymbol{r}) \ , \tag{1}$$

where H_D is the Dirac Hamiltonian in the presence of a gauge field $\boldsymbol{a}$ [4]:

$$H_D = v_\mathrm{f} \left(\frac{\partial}{i \partial x} + a_x(\boldsymbol{r}, \boldsymbol{r}_\mathrm{v}) \right) \sigma^z + v_\Delta \left(\frac{\partial}{i \partial y} + a_y(\boldsymbol{r}, \boldsymbol{r}_\mathrm{v}) \right) \sigma^x \ . \tag{2}$$

Here $\hat{\boldsymbol{x}}$ denotes any of the four nodal directions in momentum space and $\hat{\boldsymbol{y}}$ the direction perpendicular to it, $\sigma^{x,y,z}$ are Pauli matrices, v_f and v_Δ are Fermi and gap velocities respectively, and the gauge field $\boldsymbol{a}$ is given by

$$\boldsymbol{a}(\boldsymbol{r}, \boldsymbol{r}_\mathrm{v}) = \frac{\hat{\boldsymbol{z}} \times (\boldsymbol{r} - \boldsymbol{r}_\mathrm{v})}{2 |\boldsymbol{r} - \boldsymbol{r}_\mathrm{v}|^2} \ , \tag{3}$$

so that it corresponds to a π-flux centered at the instantaneous vortex position $\boldsymbol{r}_\mathrm{v}$; in the Lagrangian formulation, Eq. (3) would appear as a Chern-Simons term. An assumption is that the vortex core size can be neglected in the cuprates, especially since there are no localized quasiparticle states in the *d*-wave vortex cores. The Hamiltonian (2) is derived from the full Bogoliubov-de Gennes Hamiltonian in the limit of an extreme type-II superconductor by application of the Franz-Tešanović unitary transformation [4].

We have not displayed the 'Doppler shift' terms in H_D: their influence is considered elsewhere [1], and they do not qualitatively modify the results below. Note also that a bare vortex Hamiltonian has not been explicitly included: the needed terms are generated by integrating out the quasiparticles.

2

2 Vortex dynamics in d-wave superconductors

First, we consider the influence of the nodal quasiparticles on the vortex dynamics. This has been previously studied in a semiclassical theory [13,14]. Our results differ from this semiclassical theory which, we suspect, does not properly capture the "quantum critical" aspects of the Dirac fermion dynamics.

Instead of specifying a vortex Hamiltonian H_v, we formulate an imaginary-time path-integral for quasiparticles coupled to a single vortex at the variable position $\boldsymbol{r}_v(\tau)$, and integrate out the fermionic quasiparticle fields. The result is an effective vortex action S_v that contains only the quasiparticle contribution to vortex dynamics. We express S_v in the frequency domain as an expansion in terms of the small vortex displacement from a fictitious origin, and calculate only the terms up to the quadratic order. Then, on general symmetry grounds we can write:

$$S_v = \frac{1}{\beta} \sum_\omega \left[F_\parallel(\omega) |\boldsymbol{r}_v(\omega)|^2 + F_\perp(\omega) i\hat{\boldsymbol{z}} (\boldsymbol{r}_v^*(\omega) \times (\boldsymbol{r}_v(\omega))) \right] , \qquad (4)$$

where $F_\parallel(\omega)$ captures the "longitudinal" dynamics, such as vortex friction and inertia, while $F_\perp(\omega)$ captures the "transversal" dynamics, such as the quasiparticle contributions to the Magnus force.

The idea behind expanding the vortex action in terms of small vortex displacements is to make use of the solutions for the quasiparticle eigenstates ψ_n and eigenvalues ϵ_n in presence of a *static* vortex. Various virtual transitions between these extended quasiparticle eigenstates, caused by the vortex motion, is what gives rise to renormalization of the parameters that characterize vortex dynamics. Thus, the contribution of quasiparticles to the vortex action at finite temperatures $T = 1/\beta$ is found to have the following form:

$$S_v = \frac{1}{2} \sum_{n,n'} \frac{1}{\beta} \sum_\omega \left(f(\epsilon_n) - f(\epsilon_{n'}) \right) \frac{i\omega(\epsilon_n - \epsilon_{n'})}{\epsilon_n - \epsilon_{n'} - i\omega}$$
$$\times |\boldsymbol{r}_v(\omega)\boldsymbol{U}_{n,n'}|^2 , \qquad (5)$$

