A quantum critical trio:
solvable models of finite temperature crossovers near quantum phase transitions

Subir Sachdev
Department of Physics, P.O. Box 208120, Yale University, New Haven, CT 06520-8120
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Abstract

The physics of three simple solvable quantum models is reviewed in some detail: the spin 1/2 XX chain (and the related dilute spinless Fermi gas), the Ising chain in a transverse field, and the large N limit of the $O(N)$ quantum rotor model. A unified scaling description is presented, and the many common features among the models are highlighted. The crossovers in these systems are the simplest paradigms of finite temperature physics near quantum critical points.

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I. INTRODUCTION

Consider a quantum system on an infinite lattice described by the Hamiltonian $\mathcal{H}(g)$, with g a dimensionless coupling constant. For any reasonable g, all observable properties of the *ground state* of $\mathcal{H}$ will vary smoothly as g is varied. However, there may be special points, like $g = g_c$, where there is a non-analyticity in some property of the ground state: we identify g_c as the position of a quantum phase transition. In finite lattices, non-analyticities can only occur at level crossings; the possibilities in infinite systems are richer as avoided level crossings can become sharp in the thermodynamic limit. In this paper, I will restrict my discussion to second order quantum transitions, or transitions in which the length and time scales over which the degrees of freedom are correlated diverge as g approaches g_c. As I will discuss below, any such quantum transition can be used to define a continuum quantum field theory (CQFT): the CQFT has no intrinsic short-distance (or ultraviolet) cutoff. The main purpose of this paper is to discuss some properties of $\mathcal{H}(g)$ at *finite temperatures* (T) in the vicinity of $g = g_c$, in the context of three simple solvable, but illustrative, models. These studies are equivalent to a determination of the finite T crossovers of the associated CQFTs.

We begin by stating some basic concepts on the relationship between quantum critical points and CQFT's [1–3]; these will also be discussed in more detail in our explicit study of the three models. As correlations become long range in time in the vicinity of the critical point, every system must be characterized by an experimentally measureable energy scale, Δ which vanishes at $g = g_c$. Convenient choices are an energy gap, if one exists, or a stiffness of an ordered phase to changes in the orientation of an order parameter; we will meet several explicit examples later. (More precisely, there are two energy scales Δ_+, Δ_- corresponding to the phases with $g > g_c$, $g < g_c$.) It should be emphasizes that Δ is a dimensionful parameter, expressed in the laboratory units of energy, and directly measurable in an experiment. In all the models we shall consider here, Δ vanishes as a power-law as g approaches g_c:

$$\Delta \sim \Lambda |g - g_c|^{z\nu} \tag{1.1}$$

where Λ is an ultraviolet cutoff, measured for convenience in the units of energy too, and $z\nu > 0$ is a critical exponent; this exponent is intimately related to the behavior of the system under rescaling transformations, as will be discussed later. From the perspective of a field theorist, the CQFT associated with the quantum critical point is now defined by taking the limit $\Lambda \to \infty$ at fixed Δ; from (1.1) we see that, because $z\nu > 0$, it is possible to take this limit by tuning the bare coupling g closer and closer to the critical point as Λ increases. (A condensed matter physicist would take the complementary, but equivalent, perspective of keeping Λ fixed but moving closer to criticality by lowering his probe frequency $\omega \sim \Delta$). Assuming the $\Lambda \to \infty$ limit exits, the resulting CQFT then contains only the energy scale Δ. At finite temperatures, there is a second energy scale $k_B T$; its thermodynamic properties will then be a universal function of the only dimensionless ratio available—$\Delta / k_B T$. This paper will describe the physical properties of these universal functions in some detail for three solvable models.

Our study will find two regimes with very different physical properties, as sketched in Fig 1:

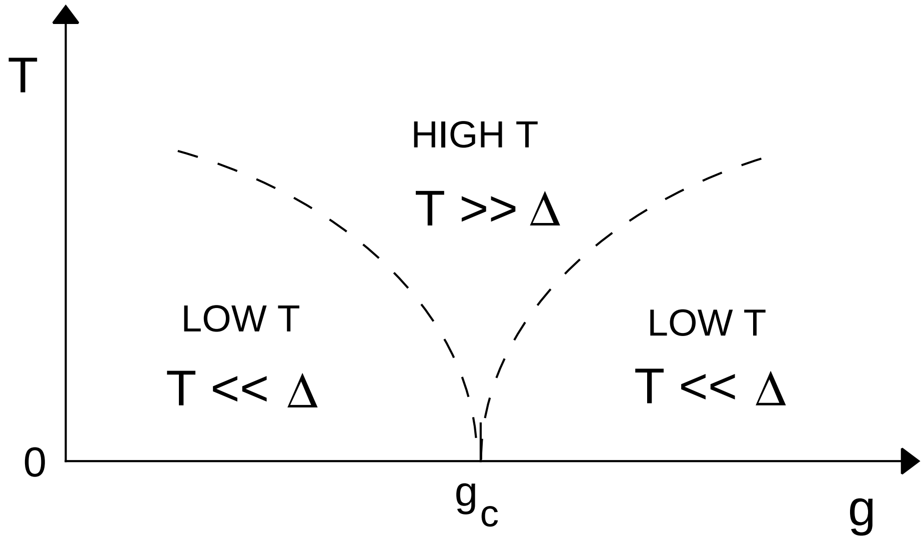

FIG. 1. Schematic phase diagram as a function of the coupling constant of the quantum Hamiltonian g, and the temperature T. The quantum critical point is at $T = 0$, $g = g_c$. The dashed lines indicate crossovers. There may finite temperature phase transitions in either of the two low temperature regimes. The symbol Δ represents a characteristic energy scale of the $T = 0$ theory which vanishes at $g = g_c$ according to 1.1. We will sometime use the symbols Δ_+, Δ_- to distinguish the ground states on either side of the $g = g_c$ point.

(i) The low temperature region $k_B T \ll \Delta$
There are actually two regimes of this type, one on either side of g_c. Correlations in this region are similar to those of the $T = 0$ ground state. The low temperature creates a small density of excitations which can sometimes have significant effects at very long scales.
(ii) The high temperature region $\Delta \ll k_B T$
As we are discussing universal properties of the CQFT, it is implicitly assumed that all energy scales, including $k_B T$, are smaller than the upper cutoff Λ which has been sent to infinity; so we also require that $k_B T \ll \Lambda$. The thermal energy, $k_B T$, sets the scale for all physical phenomena in this region, and the system behaves as if it's couplings are at the $g = g_c$ critical point. We shall devote much attention to the unfamiliar and unusual properties of this region. It is perhaps worth noting explicitly why the high T limit of the CQFT can be non-trivial. A conventional high T expansions of the lattice model $\mathcal{H}$ proceeds with the series

$$\mathrm{Tr} e^{-\mathcal{H}/k_B T} = \mathrm{Tr} 1 - \frac{1}{k_B T} \mathrm{Tr} \mathcal{H} + \frac{1}{2(k_B T)^2} \mathrm{Tr} \mathcal{H}^2 + \dots \tag{1.2}$$

The successive terms in this series are well-defined and finite because of the ultraviolet cutoffs provided by the lattice. Further, the series is well-behaved provided T is larger than all other energy scales; in particular we need $k_B T \gg \Lambda$. In contrast, the CQFT was defined by the limit $\Lambda \to \infty$ at fixed $k_B T$, Δ, and, as already stated, the high T limit of the CQFT corresponds to the intermediate temperature range $\Delta \ll k_B T \ll \Lambda$ of the lattice model. It is not possible to access this temperature range by an expansion as simple as (1.2), and more sophisticated techniques, to be discussed here, are necessary.

The three solvable models to be considered are described in the following three sections. In Section II will consider a simple spin chain with nearest neighbor exchange acting equally on only two spin components: this model is also known as the XX chain. We will decribe its relationship to the one-dimensional spinless Fermi gas, and then study the quantum phase transition in an arbitrary dilute spinless Fermi gas in general spatial dimension d. Section III will consider the second solvable model, the Ising chain in a transverse field, which is also in $d = 1$. Section IV will consider quantum transitions solely in $d > 1$: we will describe the $O(N)$ quantum rotor model, which is solvable in the $N = \infty$ limit.

II. THE QUANTUM XX CHAIN AND THE DILUTE SPINLESS FERMI GAS

We will begin our study of quantum phase transitions by examining a simple class of models, most of whose critical properties can be exactly determined. As will become clear, in many ways these models realize the "simplest" quantum phase transition; nevertheless, many of their properties are quite non-trivial.

The main model of interest here is the "XX chain", which is a particular Hamiltonian describing the interactions of spin-1/2 degrees of freedom in one dimension. As shown below, it can be solved by an exact mapping to a gas of non-interacting spinless fermions in one dimension. This will lead us to a study of a quantum phase transition exhibited by dilute spinless fermions in d dimensions [3]: its critical properties can determined exactly for *all* d. We emphasize that the two problems being studied here, the XX chain and the dilute spinless Fermi gas, are related only in $d = 1$; the critical properties of the XX model for $d > 1$ are different, and will not be discussed here.

A. The quantum XX chain

The quantum XX chain is described by the Hamiltonian

$$H_{XX} = -J \sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right) - h \sum_i \sigma_i^z \tag{2.1}$$

where σ_i^α ($\alpha = x, y, z$) are Pauli matrices describing spin-1/2 degrees of freedom on the sites, i, of an infinite chain. The spins are placed in a uniform magnetic field, $h > 0$, which couples to their z component, and interact with a ferromagnetic exchange, $J > 0$, which couples only their x and y components. This peculiar interaction is designed to ensure solvability of the model; later, we will drop the restriction on the absence of an exchange coupling between z components, and discuss universal properties of a general class of models.

1. Classical Limit

It is useful to begin by examining H_{XX} in the classical limit, in which the σ_i^α are treated as commuting vectors of unit length ($\sum_\alpha \sigma_i^{\alpha 2} = 1$); this will give us a feel for the possible ground states. At zero temperature in such a limit, there are neither quantum nor thermal fluctuations, and the ground state is specified by the static orientations of the spins on each site. As there is no frustration in H_{XX}, it is clear that the lowest energy configuration has all the spins oriented in the same direction, and the ground state always has ferromagnetic long-range order. Let the optimum spin orientation be

$$\vec{\sigma} = (\sin\theta, 0, \cos\theta), \tag{2.2}$$

where we have used the rotation invariance in the x,y plane to set the y-component to zero, and recall that we are momentarily treating the $\vec{\sigma}$ as commuting variables. The ground state energy is then

$$E_{XX} = -N(J\sin^2\theta + h\cos\theta) \tag{2.3}$$

where N is the number of sites. This is easily minimized at $\theta = \theta_0$, and gives for the expectation value of the z polarization

$$\langle\sigma^z\rangle = \cos\theta_0 = \begin{cases} h/2J & h \le 2J \\ 1 & h \ge 2J \end{cases} . \tag{2.4}$$

Thus the spins lie in the x,y plane at $h = 0$, gradually tilt upwards into the z direction with increasing field, until they lock onto a complete z polarization for $h \ge 2J$. Ground state properties of the system, when considered as functions of h at fixed J, have a *non-analyticity* at $h = 2J$. This signals the occurence of a critical point with a phase transition. As we will see shortly, the exact solution of the full quantum theory of H_{XX} has a closely related quantum phase transition, whose critical singularities are however different from those of the classical model.

2. Solution of quantum model

The essential tool in the solution of the spin-1/2 model H_{XX} is the Jordan-Wigner transformation [4]. This is a very powerful mapping between models with spin-1/2 degrees of freedom and spinless fermions. The central observation is that there is a simple mapping between the Hilbert space of a system with a spin-1/2 degree of freedom per site, and that of spinless fermions hopping between sites with single orbitals. We may associate the spin up state with an empty orbital on the site, and a spin-down state with an occupied orbital. If the canonical fermion operator c_i annhilates a spinless fermion on site i, then this simple mapping immediately implies the operator relation

$$\sigma_i^z = 1 - 2c_i^\dagger c_i \tag{2.5}$$

It is also clear that the operation of c_i is equivalent to flipping the spin from down to up, or the operation of $\sigma_i^+ = (\sigma_i^x + i\sigma_i^y)/2$; similar creating a fermion by $c_i^\dagger$ is equivalent to lowering the spin by $\sigma_i^- = (\sigma_i^x - i\sigma_i^y)/2$. While this equivalence works for a single site, we cannot yet equate the fermion operators with the corresponding spin operators for the many site problem; this is because while two fermionic operators on different sites anticommute, two spin operators commute. The solution to this dilemma was found by Jordan and Wigner, who showed that the following representation satisfied both on-site and inter-site (anti)commutation relations:

$$
\begin{aligned}
\sigma_i^+ &= \prod_{j<i} \left(1 - 2c_j^\dagger c_j\right) c_i \\
\sigma_i^- &= \prod_{j<i} \left(1 - 2c_j^\dagger c_j\right) c_i^\dagger .
\end{aligned} \tag{2.6}
$$

The naive single-site correspondence has been modified by a 'string' of operators, whose value is $+1$ (-1) if the total number of fermions on the sites to the left of site i are even (odd). Notice that the spin operators have a highly non-local representation in terms of the fermion operators. This feature is also found in the inverse of (2.6)

$$
\begin{aligned}
c_i &= \left(\prod_{j<i} \sigma_j^z\right) \sigma_i^+ \\
c_i^\dagger &= \left(\prod_{j<i} \sigma_j^z\right) \sigma_i^- .
\end{aligned} \tag{2.7}
$$

It can now be verified that (2.5,2.6,2.7) are consistent with the relations

$$
\begin{aligned}
&\left\{c_i, c_j^\dagger\right\} = \delta_{ij} \qquad \{c_i, c_j\} = \left\{c_i^\dagger, c_j^\dagger\right\} = 0 \\
&\left[\sigma_i^+, \sigma_j^-\right] = \delta_{ij}\sigma_i^z \qquad \left[\sigma_i^z, \sigma_j^\pm\right] = \pm 2\delta_{ij}\sigma_i^\pm ,
\end{aligned} \tag{2.8}
$$

where the curly brackets represent anticommutators, and square brackets are commutators.

Let us now perform the Jordan-Wigner transformation on the Hamiltonian H_{XX}. Inserting (2.5,2.6) into (2.1), we get

$$
H_{XX} = 2\sum_i \left(-J(c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}) + h c_i^\dagger c_i\right) - Nh \tag{2.9}
$$

where N is the number of sites. Notice that H_{XX} is simply a free spinless fermion Hamiltonian and its spectrum can therefore be easily determined; indeed, the original form of H_{XX} was carefully chosen to ensure this solvability. The Jordan-Wigner transformation on a general spin Hamiltonian will also lead to terms describing interactions among fermions, which usually makes exact determination of the spectrum impossible; we will discuss the consequences of such terms later. The Hamiltonian H_{XX} is diagonalized by transforming to fermions, c_k, with momentum k

$$
c_k = \frac{1}{\sqrt{N}} \sum_i c_i e^{-ikr_i}, \tag{2.10}
$$

where r_i is the spatial coordinate of site i, and H_{XX} becomes

$$
H_{XX} = \sum_k \left(2h - 4J\cos(ka)\right) c_k^\dagger c_k - Nh, \tag{2.11}
$$

where a is the lattice spacing. The ground state of H_{XX} is obtained by filling all fermion states with negative single particle energy. For $h > 2J$, all single particle states have positive energy, so the ground state is the empty state with no fermions. For $h < 2J$ the fermions

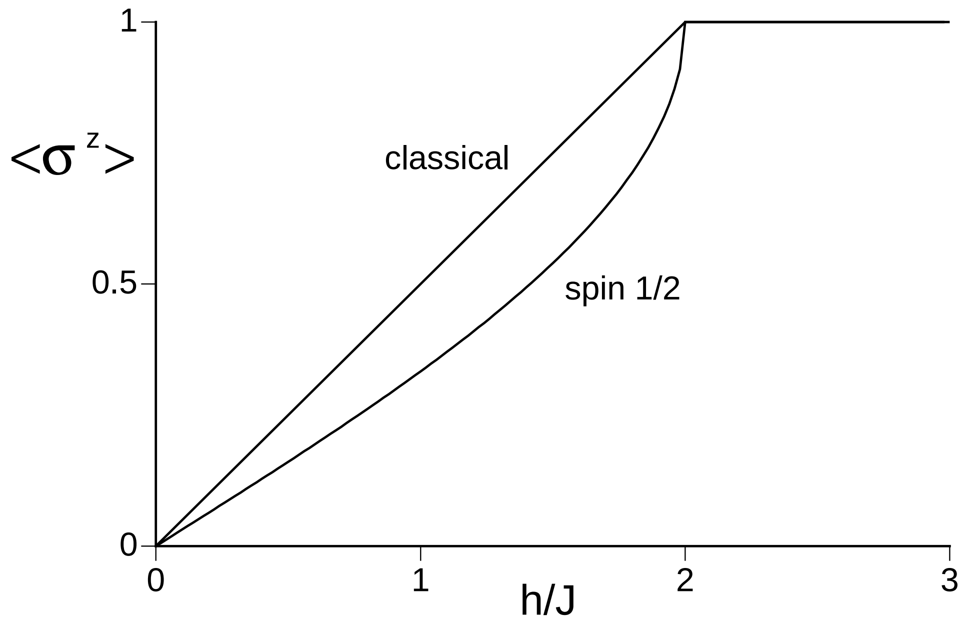

FIG. 2. The $T = 0$ value of $\langle\sigma^z\rangle$ for the Hamiltonian H_{XX} for the classical and spin-1/2 cases. We have $1 - \langle\sigma^z\rangle \sim (1 - h/2J)^{1/2}$ near the $h = 2J$ critical point for all spins less than infinity.

occupy states with momenta, k, which satisfy $2h - 4J\cos(ka) < 0$. Using (2.5), we can easily obtain now the exact result for the $T = 0$ expectation value of σ^z:

$$\langle\sigma^z\rangle = \begin{cases} 1 - (2/\pi)\cos^{-1}(h/2J) & h \leq 2J \\ 1 & h \geq 2J \end{cases} . \tag{2.12}$$

Compare this result with the classical result (2.4), as has been done in Fig 2. There is now a quantum phase transition at $h = 2J$ where the spins first move away from a saturated polarization along the z direction. For h close to $2J$, we now find for the deviation from saturated polarization that $1 - \langle\sigma^z\rangle \sim (1 - h/2J)^{1/2}$, in contrast to the linear dependence $\sim (1 - h/2J)$ in the classical model. We identify the power of 1/2 as a critical exponent of the quantum critical point at $h = 2J$.

Having identified a quantum critical point and determined one of its exponents in the Hamiltonian H_{XX}, it is now natural to ask how general these results are. In particular, we would like to know if the critical exponent would be modified by exchange interactions among the z components, or by non-nearest neighbor interactions. These questions are addressed in the next section, where we study the vicinity of the critical point in more detail by formulating the appropriate continuum theory.

B. The dilute spinless Fermi gas

This section will focus on spinless fermion models described by generalizations of (2.9). We will temporarily not make any reference to the spin system, but instead discuss properties of the quantum phase transition using the fermionic degrees of freedom. We will find it useful to consider the generalization of the transition to spinless fermions moving in an arbitrary number (d) of spatial dimensions [3]; it must be emphasized, however, that the mapping of spinless fermions to the spin chain is valid only in $d = 1$.

Notice that near the quantum critical point $h = 2J$ the fermions only occupy states with momenta near $k = 0$. This correctly suggests that a naive long wavelength expansion in spatial gradients will yield the continuum theory characterizing the critical point. We introduce the continuum spinless fermion field $\Psi_F(x)$ by

$$\Psi(x_i) = a^{-d/2} c_i. \tag{2.13}$$

The factor of $a^{-d/2}$ is in a form involving arbitrary d dimensions, anticipating our discussion of spinless fermions in d dimenions; it ensures that the anticommutation relation

$$\left\{\Psi(x), \Psi^\dagger(x')\right\} = \delta^d(x - x') \tag{2.14}$$

is satisfied, where the right hand side is a Dirac delta function in $d = 1$ spatial dimensions. We insert (2.13) into (2.9), perform an expansion in spatial gradients, and obtain the continuum fermion Hamiltonian

$$H_F = E_0 + \int d^d x \left(-\frac{\hbar^2}{2m} \Psi^\dagger(x) \nabla^2 \Psi(x) - \mu \Psi^\dagger(x) \Psi(x) \right) \tag{2.15}$$

where $E_0 = -Nh$ is the ground state energy, the chemical potential $\mu = -2h + 4J$, and the fermion mass $m = \hbar^2/(4Ja^2)$ in the mapping to the $d = 1$ model (2.9). Notice that the quantum critical point is now at $\mu = 0$. The deviation from the saturated magnetization of the quantum XX chain, $1 - \langle \sigma^z \rangle$ is proportional to the fermion density $\langle \Psi^\dagger \Psi \rangle$. At $T = 0$, this is non-zero only for $\mu > 0$, when the fermions all states with momenta less than the Fermi momentum $k_F = (2m\mu/\hbar^2)^{1/2}$. The fermion density is therefore

$$\langle \Psi^\dagger \Psi \rangle = \begin{cases} \mathcal{C}_d (2m\mu/\hbar^2)^{d/2} & \mu > 0 \\ 0 & \mu < 0 \end{cases}, \tag{2.16}$$

where $\mathcal{C}_d$ is a dimensionless *universal* number given by $\mathcal{C}_d =$. In $d = 1$, this result is clearly the continuum limit of (2.12), and the sense in which it is universal will be made clearer below.

It is convenient to perform our subsequent scaling analysis in a Lagrangean path integral representation of the dynamic of H_F. Using the standard Grassman path integral of canonical Fermi operators we obtain for the partition function $Z = \mathrm{Tr} e^{-H_F/k_B T}$

$$Z = \int \mathcal{D}\Psi \mathcal{D}\Psi^\dagger \exp\left(-\frac{1}{\hbar} \int_0^{\hbar/k_B T} d\tau d^d x \mathcal{L}_0 \right) \tag{2.17}$$

where the functional integral is over complex Grassman fields Ψ, $\Psi^\dagger$ in space (x) and imaginary time (τ), and the Lagrangean density $\mathcal{L}_0$ is

$$\mathcal{L}_0 = -\hbar \Psi^\dagger \frac{\partial \Psi}{\partial \tau} + \frac{\hbar^2}{2m} \Psi^\dagger \nabla^2 \Psi + \mu \Psi^\dagger \Psi, \tag{2.18}$$

The temperature T appears only in the length $(\hbar/k_B T)$ of the imaginary time interval, and the Grassman fields satisfy antiperiodic temporal boundary conditions $\Psi(x, \tau = \hbar/k_B T) = -\Psi(x, 0)$, and similarly for $\Psi^\dagger$.

The quantum critical point of $\mathcal{L}_0$ is at $\mu = 0$, $T = 0$. The central idea behind our analysis of this critical point is the examination of the behavior of $\mathcal{L}_0$ under a rescaling of length and time scales. In particular, let us coarse-grain the system and express its properties in terms of new length (x') and imaginary time (τ) co-ordinates [5]

$$x' = xe^{-\ell} \qquad\qquad \tau' = \tau e^{-z\ell} \tag{2.19}$$

Here $e^{-\ell}$ is the rescaling factor of length scales, and z is known as the *dynamic critical exponent* which determines the relative rescaling of time scales. The value of z will be chosen by us to ensure the scaling invariance of the quantum critical point. Now notice that if we rescale the field Ψ by

$$\Psi'(x', \tau') = e^{d\ell/2}\Psi(x, \tau), \tag{2.20}$$

and choose

$$z = 2, \tag{2.21}$$

then at $\mu = 0$, $\mathcal{L}_0$ is scale invariant; in other words it has the same form in both the primed and unprimed variables. The mass m does not change during this scale transformation: it merely plays the role of fixing the relative units of measurement of time and spatial scales.

This scale invariance has strong consequences for correlators of $\mathcal{L}_0$ which must respect the constraints imposed by (2.19) and (2.20). Although there are no particles in the ground state at $\mu = 0$, many of the correlators of $\mathcal{L}_0$ are non-zero; for instance we have at $T = \mu = 0$

$$\begin{aligned} G(x, -i\tau) \equiv \left\langle \mathcal{T}\Psi(x, \tau)\Psi^\dagger(0, 0)\right\rangle &= \int \frac{d\omega}{2\pi}\int \frac{d^d k}{(2\pi)^d}\frac{-e^{i(kx-\omega\tau)}}{i\omega - \hbar^2 k^2/(2m)} \\ &= \theta(\tau)\left(\frac{m}{2\pi\hbar\tau}\right)^{d/2}\exp\left(-\frac{mx^2}{2\hbar\tau}\right) \end{aligned} \tag{2.22}$$

where $\mathcal{T}$ is the time-ordering symbol. We have written the temporal argument of the Green's function G as $-i\tau$ to emphasize that it is a correlator in imaginary time. When the above correlator is interpreted as that of Grassmann fields under the functional integral (2.17), the $\mathcal{T}$ symbol can be ignored–it is however required when the Ψ's are treated as operators on the Hilbert space of spinless fermions. The function $\theta(\tau)$ is a step function, and so G vanishes for $\tau < 0$. Physically, (2.22) describes the quantum motion of a free particle, whose propagator has the form of the probability distribution of a diffusing particle in imaginary time.

The invariance of $\mathcal{L}_0$ at $\mu = 0$ under the scaling transformation implies that (2.22) also holds in terms of the primed variables. It can now be checked that these two forms of (2.22) are consistent with (2.19) and (2.20). Indeed the prefactor of the power of τ appearing in (2.22) is the only one consistent with the scaling transformations, and therefore could have been guessed a priori. The exponential depends only upon the combination x^2/τ which is invariant under the scaling transformation, which would also have been consistent with any function of this combination. The step function $\theta(\tau)$ is invariant under arbitrary rescalings of τ and its appearance is only determinable by an explicit calculation.

Let us now move away from the critical point $\mu = 0$. Under the rescaling (2.19,2.20) the action $\mathcal{L}_0$ remains invariant only if we introduce a new chemical potential μ',

$$\mu' = \mu e^{2\ell}. \tag{2.23}$$

Unlike at the critical point, it is now necessary to redefine a coupling constant in $\mathcal{L}_0$ to understand its behavior under scaling transformations. So at a fixed $\mu \neq 0$, the correlators of $\mathcal{L}_0$ are not scale invariant. Nevertheless, the simple behavior of μ under the rescaling transformation does place constraints on the allowed form of its correlators. We also find it useful to consider the consequences of repeated scaling transformations, in which case it is useful to define an ℓ-dependent $\mu(\ell)$ which now satisfies the differential equation

$$\frac{d\mu}{d\ell} = 2\mu \tag{2.24}$$

We see that μ grows indefinitely as one transforms to larger scales (larger ℓ), and such perturbations away from the scale-invariant quantum critical point are known as *relevant* perturbations. It is clear that they destroy scale-invariance at the largest scales and therefore must be included in any theory of the system. The parameter μ thus plays the role of the coupling g in the general discussion of Section I.

This is a convenient place to introduce the concept of the a **scaling dimension** of a coupling constant. This is simply the power to which the length rescaling factor e^{ℓ} must to raised to obtain its scaling transformation. We will denote the scaling dimension of μ by $\dim[\mu]$, and so clearly,

$$\dim[\mu] = 2 \tag{2.25}$$

It is conventional to define the exponent ν as the inverse of the scaling dimension of the most relevant perturbation about a quantum critical point; in the present case, this will turn out to be μ, and so

$$\nu = 1/2 \tag{2.26}$$

We can also talk about the scaling dimension of an operator, and clearly from (2.20) we have

$$\dim[\Psi] = d/2. \tag{2.27}$$

Finally, we may talk of scaling dimensions of space and time themselves, which are clearly

$$\dim[x] = -1 \qquad \dim[\tau] = -z \tag{2.28}$$

The knowledge of scaling dimensions allows to predict the dependences of correlators on coupling constants or space-time co-ordinates. For instance, the power of τ in (2.22) comes simply from the requirement that the right hand side have a scaling dimension equal to twice that of Ψ. Similarly, the power of μ in (2.16) follows immediately from the values of $\dim[\Psi]$ and $\dim[\mu]$.

Let us also consider the scaling dimension of the free energy density $\mathcal{F}$ of the system. From all subsequent results for $\mathcal{F}$, we will subtract out the ground state energy density at $\mu = 0$; this is always a non-universal number and not dominated by the effects of critical fluctuations (for the $d = 1$ XX chain it equals h/a, while for the continuum spinless Fermi gas it has implicitly been set equal to zero). The free energy density has dimensions of $(\text{length})^{-d}(\text{time})^{-1}$ and so

$$\dim[\mathcal{F}] = d + z \tag{2.29}$$

For $T = 0$, $\mu < 0$, there are no particles present, and so we must have $\mathcal{F} = 0$. However, for $T = 0$, $\mu > 0$, there is non-zero ground state energy, and using the scaling dimensions of $\mathcal{F}$ and μ we expect

$$\mathcal{F} \sim \mu^{(d+z)/2}\theta(\mu) \qquad \text{for } T = 0 \tag{2.30}$$

Indeed, at $T = 0$, the ground state energy can be obtained by adding up the kinetic energy of all the particles in the occupied states with $k < k_F$, and we get

$$\begin{aligned}\mathcal{F} &= \int_{k<k_F} \frac{d^d k}{(2\pi)^d} \frac{\hbar^2 k^2}{2m} \\ &= \frac{2}{(4\pi)^{d/2}(d+2)\Gamma(d/2)} \left(\frac{2m}{\hbar^2}\right)^{d/2} \mu^{(d+2)/2}\theta(\mu) \qquad \text{for } T = 0\end{aligned} \tag{2.31}$$

in agreement with the predictions of the scaling analysis.

We have so far been considering quite a trivial non-interacting fermion problem, and this elaborate machinery of scaling transformation might seem like overkill to the reader. However, we will now address the question of the robustness of the above results to perturbations with interactions to $\mathcal{L}_0$, and show that the above formalism allows us to rapidly, and simply, deduce their consequences.

Let us examine the simplest two-body interaction term that can be added to $\mathcal{L}_0$. A contact interaction like $\int dx (\Psi^\dagger(x)\Psi(x))^2$ vanishes because of the fermion anti-commutation relation, and simplest allowed term is

$$\mathcal{L}_1 = \lambda \left(\Psi^\dagger(x,\tau)\nabla\Psi^\dagger(x,\tau)\Psi(x,\tau)\nabla\Psi(x,\tau)\right) \tag{2.32}$$

where λ is a coupling constant measuring the strength of the interaction. Under the scaling transformation (2.19,2.20) we now find that $\lambda' = e^{-d\ell}\lambda$, or, equivalently

$$\frac{d\lambda}{d\ell} = -d\lambda \qquad \dim[\lambda] = -d \tag{2.33}$$

Unlike μ, we now find that the coupling λ flows to 0 under repeated scaling transformations. This is a property of couplings with negative scaling dimensions, or *irrelevant couplings*, the name suggesting their unimportance for long distance or low energy properties. The analysis of quantum critical points is usually facilitated by setting such couplings to zero at an early stage, and will do this often. Here, let us discuss the consequences of λ a little more explicitly. It is useful to consider, for example, the effect λ would have on the result (2.16). Knowledge of the scaling dimensions allows us to deduce

$$\langle \Psi^\dagger \Psi \rangle \sim \theta(\mu) \mu^{d/2} (1 + c\lambda \mu^{d/2} + \dots) \tag{2.34}$$

where c is a numerical constant, and the combination $\lambda\mu^{d/2}$ has net scaling dimension zero, as must be the case for consistency under scale transformations. Notice that λ does not modify the leading singularity as a function of μ, and therefore unimportant for small enough μ. A similar argument can be used to show the unimportance of λ for large distances, long times, or low temperatures. We can therefore safely set $\lambda = 0$ in all our subsequent discussion of critical properties of the $\mu = 0$ point.

It is not difficult to extend the above analysis to other perturbations of $\mathcal{L}_0$, and find that, in fact, all of them are irrelevant. Thus, the Lagrangean $\mathcal{L}_0$ is the complete theory of the quantum critical point of the dilute spinless Fermi gas and of all its relevant perturbations, in all spatial dimensions. The scale-invariant quantum critical point is at $\mu = 0$ and one moves away from adding to the action a relevant perturbation whose bare coupling strength is measured by μ. This bare coupling should be distinguished from the energy scale, called Δ in Section I, at which the consequences of the perturbation from the critical point are first apparent; this is the energy scale which will appear in arguments of universal scaling functions describing the deviation from the critical point. In general, this energy scale is a complicated and non-universal function of the bare coupling constant. However, the present model of spinless fermions has the potentially confusing property that this energy scale is also exactly μ. In other words, for this model, $g = \Delta = \mu$. Even the irrelevant couplings, like λ, do not modify the value of the energy scale μ. This remarkable identity between a bare coupling constant and a universal energy scale was dubbed "no scale-factor universality" [6], and is a consequence of the fact that the ground state at the critical point contains no particles. The other models considered later will have scale-factors (*i.e.* the factors appearing in the relatinship between g and Δ) which are non-universal.

Finally, it is worth noting here that the reader should not be misled by the deceptive simplicity of the above results and the apparent triviality of a quantum critical point with no particles. If we examine precisely the same situation for a spin-1/2 Fermi gas, the quantum critical point with no particles turns out to have rather non-trivial properties. In this latter situation it is possible to have a contact interaction $\Psi_\uparrow^\dagger \Psi_\uparrow \Psi_\downarrow^\dagger \Psi_\downarrow$ which is relevant about the non-interacting point for $d < 2$.