where $f(\epsilon)$ is the Fermi-Dirac distribution function. The main problem is to calculate the transition matrix elements $\boldsymbol{U}_{n,n'} = \langle \psi_n | \boldsymbol{\nabla} | \psi_{n'} \rangle$, and carry out the summations over the quantum numbers n and n'. Using the microscopic model given by (1) and (2), we find the quasiparticle eigenstates in presence of a static vortex, and substitute them into (5). In comparison with (4) we obtain at small frequencies:

$$F_\parallel(\omega) = -\eta|\omega| + A_1 \omega^2 \ln(|\omega|) + \frac{m_v \omega^2}{2} + A_2 |\omega|^3 . \qquad (6)$$

3

At zero temperature the nodal quasiparticle contribution to the vortex mass is of the order of an electron mass:

$$m_{\mathrm{v}} \approx 0.05 \left(\frac{1}{v_{\mathrm{f}}^2} + \frac{1}{v_{\Delta}^2} \right) \Lambda \ , \tag{7}$$

where Λ is the high energy cutoff of the Dirac Hamiltonian, and there is no vortex friction ($\eta = 0$, $A_1 = 0$) apart from a universal sub-Ohmic damping ($A_2(v_{\Delta}/v_{\mathrm{f}}) \neq 0$). These key results can be understood by a simple scaling argument [1]—non-analytic terms can only arise from infrared singular terms, and by power-counting, the first infrared singularity arises only at order ω^3. Ohmic friction arises either at finite temperatures:

$$\eta = \frac{\pi}{6} \left(\frac{1}{v_{\mathrm{f}}^2} + \frac{1}{v_{\Delta}^2} \right) T^2 \tag{8}$$

$$A_1 = -\frac{\ln(2)}{4} \left(\frac{1}{v_{\mathrm{f}}^2} + \frac{1}{v_{\Delta}^2} \right) T \ln(T) \ ,$$

or in presence of perturbations that create a finite density of states at zero energy $\rho(0) \neq 0$ (the scale T above is replaced by a scale $\propto \rho(0)$, which could be, for example, the strength of disorder, or Zeeman splitting). In general, quasiparticles do not contribute transversal vortex dynamics ($F_{\perp} = 0$), unless $\rho(0) \neq 0$. When the Doppler shift is included in calculations, these conclusions remain qualitatively the same in terms of their dependence on T, $\rho(0)$ and $v_{\Delta}/v_{\mathrm{f}}$, but the numerical co-efficients change significantly.

Our findings are different from the semiclassical results in two important ways. First, we find a finite renormalization of the vortex mass due to quasiparticles, while the semiclassical approach predicts a mass that diverges in small magnetic fields as $H^{-1/2}$. Second, we find that quasiparticles cannot give rise to vortex friction despite their gapless spectrum, unless they are helped by a finite temperature or perturbations such as disorder. In contrast, the semiclassical approach predicts not only that friction is possible in the limit of infinite quasiparticle scattering time, but also that quasiparticles can produce transversal forces on the vortex. We hope that future experiments will resolve these discrepancies.

3   Quasiparticle spectra in the vicinity of a vortex

Now we turn to the converse problem of the influence of the vortex motion on the quasiparticle spectrum. From the results in Section 3, we know that the effective action for the vortex acquires a mass m_{v}, and that effects of damping are negligible. We are interested in the electronic spectrum near a localized

vortex, induced either by the repulsive interaction with other vortices, or by a pinning potential associated with disorder. In both cases, for small vortex displacement, it is reasonable to make a harmonic approximation, and so we use the Hamiltonian $H \to H + H_\mathrm{v}$ where

$$H_\mathrm{v} = \frac{\boldsymbol{p}_\mathrm{v}^2}{2m_\mathrm{v}} + \frac{1}{2}m_\mathrm{v}\omega_\mathrm{v}^2\boldsymbol{r}_\mathrm{v}^2 \tag{9}$$

The effective vortex mass m_v, and the harmonic trap frequency ω_v will be treated as given parameters, which can be determined microscopically [1,15].

We will apply several simplifications that amount to removing all sources of anisotropy in our model: we will set $v_\mathrm{f} = v_\Delta$, use a single vortex harmonic frequency, and completely neglect the Doppler shift. Such simplifications are, of course, not quantitatively justified in realistic circumstances, but allow gaining a deeper physical insight and nevertheless produce quasiparticle spectra that are remarkably similar to the ones observed in experiments.