1. Non-zero temperatures

The inverse temperature appears as the "length" of the system along the imaginary time direction, and so the scaling dimension of temperature must be the negative of that of time

$$\dim[T] = z \tag{2.35}$$

Clearly, $k_B T$ is an externally imposed energy scale which breaks scale invariance and must be included in a description of the correlators of the system. The deviation from the quantum

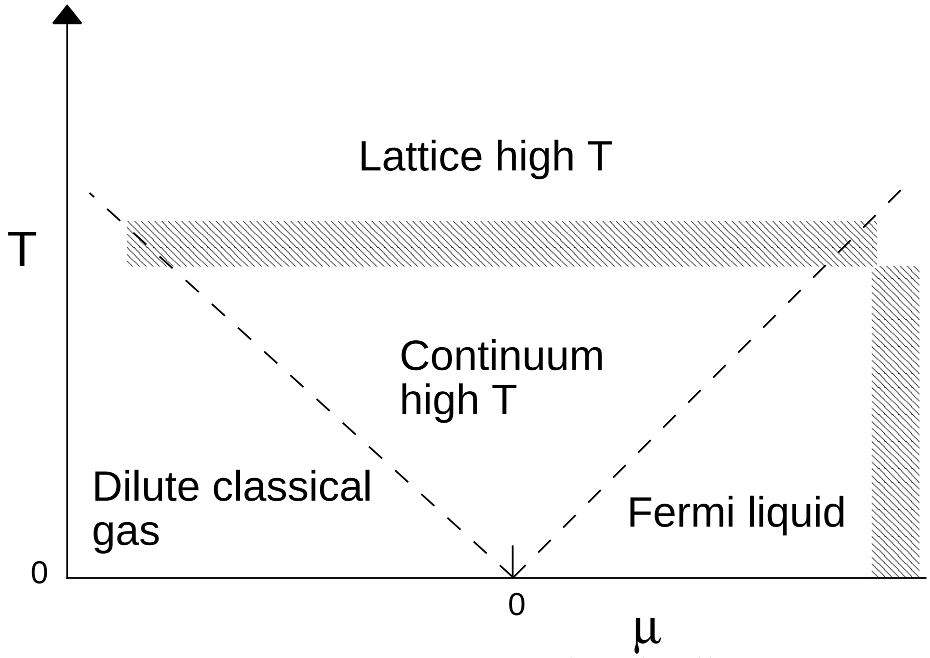

FIG. 3. Phase diagram of the dilute Fermi gas H_F (Eqn (2.15)) as a function of the chemical potential μ and the temperature T. The regions are separated by crossovers denoted by dashed lines, and their physical properties are discussed in Section II B 1. The hatched region marks the boundary of applicability of the continuum theory and occurs at $\mu, T \sim \Lambda$.

critical point is measured, as discussed above, by the energy scale μ. Note that the scaling dimension of μ is also that of energy $\dim[\mu] = 2 = z$, as it must be. By dimensional analysis it now follows that all properties are determined by the dimensionless ratio μ/k_BT, and we are then immediately able to draw the phase diagram shown in Fig 3. There are three limiting regimes, which are separated by smooth crossover denoted by the dashed lines. There is a high temperature regime ($k_BT \gg |\mu|$), and two low temperature regimes ($k_BT \ll \mu$ for $\mu > 0$, and $k_BT \ll -\mu$ for $\mu < 0$). It should be clear that the two low T regimes can be described in terms of a weak thermal perturbation of the $\mu > 0$ and $\mu < 0$ ground states. We will argue below that the high T regime is controlled by the $\mu = 0$ quantum critical point.

Before discussing the nature of the regimes in Fig 3 more explicitly, let us write down the scaling forms which contain the necessary crossovers. We will meet many such scaling forms in our analyses, and they are all deduced from a simple set of rules:
(i) write down the power of T, which has the same scaling dimension as the observable, as a prefactor;
(ii) the scaling function is a dimensionless quantity, and all of its arguments have net scaling dimension 0; again this is ensured by inserting appropriate powers of T;
(iii) finally, insert powers of non-scaling variables like m, k_B, $\hbar$ to ensure consistency in ordinary physical units of measurements.
These rules use T as the primary scaling variable which sets the scale for all other quantities; the advantages of doing this will become clear as we proceed. For the case of the fermion two-point function G, defined in (2.22), these rules give

$$G(x, -i\tau) = \left(\frac{2mk_BT}{\hbar^2}\right)^{d/2} \Phi_G\left(\frac{(2mk_BT)^{1/2}}{\hbar}x, -i\frac{k_BT}{\hbar}\tau, \frac{\mu}{k_BT}\right) \quad (2.36)$$

The function Φ_G is a scaling function, and everything about it is universal, including its overall scale and those of all its arguments. The reasons for this were discussed above: all corrections due to irrelevant couplings like λ will carry prefactors of higher powers of T and can therefore be neglected at low T near the quantum critical point. (Later we will meet more typical critical points where a non-universal renormalization of bare couplings, dependent on the strength of irrelevant couplings, is required before the scales of the scaling functions become universal).

It is, of course, quite straightforward to obtain an explicit expression for Φ_G from the propator of a single free fermion:

$$\Phi_G(\bar{x}, -i\bar{\tau}, y) = \int \frac{d^d w}{(2\pi)^d} e^{-(w^2-y)\bar{\tau}} e^{iw\bar{x}} \left(\frac{\theta(\bar{\tau})}{1+e^{-(w^2-y)}} - \frac{\theta(-\bar{\tau})}{e^{w^2-y}+1} \right) \tag{2.37}$$

Similar scaling forms hold for all other observables. The fermion density $\langle \Psi^\dagger \Psi \rangle$ is given by $-G(x=0, -i\tau = 0^-)$, and its scaling form follows from (2.36). Similarly, momentarily returning to the $d = 1$ XX model, let us note that the finite T behavior of the deviation from saturated magnetization $1 - \langle \sigma^z \rangle$ is given by

$$1 - \langle \sigma^z \rangle = -aG(x = 0, -i\tau = 0^-) \tag{2.38}$$

and so its scaling form also follows from (2.36). The free energy density $\mathcal{F}$ obeys

$$\mathcal{F} = T^{d/2+1} \Phi_{\mathcal{F}} \left(\frac{\mu}{T} \right) \tag{2.39}$$

where again the prefactor of T follows from the scaling dimension of $\mathcal{F}$ in (2.29). The scaling function $\Phi_{\mathcal{F}}$ is

$$\Phi_{\mathcal{F}}(y) = -\int \frac{d^d w}{(2\pi)^d} \ln(1 + e^{y-w^2}) \tag{2.40}$$

Let us now discuss physical properties of the above correlators in the three regimes of Fig 3.

(a) Dilute Classical Gas, $k_B T \ll |\mu|$, $\mu < 0$

The ground state for $\mu < 0$ is simply the vacuum with no particles. Turning on a non-zero temperature produces particles with a small non-zero density $\sim e^{-|\mu|/T}$. The de Broglie wavelength of the particles is of order $T^{-1/2}$ which is significantly smaller than the mean spacing between the particles which diverges as $e^{|\mu|/dT}$ as $T \to 0$. This implies that the particles behave classically, obey Boltzmann statistics, and their fermionic nature is unimportant. Because of the rather trivial nature of the ground state in this case, the spacetime dependence of correlation functions like G is rather uninteresting; it does not display the any siginificant crossovers as a function of space or time, as will be the case in the corresponding regime of more typical quantum critical points. We will therefore not discuss it in any detail here.

(b) Fermi Liquid, $k_B T \ll \mu$, $\mu > 0$

The behavior in this regime is quite complex and rich, and illustrates many common, important features of quantum critical points. It is therefore quite useful to study it in some detail.

First it can be argued, *e.g.* by studying asymptotics of the integral in (2.37), that for very short times or distances, the correlators do not notice the consequences of a non-zero T or μ and are therefore given by the $T = \mu = 0$ result in (2.22). More precisely we have

$$G(x, -i\tau) \text{ is given by (2.22) for } |x| \ll \left(\frac{\hbar^2}{2m\mu}\right)^{1/2} \quad |\tau| \ll \frac{\hbar}{\mu} \tag{2.41}$$

With increasing x or τ, the restrictions in (2.41) are eventually violated and the consequences of a non-zero μ become apparent. Notice that as μ is much larger than $k_B T$, it is the first energy scale to be noticed, and as a first approximation to understand the behavior at larger x we may ignore the effects of $k_B T$.

Let us therefore discuss the ground state for $\mu > 0$. It consists of a filled Fermi sea of particles (a Fermi liquid) with momenta $k < k_F = (2m\mu/\hbar^2)^{1/2}$. An important property of the this state that it permits excitations at arbitrarily low energies *i.e.* it is *gapless.* These low energy excitations correspond to changes in occupation number of fermions arbitrarily close to k_F. As a consequence of these gapless excitations, the line $\mu > 0$ ($T = 0$) is a *line of quantum critical points*, as will become apparent from our analysis below. Each point on the line exhibits an invariance under a scaling transformation, but this transformation is quite different from that describing the quantum-critical end point $\mu = 0$.

We now explain the statements above in some more detail. We are interested here only in x and τ which violate the constraints in (2.41), or alternatively, in excitations above the ground state with momenta near k_F. So let us parametrize, in $d = 1$

$$\Psi(x, \tau) = e^{ik_F x}\Psi_R(x, \tau) + e^{-ik_F x}\Psi_L(x, \tau) \tag{2.42}$$

where $\Psi_{R,L}$ describe right and left moving fermions, and are fields which vary slowly on spatial scales $\sim 1/k_F = (\hbar^2/2m\mu)^{1/2}$ and temporal scales $\sim \hbar/\mu$. A similar parametrization can be used for $d > 1$ but we will not explicitly discuss it here; most of the results discussed below hold, with small modifications, in all d (see the work by Shankar [7] for more details on a renormalization group analysis of fermions in $d > 1$). Inserting the above parametrization in $\mathcal{L}_0$, and keeping only terms lowest order in spatial gradients, we obtain the "effective" Lagrangian for the Fermi liquid region, $\mathcal{L}_{FL}$:

$$\mathcal{L}_{FL} = \Psi_R^\dagger\left(-\frac{\partial}{\partial\tau} - iv_F\frac{\partial}{\partial x}\right)\Psi_R + \Psi_L^\dagger\left(-\frac{\partial}{\partial\tau} + iv_F\frac{\partial}{\partial x}\right)\Psi_L \tag{2.43}$$

where $v_F = \hbar k_F/m = (2\mu/m)^{1/2}$ is the Fermi velocity. The Lagrangian $\mathcal{L}_{FL}$ also describes a massless Dirac field in one spatial dimension, and is invariant under relativistic and conformal transformations of spacetime: these facts shall be of some use to us later. Now notice that $\mathcal{L}_{FL}$ is invariant under a scaling transformation, which is rather different from that discussed earlier:

$$
\begin{aligned}
x' &= x e^{-\ell} \\
\tau' &= \tau e^{-\ell} \\
\Psi'_{R,L}(x',\tau') &= \Psi_{R,L}(x,\tau) e^{\ell/2} \\
v'_F &= v_F
\end{aligned} \tag{2.44}
$$

Notice that this transformation implies scaling dimensions which are different from those discussed earlier for the $\mu = 0$ critical point. This highlights an important point: scaling dimensions of any operator or coupling constant are not its absolute attributes, but always refer to a particular quantum critical point. The above results imply

$$
z = 1, \tag{2.45}
$$

unlike $z = 2$ (Eqn (2.21)) at the $\mu = 0$ critical point, and $\dim[\Psi] = 1/2$ which actually holds for all d and therefore differs from (2.27). Further notice that v_F, and therefore μ, are now *invariant* under rescaling, unlike the transformation (2.23) at the $\mu = 0$ critical point. Thus now v_F now plays a role rather analogous to that of m at the $\mu = 0$ critical point: it simply the physical units of spatial and length scales. The transformations (2.44) show that $\mathcal{L}_{LF}$ is scale invariant for each value of μ, and we therefore have a line of quantum critical points as claimed earlier. It should also be emphasized that the scaling dimension of interactions like λ will now also change; in particular not all interactions are irrelevant about the $\mu \neq 0$ critical points. We will ignore the ramifications of this complication here.

The action (2.43) and the scaling transformations (2.44) can be considered as defining scaling forms on their on right, independent of any derivation from the original $\mathcal{L}_0$. By complete analogy with the arguments presented earlier, we may now deduce that

$$
G_{R,L}(x, -i\tau) = \left(\frac{k_B T}{\hbar v_F}\right) \phi_{R,L}\left(\frac{k_B T}{\hbar v_F} x, -i\frac{k_B T}{\hbar}\tau\right) \tag{2.46}
$$

where $G_{R,L}$ are two point correlators of $\Psi_{R,L}$ in an obvious notation, the powers of T follow from the scaling dimensions of G, x, and τ, the factors of v_F, k_B, $\hbar$ merely keep track of physical units, and $\phi_{R,L}$ are universal scaling functions.

What is now the relationship between the original scaling function Φ_G of the $\mu = 0$ critical point in (2.36), and the scaling functions $\phi_{R,L}$ introduced above ? The answer to this question requires introduction of the very useful concept of a **reduced scaling function**; such reduced scaling functions will appear in all three models considered in this article. So it will help the reader to fully absorb this somewhat subtle idea in the present simple setting. Notice that the regime of validity of Φ_G includes that of $\phi_{R,L}$. The latter functions require the additional restrictions $k_B T \ll \mu$, and also the converse of those in (2.41). So it must be the case that the $\phi_{R,L}$ are contained in Φ_G *i.e.* they are reduced scaling functions of Φ_G. Just this requirement allows us to place important restrictions on the functional form of Φ_G in the limit of large $\mu/k_B T$. The scaling forms (2.36) and (2.46), and the mapping (2.42), imply that, for large $\mu/k_B T$, the Greens function takes the form

$$
G(x, -i\tau) = e^{ik_F x} G_R(x, -i\tau) + e^{-ik_F x} G_L(x, -i\tau), \tag{2.47}
$$

or in terms of the scaling functions Φ_G, $\phi_{R,L}$ we have for large $y = \mu/k_B T$

$$\Phi_G(\bar{x}, -i\bar{\tau}, y) = e^{i\sqrt{y}x}\Phi_{GR}(\bar{x}, -i\bar{\tau}, y) + e^{-i\sqrt{y}x}\Phi_{GL}(\bar{x}, -i\bar{\tau}, y), \tag{2.48}$$

and further that

$$\phi_{R,L}(\bar{x}, \bar{\tau}) = \lim_{y\to\infty} 2\sqrt{y}\Phi_{GR,L}(2\sqrt{y}\bar{x}, \bar{\tau}, y). \tag{2.49}$$

The reasoning we have presented above implies that a non-zero limit of the right-hand-side exists. Notice that the reduced scaling function ϕ_{RL} has one less argument than the primary function Φ_G: this is a characteristic feature of the collapse of one scaling function into another.

Let us now show the above collapse into a reduced scaling function explicitly. For simplicity, we will limit ourselves in this paragraph to $\tau > 0$. Then the full expression for the original Greens function is

$$G(x, -i\tau) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}e^{-(\hbar^2k^2/2m-\mu)\tau}}{1 + e^{-(\hbar^2k^2/2m-\mu)/k_BT}} \tag{2.50}$$

For $|x| \gg (\hbar^2/2m\mu)^{1/2}$, $\tau \gg \hbar/\mu$, and $k_BT \ll \mu$, this integral is dominated by contributions near the Fermi points $k = \pm k_F$. So near k_F let us parametrize $k = k_F + p$, expand terms in the integrand to linear order in p, and to leading order let the integral extend over all real p; a similar procedure can be carried out near $-k_F$. In this manner the above expression for G reduces to

$$G(x, \tau) = e^{ik_Fx} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{e^{p(ix-v_F\tau}}{1 + e^{-\hbar v_Fp/k_BT}} + e^{-ik_Fx} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{e^{p(-ix-v_F\tau}}{1 + e^{-\hbar v_Fp/k_BT}} \tag{2.51}$$

This result could, of course, also have been obtained direction from $\mathcal{L}_{FL}$, combined with (2.42). The integrals over p can be evaluated exactly and we obtain

$$G_{R,L}(x, \tau) = \left(\frac{k_BT}{\hbar v_F}\right) \frac{1}{2\sin(\pi k_BT(\tau \mp ix/v_F)/\hbar)} \tag{2.52}$$

This result is clearly consistent with the scaling form (2.46).