3.1 Perturbation theory

We define the operators $b_\mu^\dagger$ and b_μ ($\mu \in \{x, y\}$ that raise and lower respectively the quantum numbers n_μ of the vortex in the harmonic trap. By inserting the resolution of unity in terms of the trapped vortex eigenmodes $|n_x, n_y\rangle$ we can write the Hamiltonian perturbatively as $H = H_0 + H_1 + \cdots$, where the unperturbed Hamiltonian includes effects of the vortex zero-point quantum fluctuations:

$$H_0 = \omega_\mathrm{v}b_\mu^\dagger b_\mu + \int \mathrm{d}^2r\Psi^\dagger V_0\Psi \quad , \tag{10}$$

while the perturbation describes resonant scattering of quasiparticles from the fluctuating vortex:

$$H_1 = \int \mathrm{d}^2r\Psi^\dagger \left(V^\mu b_\mu^\dagger + h.c.\right)\Psi \quad . \tag{11}$$

Here we have introduced the following matrix elements:

$$
\begin{aligned}
V_0(\boldsymbol{r}) &= \langle 0, 0|H_D(\boldsymbol{r})|0, 0\rangle \\
V^x(\boldsymbol{r}) &= \langle 1, 0|H_D(\boldsymbol{r})|0, 0\rangle \\
V^y(\boldsymbol{r}) &= \langle 0, 1|H_D(\boldsymbol{r})|0, 0\rangle \ .
\end{aligned}
\tag{12}
$$

The remaining terms in the full Hamiltonian correspond to scattering events in which the trapped vortex undergoes two or more virtual transitions between its discrete eigenmodes; such scattering processes are also generated in the perturbation theory, and their physical effects can be qualitatively obtained from (10) and (11) alone.

All calculations are performed numerically in the basis that diagonalizes (10). After all the simplifications, the quasiparticle eigenfunctions of H_0 are characterized by the following quantum numbers: "charge" $q = \pm 1$, angular momentum $l \in \mathbb{Z}$, and radial wavevector $k > 0$. Their spectrum $\epsilon = qk$ is gapless at the gap nodes, and there are no localized states in the vortex cores of pure d-wave superconductors. The perturbation (11) describes coupling of Dirac fermions to a single bosonic two-dimensional oscillator.

The quasiparticle LDOS is obtained from the quasiparticle Green's function $G_{l_1,k_1;l_2,k_2}(\omega)$ expressed in the spinor representation:

$$\rho(\epsilon, \boldsymbol{r}) = -\frac{1}{\pi} \text{sign}(\epsilon) \cdot \text{Im} \left\{ \sum_{l_1,l_2} \int dk_1 dk_2 \right. \tag{13}$$

$$\left. \text{Tr} \left[T_{l_1,k_1}(\boldsymbol{r}) G_{l_1,k_1;l_2,k_2}(\epsilon) T^{\dagger}_{l_2,k_2}(\boldsymbol{r}) \right] \right\} \ ,$$

with $T_{l,k}(\boldsymbol{r})$ being the "Fourier" weight that translates between the position and momentum representations. Perturbative expansion of the Green's function yields an expansion of the LDOS that satisfies the following scaling form:

$$\rho(\epsilon, r) = \frac{\omega_{\text{v}}}{\hbar v_{\text{f}}^2} \sum_{n=0}^{\infty} \alpha^{2n} F_n \left(\frac{\epsilon}{\hbar \omega_{\text{v}}}, \frac{\epsilon r}{\hbar v_{\text{f}}}; \alpha \right) \ . \tag{14}$$

The small parameter α is the ratio of the perturbation energy scale and the vortex harmonic frequency:

$$\alpha = \left(\frac{m_{\text{v}} v_{\text{f}}^2}{\hbar \omega_{\text{v}}} \right)^{\frac{1}{2}} \tag{15}$$

3.2   The quasiparticle LDOS

The resonant scattering of quasiparticles from the fluctuating vortex leads to interesting effects already at the one-loop level. The quasiparticle LDOS is plotted in the Figure 1 as a function of energy at gradually increasing distances from the vortex trap center. The resonant peak height is proportional to α^2 at the trap center, and disappears over a length-scale comparable to the extent of vortex zero-point oscillations $(m_{\text{v}}\omega_{\text{v}})^{-1/2}$. This peak naturally lies at a sub-gap energy whenever the length-scale set by the superconducting gap (comparable to the vortex core size) is smaller than the amplitude of vortex quantum oscillations. Despite the simplicity of our model, the calculated LDOS has the same qualitative features as the LDOS measured in the STM experiments [10,11,12].
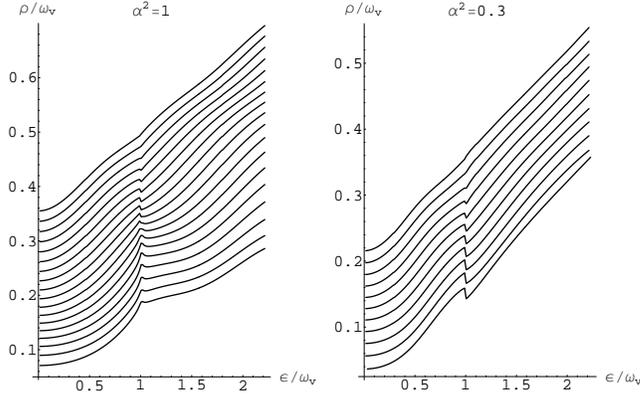
Fig. 1. Energy scans of the full LDOS at gradually increasing distances r from the vortex core. The plots are offset vertically for clarity, starting from $r = 0$ at the bottom, and moving up with increments $\Delta r = 0.2\omega_v^{-1}$ for $\alpha^2 = 1$ and $\Delta r = 0.33\omega_v^{-1}$ for $\alpha^2 = 0.3$ ($\Delta r\omega_v \approx (5\alpha)^{-1}$).

An important feature of the LDOS is the absence of a zero-energy peak appearing in earlier computations [5,6,7]. This appears to be a consequence of the smallness of the vortex core [16]. Even in our model, if the vortex core is larger than the spatial extent of vortex zero-point quantum fluctuations, the zero-energy peak appears in the quasiparticle LDOS [2]. Therefore, the STM experiments hint that it may not be the detailed structure of the vortex core that is crucial for shaping the quasiparticle spectra near the vortices in the cuprates; a viable alternative are quantum fluctuations of vortices.

The one-loop correction $\rho_1(\epsilon, r)$ of the LDOS is calculated from the Fock self-energy:

 $\qquad \rho_1(\epsilon, r) = \frac{\omega_v}{\hbar v_f^2}\alpha^2 F_1\left(\frac{\epsilon}{\hbar\omega_v}, \frac{\epsilon r}{\hbar v_f}; \alpha\right)$

where the wiggly line represents a vortex excitation, and the straight line a quasiparticle. $\rho_1(\epsilon, r)$ is shown in the Figure 2 in order to emphasize all of its features. In general, there is a discontinuity of LDOS at the energy $\epsilon = \omega_v$. A major peak appears at the energy $\epsilon \propto \alpha\omega_v$ when it does not interfere with the discontinuity. The additional secondary features of $\rho_1(\epsilon, r)$ are too small to be seen in the full quasiparticle LDOS. The main peak moves toward lower energies when the amplitude of vortex quantum fluctuations $\propto \alpha$ grows, indicating a short-lived resonant bound state between the vortex and a quasiparticle.

Future even more sensitive experiments may provide more ways to detect consequences of vortex quantum motion in d-wave superconductors. The spatial extent of the LDOS modulations allows measuring the energy scale $(\hbar\omega_v m_v v_f^2)^{1/2}$, while resolving the thermally blurred discontinuities of the LDOS would allow
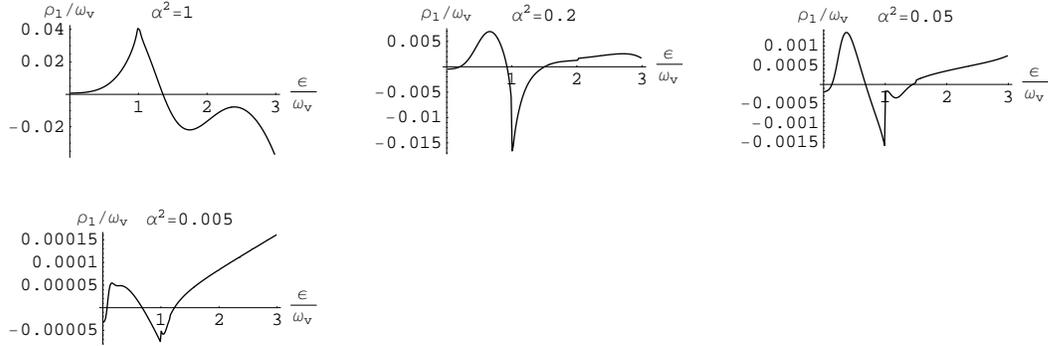
Fig. 2. The one-loop correction ρ_1 to LDOS at the vortex center as a function of energy. These plots show evolution of $\rho_1(\epsilon)$ as the small parameter α^2 changes. The magnitude of ρ_1 scales as α^2.