Let us now discuss the physical properties of the Green's functions obtained above for the region $k_BT \ll \mu$. The results are schematically indicated in Fig 4. Recall first that at the shortest scales $|x| \ll (\hbar^2/2m\mu)^{1/2}$, $\tau \ll \hbar/\mu$, we get the behavior (2.22) of the groundt state at the $\mu = 0$ critical point. At slightly larger scales, we crossover to behavior characterizing the $\mu > 0$ ground state. In particular for $(\hbar^2/2m\mu)^{1/2} \ll |x| \ll \hbar v_F/k_BT$ or $\hbar/\mu \ll \tau \ll \hbar/k_BT$, we have

$$G_{R,L}(x, \tau) = \frac{1}{2\pi\hbar v_F(\tau \mp ix/v_F)/\hbar} \tag{2.53}$$

This is the power law decay characteristic of the $\mu > 0$ critical ground state: note that it is consistent with the scaling transfromations (2.44). At the very largest space and time scales, $|x| \gg \hbar v_F/k_BT$ and real time $|t| \gg \hbar/k_BT$, the effects of a finite temperature became manifest. The gapless thermal excitations above the critical ground state damp the power

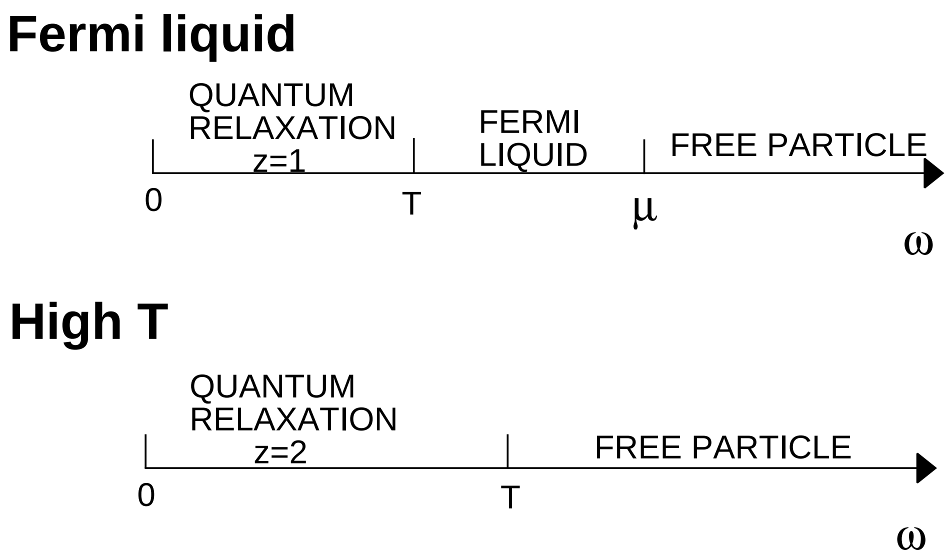

FIG. 4. Crossovers as a function of frequency for the spinless Fermi gas in two of the regimes of Fig 3. The dilute classical gas regime is not shown because it does not have any interesting crossovers in this simple model. The term 'quantum relaxational' refers to the exponential decay of correlations on a scale determined by T: the reason for this nomenclature will be discussed further in Section III D

law correlations and we get exponential decay of correlations; for the equal time correlations we have

$$G_{R,L}(x, \tau = 0) \sim \exp\left(-\frac{\pi k_B T}{\hbar v_F}|x|\right), \tag{2.54}$$

which gives us a correlation length $\xi = \hbar v_F/(\pi k_B T)$. Notice that ξ (which scales like a length) is proportional to $1/T$ (which scales as a time), and this is precisely what is expected from the scaling transformations (2.44) of the $\mu > 0$ ground state. In general, we expect the T dependent correlation length above a critical ground state to scale as $T^{-1/z}$.

Before concluding this subsection, we note an important relationship between (2.52) and (2.53). Observe that it is possible to obtain from the $T = 0$ correlator (2.53), the finite T correlator (2.52) by the mapping

$$v_F \tau \pm ix \to \frac{\hbar v_F}{\pi k_B T} \sin\left(\frac{\pi k_B T}{\hbar v_F}(v_F \tau \pm ix)\right) \tag{2.55}$$

This is actually an example of a very general and extremely powerful result. It can be shown that the mapping (2.55) in fact relates the $T = 0$ and $T > 0$ values of *any* two-point correlator of $\mathcal{L}_{FL}$; this result is a consequence of conformal symmetry of $\mathcal{L}_{FL}$, which was mentioned in passing earlier. Indeed, the mapping (2.55) relates two-point $T = 0$ and $T > 0$ correlators of any conformally invariant theory whose excitations move with the velocity v_F [8], and we will use it again on several occasions.

(c) High T limit, $k_B T \gg |\mu|$

This is the last, and in many ways the most interesting, region of Fig 3. Now k_BT is the most important energy scale controlling the deviation from the $\mu = 0$, $T = 0$ quantum critical point. It should be emphasized that while the value of k_BT is significantly larger than $|\mu|$, it cannot be so large that it exceeds the limits of applicability for the continuum action $\mathcal{L}_0$. The continuum theory fails above an upper energy scale, Λ, at which the quadratic dispersion $\hbar^2 k^2/2m$ is no longer valid, and we must have $k_BT \ll \Lambda$ (see Fig 3). For the simple tight-binding model (2.9) of the XX chain, this means that $\Lambda \sim J$.

We discuss first the behavior of the of the fermion density. In the high T limit of the continuum theory $\mathcal{L}_0$, $|\mu| \ll k_BT \ll \Lambda$ we have from (2.36,2.37) the universal result

$$\begin{aligned}\langle \Psi^\dagger \Psi \rangle &= -\left(\frac{2mk_BT}{\hbar^2}\right)^{d/2} \Phi_G(0, 0^-, 0) \\ &= \left(\frac{2mk_BT}{\hbar^2}\right)^{d/2} \int \frac{d^dw}{(2\pi)^d} \frac{1}{e^{w^2}+1} \\ &= \left(\frac{2mk_BT}{\hbar^2}\right)^{d/2} \zeta(d/2) \frac{(1-2^{d/2})}{(4\pi)^{d/2}} \end{aligned} \tag{2.56}$$

This density implies an interparticle spacing which is of order the de Brogie wavelength $= (\hbar^2/2mk_BT)^{1/2}$: thermal and quantum effects are therefore expected to be equally important, and neither dominate.

Let us also consider the fermion density for $T \gg \Lambda$ (the region above the hatched marks in Fig 3), to illustrate the limitations on the continuum description discussed above. Now the result depends upon the details of the non-universal fermion dispersion; on a hypercubic lattice with dispersion $\epsilon_k - \mu$, we obtain

$$\begin{aligned}\langle \Psi^\dagger \Psi \rangle &= \int_{-\pi/a}^{\pi/a} \frac{d^dk}{(2\pi)^d} \frac{1}{e^{(\epsilon_k-\mu)/T}+1} \\ &= \frac{1}{2a^d} - \frac{1}{4T} \int_{-\pi/a}^{\pi/a} \frac{d^dk}{(2\pi)^d} (\epsilon_k - \mu) + \mathcal{O}(1/T^2). \end{aligned} \tag{2.57}$$

The limits on the integration, which extend from $-\pi/a$ to π/a for each momentum component, had previously been sent to infinity in the continuum limit $a \to 0$. In the presence of lattice cutoff, we are able to make a naive expansion of the integrand in powers of $1/T$, and the result therefore only contains negative integer powers of T. Contrast this with the universal continuum result (2.56) where we had non-integer powers of T dependent upon the scaling dimension of Ψ.

We now return to the universal high T region, $|\mu| \ll k_BT \ll \Lambda$, and describe the behavior as a function of temporal and spatial scales. The reader will find it useful to compare the following discussion with the corresponding discussion in the Fermi liquid region discussed in Section II B 1(b). As before, we will consider the behavior of the Greens function $G(x, \tau)$; its leading behavior in the present high T region is obtained simply by setting $y = 0$ in the scaling function in (2.37). The results are schematically sketched in Fig 4. At the shortest scales we again have the free quantum particle behavior of the $\mu = 0$, $T = 0$ critical point

$$G(x,\tau) \text{ is given by (2.22) for } |x| \ll \left(\frac{\hbar^2}{2mk_BT}\right)^{1/2} \quad |\tau| \ll \frac{\hbar}{k_BT}. \tag{2.58}$$

Notice that the limits on x and τ in (2.58) are different from those in (2.41), in that they are determined by $k_B T$ and not μ. The results (2.41) and (2.58) illustrate two very important general principles of quantum critical points:
(i) At short distance and time scales the correlators are those of the quantum critical point
(ii) At larger scales, the relevant perturbations away from the critical point become apparent, but it is the largest energy scale which gets noticed first.
So here thermal effects modify the system *before* it has a chance to feel the effects of μ. In particular, even the sign of μ has a minor effect on the properties of the high T region: *the system does not look like either $\mu > 0$ the Fermi liquid ground state, or the $\mu < 0$ dilute classical gas, at any scale.* Instead for $|x| \gg \left(\hbar^2/2mk_BT\right)^{1/2}$ or $|\tau| \gg \hbar/k_BT$ it crosses over to a novel behavior characteristic of the high T region. We illustrate this by looking at the large x asymptotics of the equal time G in $d = 1$ (other d are quite similar)

$$G(x,0) = \int \frac{dk}{2\pi} \frac{e^{ikx}}{1 + e^{-\hbar^2 k^2/2mk_BT}} \tag{2.59}$$

For large x this can be evaluated by a contour integration which picks up contributions from the poles at which the denominator vanishes in the complex k plane. The dominant contributions come from the poles closest to the real axis, and gives the leading result

$$G(|x| \to \infty, 0) = -\left(\frac{\pi\hbar^2}{2mk_BT}\right)^{1/2} \exp\left(-(1-i)\left(\frac{m\pi k_BT}{\hbar^2}\right)^{1/2} x\right) \tag{2.60}$$

Thermal effects therefore lead to an exponential decay of equal-time correlations, with a correlation length $\xi = \left(\hbar^2/m\pi k_BT\right)^{1/2}$. Notice that the T dependence is precisely that expected from the exponent $z = 2$ associated with the $\mu = 0$ quantum critical point and the general scaling relation $\xi \sim T^{-1/z}$. The additional oscillatory term in (2.60) is a reminder that quantum effects are still present at the scale ξ, which is clearly of order the de Broglie wavelength of the particles. We will discuss other properties of the high T region of quantum critical points in more detail in other contexts later in this review; in particular, the dynamic correlations are particularly interesting but it appears useful to postpone their analysis to a case where the critical theory has interactions between its degrees of freedom.

We summarize the results of Section II B 1 by drawing the readers attention again to Figs 3 and 4, which show the crossovers as a function of frequency scale in the different regimes of the T, μ plane.

III. THE ISING CHAIN IN A TRANSVERSE FIELD

This section will study another simple solvable model of a quantum phase transition in one dimension. Like the XX model considered in Section II, its solvability is ensured by a mapping to free spinless fermion system via the Jordan Wigner transformation. However unlike the XX model, the present model does possess a quantum phase with true long-range order at $T = 0$, and this leads to important differences in the physics at low temperatures.

An important property of the Hamiltonian (2.1) of the XX model is that it possesses a global, continuous $U(1)$ symmetry: all of the physics is invariant under rotations by an arbitrary angle θ in the XX plane: $\sigma_i^x + i\sigma^y + i \to e^{i\theta}(\sigma_i^x + i\sigma_i^y)$. Strong low energy fluctuations associated with the orientation of the local ordering in the XY plane prevent the appearance of true long-range order at $T = 0$, $\mu > 0$; instead, correlations of the order parameter decays as $1/\sqrt{x}$ for large x. In this section we will break the continuous $U(1)$ symmetry down to a discrete Z_2 symmetry, and find phases with true long-range order.

The simplest route to accomplishing this is to introduce an XY anisotropy in the exchange couplings; this amounts to replacing the exchange terms in (2.1) by $-\sum_i (J_X \sigma_i^x \sigma_{i+1}^x + J_Y \sigma_i^y \sigma_{i+1}^y)$, while the field coupling to σ_i^z remains unchanged. Such a model with $J_X \neq J_Y$ is also completely solvable, and its properties have been discussed recently [9], following an earlier analysis [10]. This analysis shows that the properties of the special case $J_X \neq 0$, $J_Y = 0$ are in the same universality class as the general $J_X \neq J_Y$. As the computations are somewhat simpler for this special case, we will restrict our attention to it here, and consider

$$H_I' = -J \sum_i \left(g\sigma_i^z + \sigma_i^x \sigma_{i+1}^x \right) \tag{3.1}$$

where we have written it in a form in which $J > 0$ is an overall energy scale, $g > 0$ is a dimensionless coupling constant. As most of us are more accustomed to working with eigenstates of σ^z, rather than σ^x, we will therefore map

$$\sigma^z \to \sigma^x \quad , \quad \sigma^x \to -\sigma^z \tag{3.2}$$

(this preserves the commutation relations), and work instead with the completely equivalent

$$H_I = -J \sum_i \left(g\sigma_i^x + \sigma_i^z \sigma_{i+1}^z \right) \tag{3.3}$$

This Hamiltonian H_I is often referred to as the quantum Ising model, or the Ising chain in a transverse field. It is invariant under a global unitary transformation, performed by the operator $\prod_i \sigma_i^x$ under which

$$\sigma_i^z \to -\sigma_i^z \qquad \sigma_i^x \to \sigma_i^x \tag{3.4}$$

Notice that this global Ising Z_2 symmetry is present in the presence of the transverse field. A longitudinal field, coupling to σ^z would break the Z_2 symmetry.

A. Limiting cases

We begin by examining the spectrum of H_I under strong ($g \gg 1$) and weak ($g \ll 1$) coupling [11]. The analysis is relatively straightforward in these limits, and two very different physical pictures emerge. The exact solution, to be discussed later, shows that there is a critical point exactly at $g = 1$, but that the qualitative properties of the ground states for $g > 1$ ($g < 1$) are very similar to those for $g \gg 1$ ($g \ll 1$). One of the two limiting descriptions is therefore always appropriate, and only the critical point $g = 1$ has genuinely different properties.

1. Strong coupling $g \gg 1$

Let $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$ denote the eigenstates of σ_i^z. Then $|\pm\rangle_i = (|\uparrow\rangle_i \pm |\downarrow\rangle_i)/\sqrt{2}$ are the eigenstates of σ_i^x. Then at $g = \infty$ the ground state of H_I is clearly determined by the transverse field term to be

$$|0\rangle = \prod_i |+\rangle_i \tag{3.5}$$

The values of σ_i^z on different sites are totally uncorrelated in this state, and so $\langle 0|\sigma_i^z \sigma_j^z|0\rangle = \delta_{ij}$. Perturbative corrections in $1/g$ will build in correlations in σ^z which increase in range at each order in $1/g$; for g large enough these correlations are expected to remain short-ranged, and the σ^z correlator to decay exponentially with separation. There is thus no magnetic long-range order and this state is a "quantum paramagnet". Notice that this state is invariant under the Z_2 symmetry described above.

What about the excited states ? For $g = \infty$ these can also be listed exactly. The lowest excited states are

$$|i\rangle = |-\rangle_i \prod_{j \neq i} |+\rangle_j, \tag{3.6}$$

obtained by flipping the state on site i to the other eigenstate of σ^x. All such states are degenerate, and we will refer to them as the "single-particle" states. Similarly, the next degenerate manifold of states are the two-particle states $|i, j\rangle$, obtained by flipping the states at sites i and j, and so on to the general n-particle states. To leading order in $1/g$, we can neglect the mixing between states between different particle number, and just study how the degeneracy within each manifold is lifted. For the one-particle states, the exchange term in H_I leads only to the off-diagonal matrix element

$$\langle i|H_I|i+1\rangle = -J \tag{3.7}$$

which hops the 'particle' between nearest neighbor sites. As in the tight-binding models of solid state physics, the Hamiltonian is therefore diagonalized by going to the momentum space basis

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_i e^{ikx_i} |i\rangle \tag{3.8}$$

where N is the number of sites. This eigenstate has energy (we have choosen an overall constant in H_I to make the energy of the ground state zero)

$$\epsilon_k = Jg \left(2 - (2/g)\cos ka + \mathcal{O}(1/g^2)\right) \tag{3.9}$$

where a is the lattice spacing. The lowest energy one-particle state is therefore at $\epsilon_0 = 2g - 2J$

Now consider the two-particle states. As long as the two particles are well separated from each other, the eigenstate is formed simply by taking the tensor product of two single particle eigenstates. However these particles will collide, and will then have a non-trivial S

matrix which mixes the states in the single particle basis. If we exclude the possibility of two-particle bound states (which do not occur here), the total energy of the state is determined by the configuration where the particles are well separated, and is simply the sum of the single particle energies. Thus the energy of a two-particle state with total momentum p is given by $E_p = \epsilon_{p_1} + \epsilon_{p_2}$ where $p = p_1 + p_2$. Notice that for a fixed p, there is still an arbitrariness in the single particle momenta $p_{1,2}$ and so the total energy E_p can take a range of values. There is thus no definite energy momentum relation, and we have instead a 'two-particle continuum'. It should be clear, however, that the lowest energy two-particle state in the infinite system (its "threshold") is at $2\epsilon_0$. Similar considerations apply to the n-particle continua, which have thresholds at $n\epsilon_0$.

At next order in $1/g$ we have to account for the mixing between states with differing numbers of particles. Non-zero matrix elements like

$$\langle 0|H_I|i, i+1\rangle = -J \tag{3.10}$$

lead to a coupling between n and $n+2$ particle states. It is clear that these will renormalize the one-particle energies ϵ_p. However qualitative features of the spectrum will not change, and we will still have renormalized one-particle states with a definite energy-momentum relationship, and renormalized $n \geq 2$ particle continua with thresholds at $n\epsilon_0$.

The spectrum described above has simple, but important, consequences for the dynamic spin susceptibility $\chi(p, \omega_n)$. This is defined in imaginary time as the Fourier transform into momentum and frequency of the σ^z correlator:

$$\chi(k, i\omega_n) = \int_0^{1/T} d\tau \int dx \, \langle \sigma^z(x,\tau)\sigma^z(0,0)\rangle \, e^{i(kx - \omega_n \tau)} \tag{3.11}$$

Its spectral density $\chi''(p, \omega)$ is the imaginary part of the real frequency $\chi(k, \omega)$, and it given by

$$\chi''(k, \omega) = \pi \sum_a |\, \langle 0|\sigma^z(k)|a\rangle \,|^2 \delta(\omega - E_a) \tag{3.12}$$

where the sum over a extends over all the eigenstates of H_I with energy E_a. The eigenstates and energies described above allow us to simply deduce the qualitative form of $\chi''(p, \omega)$ which is sketched in Fig 5. The operator σ^z flips the state at a single site, and so the matrix element in (3.12) is non-zero for the single particle states: only the state with momentum p will contribute, and so there is an infinitely sharp delta function contribution to $\chi''(k, \omega) \sim \delta(\omega - \epsilon_k)$. This delta function is the "quasiparticle peak" and its co-efficient is the quasiparticle amplitude. At $g = \infty$ this quasiparticle peak is the entire spectral density, but for smaller g the quasiparticle amplitude decreases and the multiparticle states also contribute to the spectral density. The mixing between the one and three particle states discussed above, means that the next contribution to $\chi''(p, \omega)$ occurs above the 3 particle threshold $\omega > 3\epsilon_0$; because there are a continuum of such states, their contribution is no longer a delta function, but a smooth function of omega (apart from a threshold singularity), as shown in Fig 5. Similarly there are continua above higher odd number particle thresholds; only states with odd numbers of particles contribute because the matrix element in (3.12) vanishes for even numbers of particles.

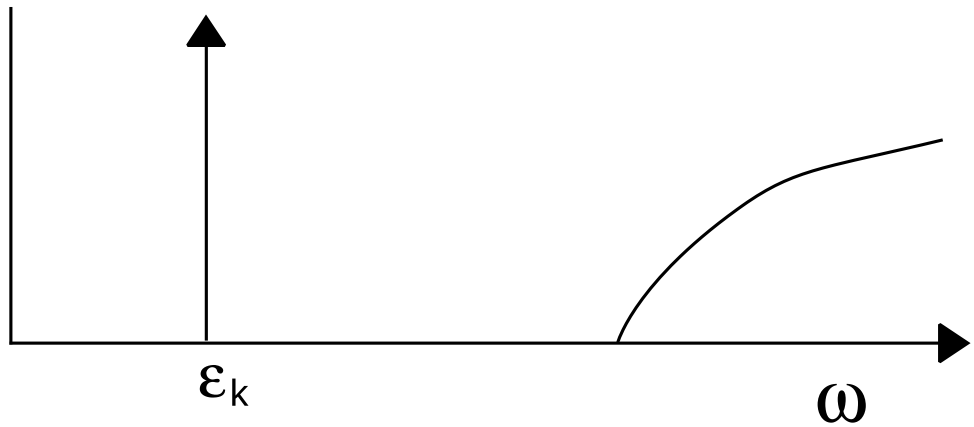

FIG. 5. Schematic of the spectral density $\chi''(k,\omega)$ of H_I as a function of ω at $T = 0$ and a small k. There is a quasiparticle delta function at $\omega = \epsilon_k$, and a three-particle continuum at higher frequencies.

2. Weak coupling $g \ll 1$

Now the energy is dominated by the exchange term. There are two degenerate ground states at $g = 0$ with the spins either all up or down (in eigenstates of σ^z):

$$|\Uparrow\rangle = \prod_i |\uparrow\rangle_i \qquad |\Downarrow\rangle = \prod_i |\downarrow\rangle_i \tag{3.13}$$

Turning on a small g will mix in a small fraction of spins of the opposite orientation, but the degeneracy will survive as the two states are related to each other by the global Z_2 symmetry noted above (3.4). A thermodynamic system will always choose one or the other of the states as its ground states (which may be preferred by some infinitesimal external perturbation), and hence the Z_2 symmetry will be spontaneously broken. The correlations of the magnetization σ^z have an infinite range in either state as

$$\lim_{|x|\to\infty} \langle \sigma^z(x,0)\sigma^z(0,0)\rangle = N_0^2 \neq 0 \tag{3.14}$$

The quantity N_0 is the spontaneous magnetization, and equals $\langle\sigma^z\rangle$ in either of the two ground states. All of the statements made in this paragraph clearly hold for $g = 0$, and will hold for some $g > 0$ provided the perturbation theory in g has a non-zero radius of convergence. The exact solution of the model to be discussed later will verify that this is indeed the case.

The excited states can be described in terms of an elementary domain wall (or kink) excitation. For instance the state

$$\cdots |\uparrow\rangle_i\, |\uparrow\rangle_{i+1}\, |\downarrow\rangle_{i+2}\, |\downarrow\rangle_{i+3}\, |\downarrow\rangle_{i+4}\, |\uparrow\rangle_{i+5}\, |\uparrow\rangle_{i+6} \cdots$$

has domain walls, or nearest neighbor pairs of antiparallel spins, between sites $i+1$, $i+2$ and sites $i+4$, $i+5$. At $g = 0$ the energy of such a state is clearly $2J\times$number of domain walls. The consequences of a small non-zero g are now very similar to those due to $1/g$

corrections in the complementary large g limit: the domain walls become "particles" which can hop and form momentum eigenstates with excitation energy

$$\epsilon_k = J\left(2 - 2g\cos(ka) + \mathcal{O}(g^2)\right). \tag{3.15}$$

The spectrum can be interpreted in terms n-particle scattering states, although it must be emphasized that the interpretation of the particle is now very different from that in the large g limit. Again, the perturbation theory in g only mixes states which differ by even numbers of particles, although now the matrix element in (3.12) is non-zero only for states a with an *even* number of particles; these assertions can easily be checked to hold in a perturbation theory in g. So $\chi''(p,\omega)$ will now have a pole at $p = 0$, $\omega = 0^+$, from the term in (3.12) where $a =$ one of the ground states, indicating the presence of long-range order. Further, there is now no single particle contribution, and the first finite ω spectral density is the continuum above the two particle threshold. The absence of a single particle delta function in this case is a very special feature of the $d = 1$ quantum Ising model, and is not expected to hold for Ising models in higher d.

B. Exact spectrum

The qualitative considerations of the previous section are quite useful in developing an intuitive physical picture. We will now take a different route, and set up a formalism that will eventually lead to an exact determination of many physical correlators; these results will vindicate the approximate methods for $g > 1$, $g < 1$, and also provide an understanding of the novel physics at $g = 1$.

The central idea, as in Section II, is the application of the Jordan-Wigner transformation [4]. We apply (2.5,2.6), while recalling the mapping (3.2), to obtain H_I in the form

$$H_I = -J\sum_i \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1}c_i - 2g c_i^\dagger c_i - g\right) \tag{3.16}$$

This fermionic Hamiltonian differs from the one for the XX model (Eqn (2.9)) by terms like $c^\dagger c^\dagger$ which violate the fermion conservation number. So the eigenstates of H_I will not have a definite fermion number. Nevertheless, the new terms are still quadratic in the fermion operators, and H_I can be diagonalized by elementary means. First, use the momentum eigenstates (2.10) to get

$$H_I = J\sum_k \left(2(g - \cos(ka))c_k^\dagger c_k - i\sin(ka)(c_{-k}^\dagger c_k^\dagger + c_{-k}c_k) - g\right) \tag{3.17}$$

Next, use the Bogoluibov transformation to map into a new set of fermionic operators (γ_k) whose number is conserved. These new operators are defined by a unitary transformation on the pair $c_k, c_{-k}^\dagger$

$$\gamma_k = u_k c_k - i v_k c_{-k}^\dagger, \tag{3.18}$$

where u_k, v_k are real numbers satisfying $u_k^2 + v_k^2 = 1$, $u_{-k} = u_k$, and $v_{-k} = -v_k$. It can now be checked that canonical fermion anticommutation relations for the c_k imply that the same relations are also satisfied by the γ_k *i.e.*

$$\left\{\gamma_k, \gamma_{k'}^\dagger\right\} = \delta_{k,k'} \quad \left\{\gamma_k^\dagger, \gamma_{k'}^\dagger\right\} = \{\gamma_k, \gamma_{k'}\} = 0 \tag{3.19}$$

We also note the inverse of (3.18)

$$c_k = u_k \gamma_k + i v_k \gamma_{-k}^\dagger \tag{3.20}$$

We now insert (3.20) into (3.17), and demand that H_I not contain any terms like $\gamma^\dagger\gamma^\dagger$ which violate conservation of the γ fermions. The as yet undefined constants, u_k, v_k can always be chosen to ensure this: we define $u_k = \cos(\theta_k/2)$, $v_k = \sin(\theta_k/2)$, and a simple calculation then shows that the choice

$$\tan\theta_k = \frac{\sin(ka)}{\cos(ka) - g} \tag{3.21}$$

satisfies our requirements. The final form of H_I is now

$$H_I = \sum_k \epsilon_k (\gamma_k^\dagger \gamma_k - 1/2) \tag{3.22}$$

where

$$\epsilon_k = 2J\left(1 + g^2 - 2g\cos k\right)^{1/2} \tag{3.23}$$

is the single particle energy. As $\epsilon_k \geq 0$, the ground state, $|0\rangle$, of H_I has has no γ fermions and therefore satisfies $\gamma_k|0\rangle = 0$ for all k. The excited states are created by occupying the single particle states; they can clearly be classified by the total number of occupied states, and a n-particle state has the from $\gamma_{k_1}^\dagger \gamma_{k_2}^\dagger \cdots \gamma_{k_n}^\dagger |0\rangle$, with all the k_i distinct.

The above structure of the spectrum confirms the approximate considerations of Section III A. We have now found that the particles are in fact free fermions, and two fermions will not scatter even when they are close to each other; alternatively they can be considered as hard-core bosons which have an S matrix of -1 for two-particle scattering; this remarkable feature was not anticipated in the simple approach of Section III A, where we constructed general features of the spectrum by only considering well-separated particles. It is also reassuring to see that the exact single-particle excitation energy (3.23) agrees with (3.9) in the limit $g \gg 1$, and with (3.15) in the limit $g \ll 1$.

C. Continuum theory

The excitation energy ϵ_k in (3.23) is non-zero and positive for all k provided $g \neq 1$. The energy gap, or the minimum excitation energy is always at $k = 0$, and equals $2J|1-g|$. This gap vanishes at $g = 1$, and it is natural to expect that $g = 1$ is the phase boundary between the two qualitatively different phases discussed in Section III A. Precisely at $g = 1$, fermions with low momenta can carry arbitrarily low energy, and therefore must dominate the low temperature properties. These properties suggest that the state at $g = 1$ is critical, and there is a universal continuum quantum field theory which decribes the critical properties in its vicinity.

We shall now obtain this critical theory using a strategy very similar to that followed in Section II B. As the important excitations are near $k = 0$, we define the continuum Fermi field

$$\Psi(x_i) = \frac{1}{\sqrt{a}} c_i \tag{3.24}$$

We express H_I in terms of Ψ, and expand in spatial gradients. We present the final answer in terms of the Lagrangean $\mathcal{L}_I$, which will enter a Grassman path integral like (2.18)

$$\mathcal{L}_I = -\Psi^\dagger \frac{\partial \Psi}{\partial \tau}^\dagger + \frac{c}{2}\left(\Psi^\dagger \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi}{\partial x}\right) + \Delta \Psi^\dagger \Psi \tag{3.25}$$

The field Ψ is now implicitly assumed to be a function of space and imaginary time (τ). The coupling constants in (3.25) are given for H_I by

$$\Delta = 2J(1-g) \qquad c = 2Ja. \tag{3.26}$$

However, the relations (3.26) are very specific to the solvable model H_I. For more complicated, non-solvable, Ising models which have a similar naive continuum limit (*e.g.* models with second neighbor exchange), the values of Δ and c appearing in the continuum quantum field theory cannot be determined exactly. Indeed Δ and c are parameters which depend upon details of the microscopic Hamiltonian, *i.e.* they have a non-universal dependence upon a coupling constant like g, as can be checked by a simple estimate of perturbative fluctuation corrections due to irrelevant operators. This is to be contrasted by the behavior of the XX model, where the values of physical parameters like m and μ were given exactly by their naive continuum limit values for a arbitrary microscopic Hamiltonians; recall that this was called "no scale factor" universality, which is absent here.

In general then, determination of Δ and c requires relating them to a physically measurable observable. The continuum theory $\mathcal{L}_I$ can be diagonalized much like the lattice model H_I, and the excitation energy now takes a "relativistic" form

$$\epsilon_k = \left(\Delta^2 + c^2 k^2\right)^{1/2} \tag{3.27}$$

which shows that $|\Delta|$ is the $T = 0$ energy gap (we will choose the sign of Δ to be different on the two sides of the critical value of g), and c is the velocity of the excitations, both measurable quantities. The form of ϵ_k correctly suggests that $\mathcal{L}_I$ is invariant under Lorentz transformations. This can be made explicit by writing the complex Grassman field Ψ in terms of two real Grassman fields, when the action becomes what is known as the field theory of Majorana fermions of mass Δ/c^2 [12]; we will not explicitly display this here.

It is now easy to see that $\mathcal{L}_I$ is scale-invariant at $\Delta = 0$. The rescaling transformation

$$\begin{aligned} x' &= x e^{-\ell} \\ \tau' &= \tau e^{-\ell} \\ \Psi' &= \Psi e^{\ell/2} \end{aligned} \tag{3.28}$$

leaves $\mathcal{L}_I$ unchanged in form. The velocity c is taken to be invariant under rescaling, and its role is merely to specify the relative units of space and time; this role is similar to that of m

at the $\mu = 0$ critical point of dilute spinless Fermi gas in Section II B. The transformation (3.28) specifies the scaling dimensions

$$\begin{aligned} z &= 1 \\ \dim[\Psi] &= 1/2. \end{aligned} \tag{3.29}$$

The coupling Δ is a *relevant* perturbation as it transforms like $\Delta' = e^{\ell}\Delta$, or

$$\dim[\Delta] = 1 \tag{3.30}$$

As shown below, Δ is the only relevant perturbation, and therefore the exponent ν (defined below (2.25) is given by

$$\nu = 1 \tag{3.31}$$

We now show that no other relevant perturbations to $\mathcal{L}_I$ which respect the symmetry (3.4) of the Ising model, and the simplicity of the argument is another illustration of the power of the scaling analysis. There are two different types of perturbations to $\mathcal{L}_I$ that are possible. The first type arises higher spatial gradients in the mapping from the particular Hamiltonian H_I, and the simplest of these is

$$\lambda_1 \Psi^\dagger \frac{\partial^2 \Psi}{\partial x^2}. \tag{3.32}$$

The second type comes from additional terms we could add to H_I, like $\sigma_i^x \sigma_{i+1}^x$, which respect the symmetry (3.4) and are therefore not expected to modify qualitative features of the transition; after the Jordan-Wigner transformation, and expansion in spatial gradients, such a term induces in the continuum

$$\lambda_2 \Psi^\dagger \frac{\partial \Psi^\dagger}{\partial x} \frac{\partial \Psi}{\partial x} \Psi. \tag{3.33}$$

A simple computation now shows that

$$\dim[\lambda_1] = -1 \qquad \dim[\lambda_2] = -2 \tag{3.34}$$

and hence both are irrelevent.

The absence of other relevant perturbations at $\Delta = 0$ implies that $\mathcal{L}_I$ is the universal continuum quantum field theory describing crossovers near the $\Delta = 0$, $T = 0$ quantum-critical point. It is fortunate that this universal theory happens to be expressible as a free fermion model. Although our original motivation for examining H_I was its solvability, the arguments of this section have shown that this choice also happily co-incides with that required for obtaining a universal critical theory.