direct measurement of $\hbar\omega_v$. Combined knowledge of both energy scales reveals the vortex mass m_v and the strength of the vortex trapping force (inter-vortex interactions), which in turn can be related to other parameters, such as the superconducting gap and magnetic field. Effects of the moderately strong Magnus force merely reduce to quantitative modifications of these energy scales.

In closing, we reiterate that the above model can also explain [8] periodic LDOS modulations observed in recent STM experiments [11,12]. Such modulations appear in the LDOS over the region of vortex motion, and consequently the experiments lead to an estimate of m_v [15].

4   Conclusions

Contribution of the nodal quasiparticles to the vortex mass is of the order of an electron mass. Vortex friction arises only at finite temperatures, or in presence of perturbations that create a finite density of states at zero energy, such as disorder. Similarly, quasiparticles whose density of states vanishes at zero energy do not contribute any transversal forces on the vortex. Smallness of vortex friction and mass at low temperatures have important implications for the flux-flow in the "normal state" of d-wave superconductors.

The quasiparticle LDOS in the vicinity of a localized quantum vortex shows remarkable similarity to the experimental STM observations. Absence of a zero-energy LDOS peak is an indication of the small vortex core. Small sub-gap peaks appear in the LDOS as a result of resonant scattering of quasiparticles from the quantum vortex. Some secondary weak features also appear in the LDOS, including discontinuities at energies that reflect the discrete spectrum of the localized vortex. The future STM experiments might be sensitive enough

to detect these features, and allow measurements of the vortex mass and vortex trapping forces.

5   Acknowledgments

References

[1] P. Nikolić and S. Sachdev, Phys. Rev. B **73**, 134511 (2006).

[2] L. Bartosch and S. Sachdev, cond-mat/0604105.

[3] P. Nikolić, S. Sachdev and L. Bartosch, cond-mat/0606001.

[4] A. Melikyan and Z. Tešanović, cond-mat/0605314.

[5] Y. Wang and A. H. MacDonald, Phys. Rev. B **52**, R 3876 (1995).

[6] M. Ichioka, N. Hayashi, N. Enomoto, and K. Machida, Phys. Rev. B **53**, 15316 (1996).

[7] M. Franz and Z. Tešanović, Phys. Rev. Lett. **80**, 4763 (1998).

[8] L. Balents, L. Bartosch, A. Burkov, S. Sachdev, and K. Sengupta, Phys. Rev. B **71**, 144508 (2005).

[9] Z. Tešanović, Phys. Rev. Lett. **93**, 217004 (2004); A. Melikyan and Z. Tešanović, Phys. Rev. B **71**, 214511 (2005).

[10] I. Maggio-Aprile, Ch. Renner, A. Erb, E. Walker and Ø. Fischer, Phys. Rev. Lett. **75**, 2754 (1995).

[11] J. E. Hoffman, E. W. Hudson, K. M. Lang, V. Madhavan, S. H. Pan, H. Eisaki, S. Uchida, and J. C. Davis, Science **295**, 466 (2002).

[12] G. Levy, M. Kugler, A. A. Manuel, Ø. Fischer, and M. Li, Phys. Rev. Lett. **95**, 257005 (2005).

[13] G. E. Volovik, Pisma Zh. Eksp. Teor. Fiz. **65** 201 (1997) [JETP Lett. **65** 217 (1997)].

[14] N. B. Kopnin, *Theory of Nonequilibrium Superconductivity*, Cambridge University Press, Cambridge (2001).

[15] L. Bartosch, L. Balents, and S. Sachdev, Annals of Physics **321**, 1528 (2006).

[16] H. Tsuchiura, M. Ogata, Y. Tanaka, and S. Kashiwaya, Phys. Rev. B **68**, 012509 (2003).