Let us now compute finite temperature correlators of the free fermion field Ψ. These correlators are not related to any local observable of the Ising chain, and therefore cannot be measured experimentally. Our main purpose in discussing them is to present scaling ideas in a simple context; for simplicity we will restrict ourselves to the critical point $\Delta = 0$, although the results for $\Delta \neq 0$ are easy to obtain. The two-point Ψ correlators can be computed by performing the analog of the lattice Bogoluibov transformation on the continuum theory. We found for imaginary time $\tau > 0$

$$\begin{aligned}\left\langle \Psi(x,\tau)\Psi^\dagger(0,0)\right\rangle &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{ikx}}{e^{c|k|/T}+1}\left(e^{c|k|(1/T-\tau)}+e^{c|k|\tau}\right)\\ &= \left(\frac{T}{4c}\right)\left(\frac{1}{\sin(\pi T(\tau - ix/c))}+\frac{1}{\sin(\pi T(\tau + ix/c))}\right).\end{aligned} \tag{3.35}$$

We are now using units in which $\hbar = k_B = 1$, and will continue to do so in the remainder of the paper. In a similar manner, we can find

$$\left\langle \Psi(x,\tau)\Psi(0,0)\right\rangle = \left(\frac{iT}{4c}\right)\left(\frac{1}{\sin(\pi T(\tau - ix/c))}-\frac{1}{\sin(\pi T(\tau + ix/c))}\right). \tag{3.36}$$

The results (3.35,3.38) have precisely the scaling forms that would have been expected under the scaling dimensions in (3.29). At $T = 0$, (3.35) simplifies to

$$\left\langle \Psi(x,\tau)\Psi^\dagger(0,0)\right\rangle = \frac{1}{4\pi}\left(\frac{1}{c\tau - ix}+\frac{1}{c\tau + ix}\right), \tag{3.37}$$

when we notice that the analog of the transformation (2.55)

$$c\tau \pm ix \rightarrow \frac{c}{\pi T}\sin\left(\frac{\pi T}{c}(c\tau \pm ix)\right) \tag{3.38}$$

connects the $T = 0$ and $T > 0$ results. This is again due to the conformal invariance of $\mathcal{L}_I$. We will use (3.38) in an important way later in this section.

Correlators of σ^x can be constructed out of those of simple bilinears of the fermion operators, and we will not display them explicitly. More interesting, however, are the correlators of the order parameter σ^z. Computing just the equal time two-point correlator, or even simply the value of $\dim[\sigma^x]$ at the $\Delta = 0$ critical point, involves a rather lengthy and involved computation, which will not be discussed here. Rather, in the next section, we will quote some recently obtained technical results and then proceed to a physical discussion of the crossovers at finite temperature.

D. Finite temperature crossovers

A useful way to begin our of study finite T crossovers in the correlations of the order parameter σ^z is to examine the equal time correlators. These were studied recently in Ref [13], where some old results of McCoy [14] were used to obtain their exact long-distance behavior in the continuum limit. As the computations are somewhat technical, we will not go over them here; we simply quote the final result:

$$\left\langle \sigma^z(x,\tau=0)\sigma^z(0,0)\right\rangle = ZT^{1/4}G_I(\Delta/T)\exp\left(-\frac{T|x|}{c}F_I(\Delta/T)\right) \quad \text{as } |x|\rightarrow\infty \tag{3.39}$$

where Z is non-universal constant, and $F_I(s)$ and $G_I(s)$ are universal scaling functions.

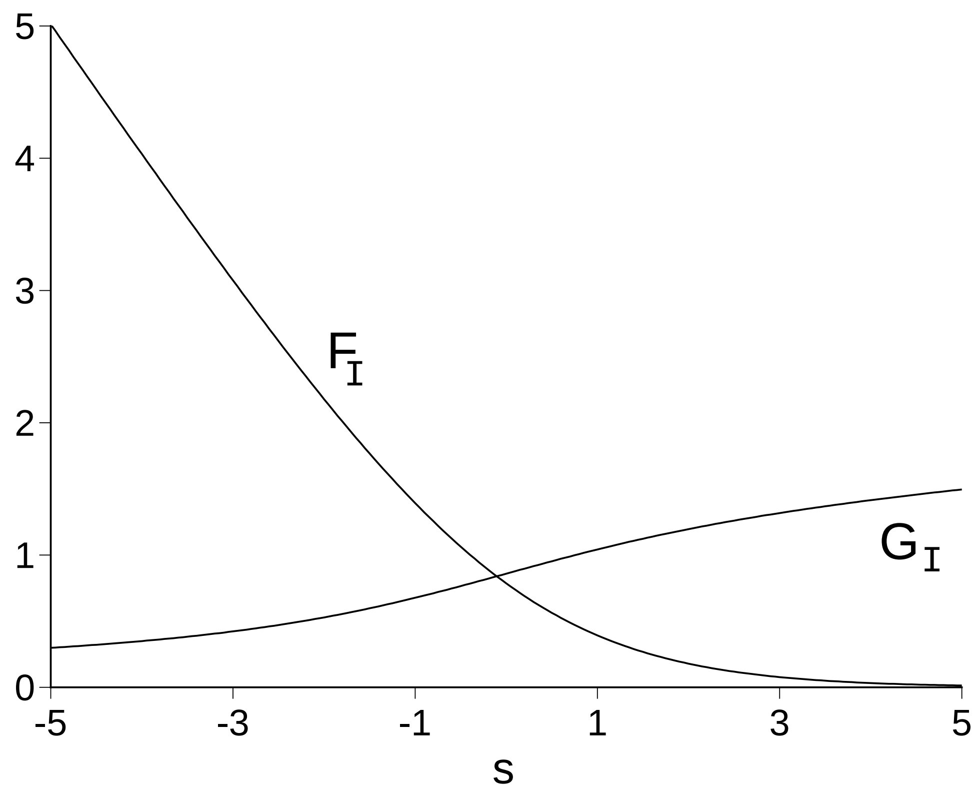

FIG. 6. The crossover functions for the correlation length (F_I) and the amplitude (G_I) as a function of $s = \Delta/T$.

The result (3.39) implies that the correlation length ξ obeys

$$\xi^{-1} = \frac{T}{c} F_I\left(\frac{\Delta}{T}\right) \tag{3.40}$$

The form of this result is precisely what could have been expected on general scaling grounds with $\dim[T] = z = 1$, and the length (as for all lengths) $\dim[\xi] = -1$. The velocity c has scaling dimension 0, but is present to ensure the correct engineering dimensions. The function $F_I(s)$ was determined exactly and is given by

$$F_I(s) = \frac{1}{\pi} \int_0^\infty dy \ln \coth \frac{(y^2 + s^2)^{1/2}}{2} + |s|\theta(-s) \tag{3.41}$$

Despite appearances, the function $F_I(s)$ is smooth as a function of s for all real s, and is analytic at $s = 0$. The analyticity at $s = 0$ is required by the absence of any thermodynamic singularity at finite T for $\Delta = 0$. This is a key property, which was in fact used to obtain the answer in (3.41). The exact expression for the function $G_I(s)$ is also known

$$\ln G_I(s) = \int_s^1 \frac{dy}{y} \left[\left(\frac{dF_I(y)}{dy} \right)^2 - \frac{1}{4} \right] + \int_1^\infty \frac{dy}{y} \left(\frac{dF_I(y)}{dy} \right)^2 , \tag{3.42}$$

and its analyticity at $s = 0$ follows from that of F_I. We show a plot of F_I and G_I in Fig 6.

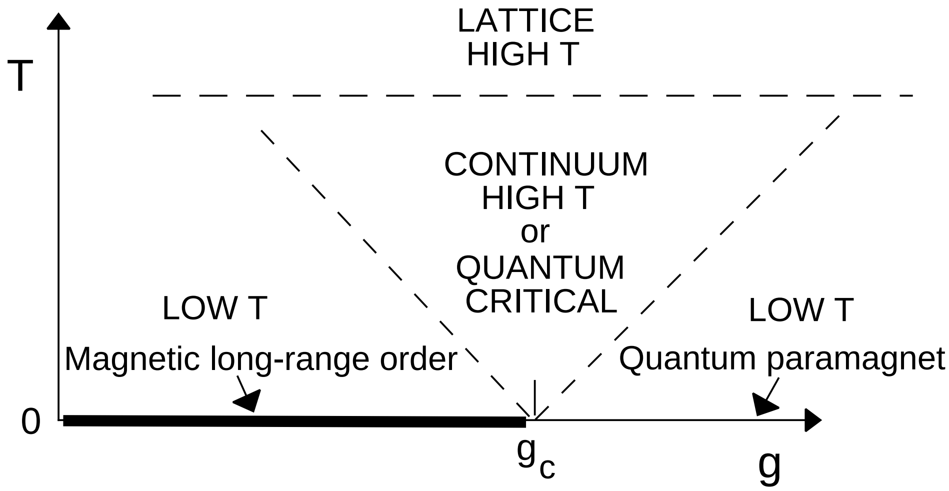

FIG. 7. Finite T phase diagram of the $d = 1$ quantum Ising model, H_I, as a function of the coupling g and temperature T. Long-range order ($N_0 = \langle \sigma_z \rangle \neq 0$) is present only for $T = 0$ and $g < g_c$. The energy scale $\Delta \sim g_c - g$, and dashed lines are crossovers at $|\Delta| \sim T$

The prefactor Z in (3.39) is a non-universal constant, as noted earlier. For the solvable H_I, we chose the overall normalization of G_I such that $Z = J^{-1/4}$. In general, the value of Z is set by relating it to an observable, as we will show below. Also note that Z has no dependence on Δ, and is therefore non-singular at the quantum critical point.

The power of T is front of (3.39), and the value $z = 1$, tell us

$$\dim[\sigma^z] = 1/8. \tag{3.43}$$

Armed with the knowledge of the scaling dimensions, we can write down the full scaling form for the time-dependent σ^z correlator, which applies to the lattice model in the limits $\Lambda \sim\!\!\rightarrow \infty$, $a \rightarrow 0$ at fixed Δ, c and T

$$\langle \sigma^z(x,\tau)\sigma^z(0,0)\rangle = ZT^{1/4}\Phi_I\left(\frac{Tx}{c}, T\tau, \frac{\Delta}{T}\right) \tag{3.44}$$

where Φ_I is a universal function which is analytic as a function of its third argument $s = \Delta/T$ on the real s axis. The result (3.39) obviously specifies Φ_I for large Tx/c and $\tau = 0$.

We will now describe the physical properties of Φ_I in different regions of the phase diagram as a function of g and T, sketched in Fig 7.

1. Low T on the ordered side, $\Delta > 0$, $T \ll \Delta$

This is the "renormalized classical" [15] region of Fig 7, and the reasons for this name will become clear below.

Assuming that it is valid to interchange the limits $T \rightarrow 0$ and $x \rightarrow \infty$ in (3.39), we can use the limiting values $F_I(\infty) = 0$, $G_I(s \rightarrow \infty) = s^{1/4}$ to deduce that (recall (3.14):

$$N_0^2 \equiv \lim_{|x|\rightarrow\infty} \langle \sigma^z(x,0)\sigma^z(0,0)\rangle = Z\Delta^{1/4} \qquad \text{at } T = 0. \tag{3.45}$$

Thus, as claimed earlier, there is long-range order in the $g < 1$ ground state of H_I, with the order parameter $N_0 \langle \sigma^z \rangle = Z^{1/2} \Delta^{1/8}$ (this relates the value of Z to a physical observable). For small $T \ll \Delta$, we obtain from the large s behavior of $F_I(s)$ (see (3.41)) that

$$\langle \sigma^z(x,0) \sigma^z(0,0) \rangle = N_0^2 e^{-|x|/\xi_c} \quad \text{large } |x|, \tag{3.46}$$

where the correlation length

$$\xi_c^{-1} = \left(\frac{2\Delta T}{\pi c^2} \right)^{1/2} e^{-\Delta/T}. \tag{3.47}$$

We have put a subscript c on the correlation length to emphasize that the system is expected to behave *classically* in this low temperature region [13]. The excitations above the ground states consists of particles (the kinks and anti-kinks of Section III A) whose mean separation ($\sim \xi_c$) is much larger than their de Broglie wavelengths ($\sim (c^2/\Delta T)^{1/2}$, as the mass of these particles $= \Delta/c^2$), which is precisely the canonical condition for the applicability of classical physics. More generally, we expect classical models to apply near any phase with true long-range order. It is also reassuring to note that (3.46) is precisely the form of equal-time correlations in the classical Ising model at low T. The prefactor N_0^2 is the true ground state magnetization including the effects of quantum fluctuations, and this is the reason for the adjective "renormalized" in the name for this region.

We now conjecture that the mathematical structure of the emergence of classical physics in the scaling form (3.44), is analogous to the appearance of Fermi liquid behavior in the dilute spinless Fermi gas, as discussed in Section II B(b). This means that the function Φ_I should collapse into a *reduced scaling function*, Φ_{Ic}, in the limit $\Delta \ll T$, and further that Φ_{Ic} should be obtainable completely from an effective classical model. Using the form (3.46), it is easy to see that

$$\langle \sigma^z(x,t) \sigma^z(0,0) \rangle = N_0^2 \Phi_{Ic} \left(\frac{x}{\xi_c}, \gamma \frac{t}{\xi_c^{z_c}} \right), \tag{3.48}$$

where t now denotes *real* physical time. The dynamics of the classical Ising model determines the dynamic exponent z_c; there is no fundamental scaling constraint which states that z_c must equal the dynamic exponent $z = 1$ of the $\Delta = 0$ quantum critical point, just as the dynamic exponent of the Fermi liquid scaling ($z = 1$, Eqn (2.45)) was different from that the $\mu = 0$ critical point of the dilute spinless Fermi gas ($z = 2$, Eqn (2.21)) in Section II B. The function Φ_{Ic} is contained wholly within the universal function Φ_I, and so the prefactor γ must also be universal; on purely dimensional grounds we expect

$$\gamma = c \left(\frac{c}{\Delta} \right)^{z_c - 1} \left(\frac{\Delta}{T} \right)^{\rho} \mathcal{R} \tag{3.49}$$

where $\mathcal{R}$ and ρ are universal numbers.

This is as far as general scaling ideas will take us; further progress requires specification of the classical dynamics of the effective model and its solution. This has very recently been done by A.P. Young and the author, who obtained the full space-time dependent function Φ_{Ic} and the values of $\mathcal{R}$ and ρ exactly. The reader is referred to Ref [16] for more details. The effective classical dynamics of the spins is *relaxational*, but an earlier conjecture by the author [13] that the relaxational dynamics is described by the Glauber model [17] is incorrect.

2. Low T on the disordered side, $\Delta < 0$, $T \ll |\Delta|$

This is the "quantum disordered" region of Fig 7.

Now we need to take the $s \to -\infty$ limit of the functions $F_I(s)$, $G_I(s)$; from these limits we find

$$\langle \sigma^z(x,0)\sigma^z(0,0)\rangle = \frac{ZT}{|\Delta|^{3/4}} e^{-|r|/\xi} \quad |x| \to \infty \text{ at fixed } 0 < T \ll |\Delta|, \tag{3.50}$$

with the correlation length ξ given by

$$\xi^{-1} = \frac{|\Delta|}{c} + \left(\frac{2|\Delta|T}{\pi c^2}\right)^{1/2} e^{-|\Delta|/T} \tag{3.51}$$

So correlations decay exponentially on a scale $\sim c/|\Delta|$, and there is no long-range order. We have a similar result for equal time correlations at $T = 0$, although the limits $T \to 0$ and $|x| \to \infty$ do not commute for the prefactor of the exponential decay. The $T = 0$ result is determined by the form-factor expansion technique [18], which we will not discuss here; it yields [19,20]

$$\langle \sigma^z(x,0)\sigma^z(0,0)\rangle = Z|\Delta|^{1/4} \left(\frac{c}{2\pi|\Delta||x|}\right)^{1/2} e^{-\Delta|r|/c} \quad |x| \to \infty \text{ at } T = 0. \tag{3.52}$$

The form factor expansion also yields the $T = 0$ dynamic susceptibility (defined in (3.11)). The leading term is in fact precisely the quasiparticle pole at energy $\epsilon_k = (c^2k^2 + \Delta^2)^{1/2}$ that was argued to exist in this phase in Section III A. We have

$$\chi(k,\omega) = \frac{2Z\Delta^{1/4}}{c^2k^2 + \Delta - (\omega + i\epsilon)^2} + \ldots \quad T = 0 \tag{3.53}$$

where ϵ is a positive infinitesimal. The imaginary part of this gives the delta function sketched in Fig 5; the continuum of excitations above the three particle threshold come from higher order terms in the form factor expansion, and are respresented by the ellipses in (3.53). It can now be checked that the Fourier transform of (3.53) yields the leading term (3.52) in the equal time correlation function.

The result (3.53) shows that the *quasiparticle residue* is $Z\Delta^{1/4}$ (this is another relation between Z and a physical measureable, and along with (3.45), it implies a relationship between the values of the residue and N_0 as we approach the critical point from either side). The residue vanishes at the critical point $\Delta = 0$, where the quasiparticle picture breaks down, and we will have a completely different structure of excitations.

The above is an essentially complete description of the correlations and excitations of the quantum paramagnetic ground state. At finite T, there will be a small density of quasi-particle excitations which will behave classical for the same reasons as in Section III D 1: their mean spacing is much larger than their de Broglie wavelength. The motion of these thermally excited particles leads to a small dissipative broadening for the quasi-particle delta function at a spatial scales of order the mean spacing between the particles. The details of the classical relaxational dynamics leading to this broadening have recently been worked out by A.P. Young and the author, and the reader is referred to Ref [16] for further details.

3. High T, $T \gg |\Delta|$

Right at the critical point, $\Delta = 0$, this regime extends all the way down to $T = 0$. We begin by writing the $T = 0$ equal-time correlator of the continuum theory; from the scaling dimension of σ^z this must have the form

$$\langle \sigma^z(x,0)\sigma^z(0,0)\rangle \sim \frac{1}{(|x|/c)^{1/4}} \qquad \text{at } T=0,\ \Delta=0, \tag{3.54}$$

We will now fix the prefactor in (3.54) using our earlier results. The key ingredient is our knowledge that the underlying continuum model $\mathcal{L}_I$ is is relativistically and conformally invariant at the $T = 0$ critical point. Further we assume that the mapping (3.38) between correlators at $T \neq 0$ from those at $T = 0$ holds also for the two point correlator. of σ^z Then from (3.54) we must have at $T \neq 0$

$$\langle \sigma^z(x,\tau)\sigma^z(0,0)\rangle \sim T^{1/4}\frac{1}{\left[\sin(\pi T(\tau - ix/c))\sin(\pi T(\tau + ix/c))\right]^{1/8}} \qquad \text{at } \Delta = 0. \tag{3.55}$$

Let us now use this result in the equal-time case in the regime $xT/c \gg 1$. Precise results for this regime where quoted earlier in (3.39), where using the values $F_I(0) = \pi/4$ (from evaluation of (3.41)) and $G_I(0) = 0.858714569\ldots$ we have

$$\langle \sigma^z(x,\tau=0)\sigma^z(0,0)\rangle = ZT^{1/4}G_I(0)\exp\left(-\frac{\pi T|x|}{4c}\right) \qquad \text{as } |x|\to\infty \text{ at } \Delta=0 \tag{3.56}$$

Finally, comparing with (3.55) we obtain

$$\langle \sigma^z(x,\tau)\sigma^z(0,0)\rangle = ZT^{1/4}\frac{2^{-1/8}G_I(0)}{\left[\sin(\pi T(\tau - ix/c))\sin(\pi T(\tau + ix/c))\right]^{1/8}} \qquad \text{at } \Delta = 0. \tag{3.57}$$

As expected, this result is of the scaling form (3.44), and indeed completely determines the function Φ_I for the case where its last argument is zero.

Now let us turn to a physical interpretation of the main result (3.57). Consider first the case $T = 0$. By a Fourier transformation of the $T = 0$ limit of (3.57) we obtain the dynamic susceptibility

$$\chi(k,\omega) = Z(4\pi)^{3/4}G_I(0)\frac{\Gamma(7/8)}{\Gamma(1/8)}\frac{c}{(c^2k^2-(\omega+i\epsilon)^2)^{7/8}} \qquad T=0,\ \Delta=0 \tag{3.58}$$

We plot $\text{Im}\chi(k,\omega)/\omega$ in Fig 2. Notice that there are no delta functions in the spectral density like there were in the quantum disordered phase (Fig 5), indicating the absence of any well-defined quasiparticles. Instead, we have a critical continuum of excitations.

Now let us turn to non-zero T. We Fourier transform (3.57) to obtain $\chi(k, i\omega_n)$ at the Matsubara frequencies ω_n and then analytically continue to real frequencies (there are some interesting subtleties in the Fourier transform to $\chi(k, i\omega_n)$ and its analytic structure in the complex ω plane, which are discussed elsewhere [6]). This gives us the leading result for $\chi(k,\omega)$ in the high T region

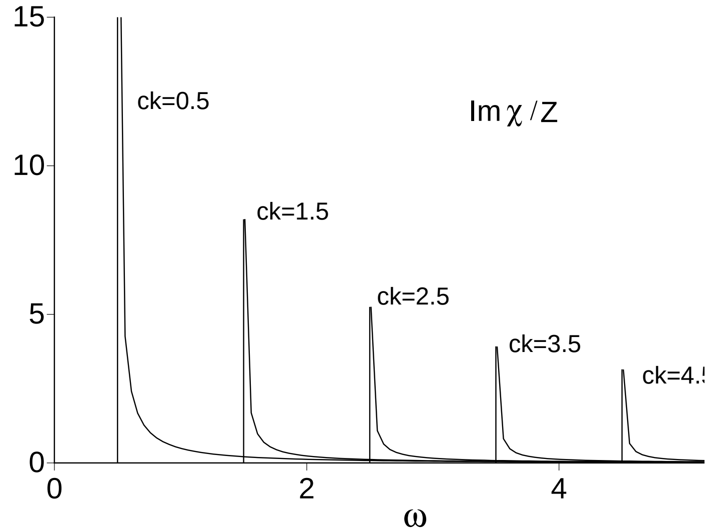

FIG. 8. Spectral density, $\text{Im}\chi(k,\omega)/\omega Z$, of H_I at its critical point $g = 1$ $(\Delta = 0)$ at $T = 0$, as a function of frequency ω, for a set of values of k.

$$\chi(k,\omega) = \frac{Zc}{T^{7/4}} \frac{G_I(0)}{4\pi} \frac{\Gamma(7/8)}{\Gamma(1/8)} \frac{\Gamma\left(\frac{1}{16} + i\frac{\omega + ck}{4\pi T}\right) \Gamma\left(\frac{1}{16} + i\frac{\omega - ck}{4\pi T}\right)}{\Gamma\left(\frac{15}{16} + i\frac{\omega + ck}{4\pi T}\right) \Gamma\left(\frac{15}{16} + i\frac{\omega - ck}{4\pi T}\right)} \qquad \Delta = 0. \qquad (3.59)$$

We show a plot of $\text{Im}\chi/\omega$ in Fig 9. This result is the finite T version of Fig 8. Notice that the sharp features of Fig 8 have been smoothed out on the scale T, and there is non-zero absorption at all frequencies. For $\omega, k \gg T$ there is a well-defined peak in $\text{Im}\chi/\omega$ (Fig 9) rather like the $T = 0$ critical behavior of Fig 8. However, for $\omega, k \ll T$ we cross-over to the **quantum relaxational** regime [21] and the spectral density $\text{Im}\chi/\omega$ is similar to (but not identical) a Lorentzian around $\omega = 0$. This relaxational behavior can be characterized by a relaxation rate Γ_R defined as

$$\Gamma_R^{-1} = -i \left. \frac{\partial \ln \chi(0,\omega)}{\partial \omega} \right|_{\omega=0} \qquad (3.60)$$

(this is motivated by the phenomenological relaxational form $\chi(0,\omega) = \chi_0/(1 - i\omega/\Gamma_R + \mathcal{O}(\omega^2))$). From (3.59) we determine:

$$\Gamma_R = \left(2\tan\frac{\pi}{16}\right)\frac{k_B T}{\hbar}, \qquad (3.61)$$

where we have returned to physical units. At the scale of the characteristic rate Γ_R, the dynamics of the system involves intrinsic quantum effects (responsible for the non-Lorentzian lineshape) which cannot be neglected; description by an effective classical model (as was appropriate in both the renormalized classical and quantum disordered regions of Fig 7) would require that $\Gamma_R \ll k_B T/\hbar$, which is thus not satisfied in the high T region of Fig 7. The ease with which (3.61) was obtained belies its remarkable nature. Notice that we are working in a closed Hamiltonian system, evolving unitarily in time with the operator $e^{-iH_I t/\hbar}$,

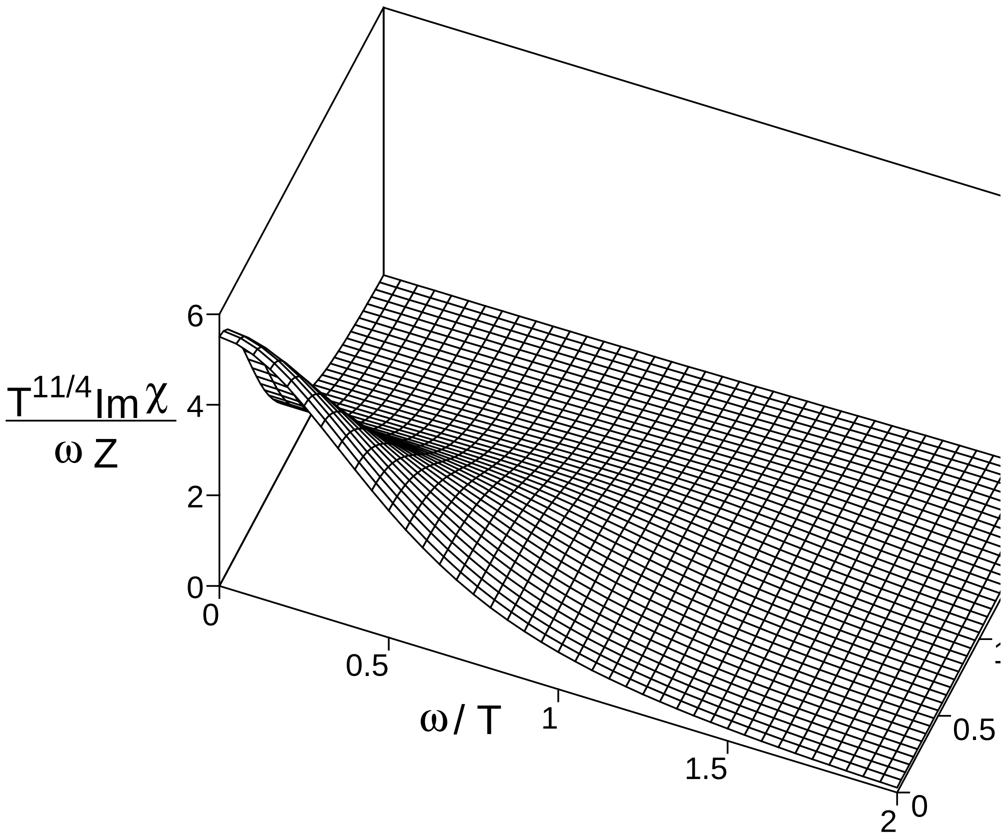

FIG. 9. The same observable as in Fig 8, $T^{11/4}\text{Im}\chi(k,\omega)/\omega Z$, but for $T \neq 0$. This is the leading result for Imχ for $T \gg |\Delta|$ *i.e.* in the high T region of Fig 7. All quantities are scaled appropriately with powers of T, and the absolute numerical values of both axes are meaningful.

from an initial density matrix given by the Gibbs ensemble at a temperature T. Yet, we have obtained relaxational behavior at low frequencies, and determined an exact value for a dissipation constant. Such behavior is more typically obtained in phenomenological models which couple the system to an external heat bath and postulate an equation of motion of the Langevin type.

4. Summary

The main features of the finite temperature physics of the quantum Ising model are summarized in Figs 7 and Fig 10. At short enough times or distances in all three regions of Fig 7, the systems displays critical fluctuations characterized by the dynamic susceptibility (3.58). The regions are distinguished by their behaviors at the low frequencies and momenta. In both the low T regimes of Fig 7 (renormalized classical and quantum disordered), the long time dynamics is relaxational and is described by effective classical models. In contrast, the dynamics in the high T region is also relaxational, but involves quantum effects in an essential way, as was described above.

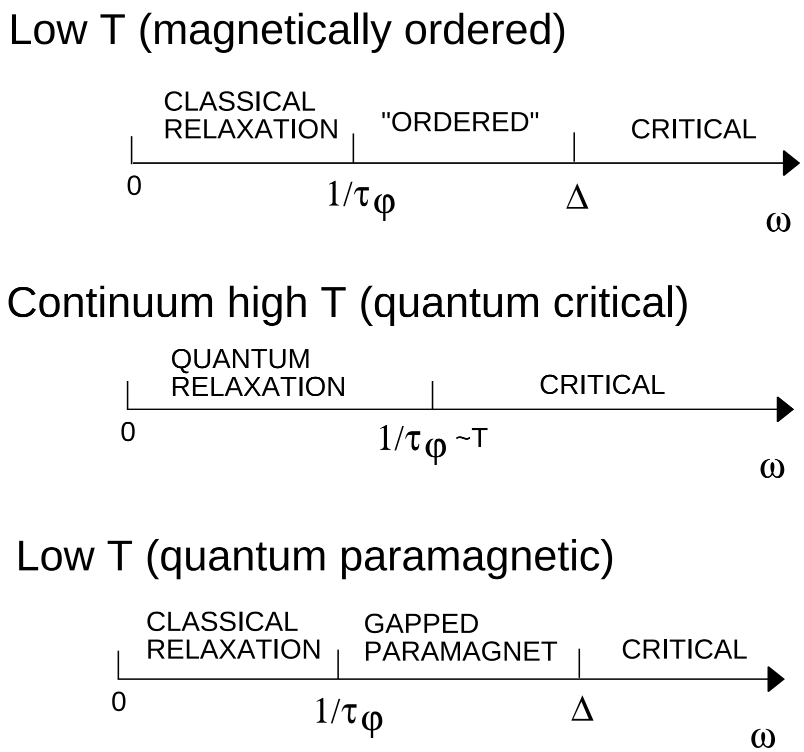

FIG. 10. Crossovers as a function of frequency for the Ising model in the different regimes of Fig 7. The correlations in the two classical relaxational regimes are quite different from each other. The "ordered" regime is in quotes, because there is no long-range order, and the system only appears ordered between spatial scales c/Δ and ξ_c. The reader is referred to Ref [16] for a precise specification of the frequency scales $\tau_R \sim e^{|\Delta|/T}$.

IV. QUANTUM ROTORS BETWEEN ONE AND THREE DIMENSIONS

This section is our first foray into the examination of quantum critical points in greater than one dimension. Much of the technology and the physical ideas introduced earlier for $d = 1$ will generalize rather straightforwardly to higher dimensions, although we will no longer be able to obtain exact results for crossover functions. The characterization of the physics in terms of three regions separated by smooth crossovers, the high T and the two low T regions on either side of the quantum critical point, will continue to be extremely useful in higher dimensions, and will again be the basis of our discussion. We will however, meet several genuinely new features. It is now possible for systems to have a thermodynamic phase transition at a non-zero temperature. We shall be particularly interested in the interplay between the critical singularties of the finite temperature transition and those of the quantum critical point. The concept of a 'reduced scaling function', introduced earlier in the discussion of the free Fermi gas, and used again to describe renormalized classical behavior in Section III D 1, will turn out to be precisely what is needed to deal with this intricate phenomenon.

The $O(N)$ quantum rotor model is defined by the following Hamiltonian on the sites i of a regular d dimensional lattice:

$$H_R = \frac{Jg}{2} \sum_i \overleftrightarrow{L}^2{}_i - J \sum_{\langle ij \rangle} \vec{n}_i \cdot \vec{n}_j, \tag{4.1}$$

where the sum $\langle ij \rangle$ is over nearest neighbors, $J > 0$ is an overall energy scale, and $g > 0$ is a dimensionless coupling constant. The N component vectors The N-component vectors $\vec{n}_i$, with $N \geq 2$, are of unit length, $\vec{n}_i^2 = 1$, and represent the orientation of the rotors on the surface of a sphere in N-dimensional rotor space. The operators $L_{i\mu\nu}$ ($\mu < \nu$, μ, $\nu = 1 \ldots N$) are the $N(N-1)/2$ components of the angular momentum $\overleftrightarrow{L}_i$ of the rotor: the first term in H_R is the kinetic energy of the rotor with $1/g$ the moment of inertia. The different components of $\vec{n}_i$ constitute a complete set of commuting observables and the state of the system can be described by a wavefunction $\Psi(\vec{n}_i)$. The action of $\overleftrightarrow{L}_i$ on Ψ is given by the usual differential form of the angular momentum

$$L_{i\mu\nu} = -i \left(n_{i\mu} \frac{\partial}{\partial n_{i\nu}} - n_{i\nu} \frac{\partial}{\partial n_{i\mu}} \right). \tag{4.2}$$

The commutation relations among the $\overleftrightarrow{L}_i$ and $\vec{n}_i$ can now be easily deduced. We emphasize the difference of the rotors from Heisenberg-Dirac quantum spins: the components of the latter at the same site do not commute, whereas the components of the $\vec{n}_i$ do.

There is a strong analogy between the rotor Hamiltonian H_R in (4.1) and the Ising Hamiltonian H_I in (3.3). We will be looking at the transition between a magnetically ordered state with $\langle \vec{n} \rangle \neq 0$ and $O(N)$ symmetry broken, and a quantum paramagnet in which correlations of $\vec{n}$ are short ranged. As in the Ising model, it is the exchage term, proportional to J, that favors the ordered state, while the 'kinetic energy', proportional to Jg leads to fluctuations in the orientation of the order parameter and eventually to loss of long-range order. The similarity between the two models will also be apparent in the strong (large g) and weak coupling (small g) analyses in the following section.

A. Limiting cases

The pictures which emerge in the following two perturbative analyses are expected to hold on either side of a quantum critical point at $g = g_c$.

1. Strong coupling $g \gg 1$

At $g = \infty$, the exchange term in H_R can be neglected, and the Hamiltonian decouples into independent sites, and can be diagonalized exactly. The eigenstates on each site are the eigenstates of $\overleftrightarrow{L}^2$; for $N = 3$ these are the states

$$|\ell, m\rangle_i \quad \ell = 0, 1, 2, \ldots, \quad -\ell \leq m \leq \ell \tag{4.3}$$

and have eigenenergy $Jg\ell(\ell+1)/2$. Compare this single site spectrum with that of a pair of Heisenberg-Dirac spin S quantum spins with an antiferromagnetic exchange K; for a suitably chosen K we get the same sequence of levels and energies but with a maximum allowed value of $\ell = 2S$. Assuming this upper cutoff is not crucial for the low energy physics, we can use a single quantum rotor is an effective model for a *pair* of spins.

The ground state of H_R in the large g limit consists of the quantum paramagnetic state with $\ell = 0$ on every site:

$$|0\rangle = \prod_i |\ell = 0, m = 0\rangle_i \tag{4.4}$$

Compare this with strong coupling ground state (3.5) of the Ising model. Indeed, the remainder of the strong coupling analysis of Section III A can borrowed here for the rotor model, and we can therefore be quite brief. The lowest excited state is a 'particle' in which a single site has $\ell = 1$, and this excitation hops from site to site. An important difference from the Ising model is that this particle is three-fold degenerate, corresponding to the three allowed values m. The dynamic susceptibility has a quasiparticle pole at the energy of this particle, and odd particle continua above the three particle threshold.

2. Weak coupling, $g \ll 1$

At $g = 0$, the ground state breaks $O(N)$ symmetry, and all the $\vec{n}_i$ vectors orient themselves in a common, but arbitrary direction. Excitations above this state consist of 'spin waves' which can have an arbitrarily low energy. This is a crucial difference from the Ising model, in which there was an energy gap above the ground state. The presence of gapless spin excitations is a direct consequence of the continuous $O(N)$ symmetry of H_R: we can make very slow deformations in the orientation of $\langle \vec{n} \rangle$, and get an orthogonal state whose energy is arbitrarily close to that of the ground state. Explicitly, for $N = 3$, and a ground state polarized along $(1, 0, 0)$ we parametrize

$$\vec{n}(x, t) = (1, \pi_1(x, t), \pi_2(x, t)) \tag{4.5}$$

where $|\pi_1|, |\pi_2| \ll 1$, and look at the linearized equations of motion for π_1, π_2. A standard calculation then gives harmonic spin waves with energy $\omega = ck$ (c is the spin wave velocity); their wavefunctions can then be contructed using harmonic oscillator states.

B. Continuum theory and large N limit

To obtain the path integral representation of the quantum mechanics of H_R, we interpret the $\vec{n}_i$ as the co-ordinates of particles constrained to move on the surface of a sphere in N dimensions: we can then simple use the standard Feynman path integral representation of single-particle quantum mechanics. After taking the continuum limit, such a procedure gives

$$Z = \int \mathcal{D}\vec{n}(x,\tau)\delta(\vec{n}^2(x,\tau) - 1) \exp\left(- \int_0^{1/T} d\tau \int d^d x \mathcal{L} \right)$$

$$\mathcal{L} = \frac{N}{2c\tilde{g}} \left[c^2 \left(\frac{\partial \vec{n}}{\partial x_i} \right)^2 + \left(\frac{\partial \vec{n}}{\partial \tau} \right)^2 \right] \tag{4.6}$$

Here $c \sim Ja$ is a velocity which will turn out to be the spin wave velocity, and we have set $\hbar = k_B = 1$. The prefactor of N is for future convenience. The coupling constant $\tilde{g} \sim g a^{d-1}$ has the dimensions of (length)$^{d-1}$; we will not use the original g in H_R further in this discussion, and we will drop the tilde in $\tilde{g}$ from now. The above action is valid only at long distances and times, so there is an implicit cutoff above momenta of order $\Lambda \sim 1/a$ and frequencies of order $c\Lambda$. Our main interest here shall be the universal physics at scales much smaller than Λ.

This section shall present a study of the model (4.6) in the limit of a large number of components of $\vec{n}$ *i.e.* at $N = \infty$ [21]. The model is exactly soluble in this limit, and displays an interesting quantum phase transition. Most features of the finite temperature crossovers turn out not to be artifacts of the $N = \infty$ point, and thus provide a useful introduction to quantum critical points in dimensions $d > 1$. Some dynamic properties are however not correctly captured at $N = \infty$, and require loop corrections which are discussed in the literature [21].

The framework of the $N = \infty$ solution is quite easy to set up, at least in the phase without long range order in the order parameter $\vec{n}$; we will consider the case with long range order later in this chapter. We rescale the $\vec{n}$ field to

$$\vec{\tilde{n}} = \sqrt{N}\vec{n}, \tag{4.7}$$

and impose the $\vec{\tilde{n}}^2 = N$ constraint by a Lagrange multiplier, λ. The action is then quadratic in the $\vec{\tilde{n}}$ field, which can then be integrated out to yield

$$Z = \int \mathcal{D}\lambda(x,\tau) \exp\left[-\frac{N}{2}\left(\mathrm{Tr}\ln(-c^2\partial_i^2 - \partial_\tau^2 + i\lambda) - \frac{i}{cg}\int_0^{1/T} d\tau \int d^d x \lambda\right)\right] \tag{4.8}$$

The action now has a prefactor of N, and the $N = \infty$ limit of the functional integral is therefore given exactly by its saddle point value. We assume that the saddle-point value of λ is space and time independent, and given by $i\lambda = \sigma^2$. The saddle-point equation determining the value of the parameter σ^2 is

$$\int^{\Lambda} \frac{d^d k}{(2\pi)^d} T \sum_{\omega_n} \frac{1}{c^2 k^2 + \omega_n^2 + \sigma^2} = \frac{1}{cg}, \tag{4.9}$$

where the sum over ω_n extends over the Matsubara frequencies $\omega_n = 2n\pi T$, n integer. It is also not difficult to see that the retarded dynamic susceptibility $\chi(k,\omega)$, which is the two-point correlator of the n field, is given by

$$\chi(k,\omega) = \frac{cg/N}{c^2k^2 - (\omega + i\epsilon)^2 + \sigma^2}, \tag{4.10}$$

at $N = \infty$. The Eqns (4.9,4.10) are the central results of the $N = \infty$ theory, and most of the remainder of this chapter will be spent on analyzing their consequences. In spite of their extremely simple structure, these equations contain a great deal of information, and it takes a rather subtle and careful analysis to extract the universal information contained in them [21]. We will display all the details, as similar ideas can be used in a variety of contexts. We will begin by characterizing the $T = 0$ ground states, and then turn to the finite temperature crossovers.

C. Zero temperature

At $T = 0$, we can make use of the relativistic invariance of the action (4.6) to simplify our analysis. The summation over Masubara frequencies in (4.9) turns into an integral, and after introducting spacetime momentum $p \equiv (k, \omega/c)$, the constraint equation (4.9) becomes

$$\int^{\Lambda} \frac{d^{d+1}p}{(2\pi)^d} \frac{1}{p^2 + (\sigma/c)^2} = \frac{1}{g} \tag{4.11}$$

The integral on the left hand side increases monotonically with decreasing σ, and has its maximum finite value at $\sigma = 0$. It is then clear that there is no solution to (4.11) for $g < g_c$ where

$$\int^{\Lambda} \frac{d^{d+1}p}{(2\pi)^d} \frac{1}{p^2} = \frac{1}{g_c} \tag{4.12}$$

For $g \geq g_c$ there is a unique solution of the saddle-point equation (4.11), which describes a quantum paramagnetic ground state: we will study its properties in the following subsections for $g > g_c$ and $g = g_c$, and find that they are quite similar to those of the Ising chain. Determination of the ground state for $g \leq g_c$ requires a reanalysis of the derivation of the large N saddle equation. This will be done in Section IV C 3, where we find a state with magnetic long-range order and spontaneous breakdown of the $O(N)$ symmetry.

1. *Quantum paramagnet, $g > g_c$*

Subtract (4.11) from (4.12), and obtain

$$\begin{aligned} \frac{1}{g_c} - \frac{1}{g} &= \int \frac{d^{d+1}p}{(2\pi)^d} \left(\frac{1}{p^2} - \frac{1}{p^2 + (\sigma/c)^2} \right) \\ &= X_{d+1} (\sigma/c)^{d-1}, \end{aligned} \tag{4.13}$$

where the constant $X_d \equiv 2\Gamma((4-d)/2)(4\pi)^{-d/2}/(d-2)$. In the last equation, we have used the fact that the integral was convergent at large momenta to send the cut-off Λ to infinity. This result gives us the value of σ for all $g \geq g_c$.

A key step in the analysis of any ground state of a continuum theory, is the determination of an energy scales which characterizes it. In this case, the quantum paramagnet has a gap, Δ_+, given by

$$\Delta_+ = \sigma \qquad T = 0. \tag{4.14}$$

We emphasize that, by definition, the gap Δ_+ is a temperature independent quantity, and equals the temperature dependent value of σ only at $T = 0$. The presence of a gap is apparent in the structure of the spectral density $\chi''(k, \omega)$, which from (4.10) is given by

$$\chi''(k, \omega) = \mathcal{A} \frac{\pi}{2\sqrt{c^2k^2 + \Delta_+^2}} \left(\delta(\omega - \sqrt{c^2k^2 + \Delta_+^2}) - \delta(\omega + \sqrt{c^2k^2 + \Delta_+^2}) \right) \tag{4.15}$$

which has weight only at frequencies greater than Δ. The spectral weight appears entirely in the form of delta functions which indicate the presence of magnon quasiparticles; the quantity $\mathcal{A} = cg/N$ is the quasi-particle residue. This magnon is obviously the same as the three-fold degenerate particle that appeared earlier in the strong-coupling analysis of the $O(3)$ model. The n-particle continua ($n \geq 3$, odd) are absent here in the $N = \infty$ theory, but appear at higher orders in $1/N$.

Also justifying our identification of this phase as a quantum paramagnet, is that equal-time n correlations decay exponentially in space

$$\begin{aligned}\frac{1}{N}\langle \vec{n}(x,0)\cdot\vec{n}(0,0)\rangle &= \mathcal{A}\int \frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{e^{ip\cdot x}}{p^2 + (\Delta_+/c)^2} \\ &= \frac{\mathcal{A}}{2(2\pi)^{d/2}(\Delta_+/c)^{(2-d)/2}}\frac{e^{-x\Delta/c}}{x^{d/2}}\end{aligned} \tag{4.16}$$

which identifies Δ/c as the inverse correlation length. Notice the close similarity of these results to those in Section III D 2 on the Ising model, where $2Z\Delta^{1/4}$ played the role of the quasiparticle residue, $\mathcal{A}$.

2. Critical point, $g = g_c$

As g approaches g_c from above, the energy gap, Δ_+, vanishes as

$$\Delta_+ \sim (g - g_c)^{1/(d-1)} \tag{4.17}$$

The critical state at $g = g_c$ turns out to be scale-invariant at scales much longer than Λ^{-1}, as expected by analogy with the Ising model. The coupling g is the parameter which tunes the system away from this scale-invariant point, and as Δ is an energy (inverse time) scale, the result (4.17) identifies the exponent

$$z\nu = \frac{1}{d-1} \tag{4.18}$$

The equal-time correlations now decay as

$$\begin{aligned}\langle \vec{n}(x,0)\cdot\vec{n}(0,0)\rangle &\sim \int \frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{e^{ikx}}{p^2} \\ &\sim \frac{1}{x^{d-1}}\end{aligned} \tag{4.19}$$

which is a power-law, as expected for a scale-invariant theory; the decay as a function of time has the same exponent, and so

$$z = 1, \tag{4.20}$$

as must be the case for a Lorentz-invariant theory. The application of the scaling transformation on (4.19), also tells us that

$$\dim[\vec{n}] = \frac{d-1}{2} \tag{4.21}$$

The value of the exponent z is exact, as it is fixed by Lorentz invariance of the critical theory, but the values of ν and $\dim[\vec{n}]$ will have corrections at order $1/N$. In general, it is conventional to parametrize

$$\dim[\vec{n}] = \frac{d+z-2+\eta}{2}, \tag{4.22}$$

with η the "anomalous dimension" of the field. Comparing with (4.21) we see that $\eta = 0$ in the $N = \infty$ theory. The exact solution of the Ising chain had $\eta = 1/4$, as that gives $\dim[\sigma^z] = 1/8$. A non-zero, positive, value of η will appear upon consideration of fluctuation corrections, and has important physical consequences. In particular, notice that in the present $N = \infty$ theory, the quasiparticle residue $\mathcal{A}$ was non-zero all the way upto $g = g_c$. A standard scaling argument, which demands the consistency of an expression like (4.15) with the scaling dimension of $\vec{n}$, tells us that we must have

$$\mathcal{A} \sim (g - g_c)^{\eta\nu}, \tag{4.23}$$

i.e. the quasiparticle residue vanishes the system approaches the critical point. Again this scaling is consistent with the Ising model in which $\mathcal{A} = 2Z\Delta^{1/4} \sim (g - g_c)^{1/4}$.

If there are no quasiparticles, what do the excitations look like ? As in the Ising chain, there is a critical continuum of excitations, whose spectral density is determined by η. The dynamic susceptibility has the form

$$\chi(k, \omega) \sim \frac{1}{(c^2k^2 - \omega^2)^{1-\eta/2}} \tag{4.24}$$

and its imaginary part looks much like Fig 8. The $\eta = 0$ case is of course special, in that the spectral density has a single delta function at $\omega = ck$, and the critical excitations have a particle-like nature: this is clearly an artifact of the $N = \infty$ theory, and is its major failing.

3. Magnetically ordered ground state, $g < g_c$

Our analysis so far has shown no meaningful solution of the saddle-point equations in the large N limit for $g < g_c$. The culprit for this shortcoming lies in the step before (4.8), where we indiscriminately integrated out all N components of the $\vec{n}$ field [22]. As we expect a magnetically ordered phase to appear for $g < g_c$, it seems sensible to allow for the possibility that fluctuations of $\vec{n}$ along the direction of the ordered ground state will be different from those orthogonal to it. So we write

$$\vec{\tilde{n}} = (\sqrt{N} r_0, \pi_1, \pi_2 \ldots \pi_{N-1}), \tag{4.25}$$

where it is assumed that the order parameter is polarized along the 1 direction. Inserting this and (4.7) into (4.6), imposing the constraint with a Lagrange multiplier λ, and integrating out *only* the $\pi_{1\ldots N-1}$ fields, we find

$$Z = \int \mathcal{D}\lambda \mathcal{D} r_0 \exp\left[-\frac{N-1}{2} \mathrm{Tr} \ln(-c^2 \partial_i^2 - \partial_\tau^2 + i\lambda) + \frac{iN}{cg} \int_0^{1/T} d\tau \int d^d x \lambda (1 - r_0^2) \right] \quad (4.26)$$

In the large N limit, we can ignore the difference between $N-1$ and N, and obtain the saddle point equations with respect to variations in λ *and* r_0. As before, σ^2 is taken to be the saddle-point value of $i\lambda$. The mean value of r_0 will be the spontaneous magnetization at $N = \infty$, which we denote by N_0; so

$$N_0 = \langle n_1 \rangle . \quad (4.27)$$

The saddle point equations are

$$\begin{aligned} N_0^2 + g \int^{\Lambda} \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{p^2 + (\sigma/c)^2} &= 1 \\ \sigma^2 N_0 &= 0 \end{aligned} \quad (4.28)$$

where we have set $T = 0$. One solution of the second equation is $N_0 = 0$, but then the first equation for σ becomes identical to the one considered earlier, and is known to fail for $g < g_c$. So we choose the other solution, where

$$\begin{aligned} \sigma &= 0 \\ N_0^2 &= 1 - g \int^{\Lambda} \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{p^2} \\ &= 1 - \frac{g}{g_c} \end{aligned} \quad (4.29)$$

It is satisfying to find that N_0 is non-zero precisely for $g < g_c$, reinforcing our belief in the correctness of our procedure in finding the saddle point. Notice that N_0 vanishes as $(g_c - g)^{1/2}$ as g approaches g_c. It is conventional to define the critical exponent β by the dependence $N_0 \sim (g_c - g)^{\beta}$, and we therefore have $\beta = 1/2$ in the present $N = \infty$ theory. More generally, the scaling dimension of N_0 must be the same as the scaling dimension of $\vec{n}$, and we therefore have from (4.22) that

$$2\beta = (d + z - 2 + \eta)\nu, \quad (4.30)$$

an exponent relation that is satisfied by the $N = \infty$ theory.

The above approach also determines the two-point correlator of spin components orthogonal to the axis of the spontaneous magnetization. We denote the corresponding susceptibility by $\chi_\perp(k, \omega)$, and it is the Fourier transform of the n_2, n_2 correlator (say); we have at $N = \infty$

$$\chi_\perp(k, \omega) = \frac{cg/N}{c^2 k^2 - (\omega + i\epsilon)^2} \quad (4.31)$$

Notice that there is a quasiparticle pole at $\omega = ck$, and the energy of this excitation vanishes as $k \to 0$. These are the spin-wave excitations discussed earlier in the weak-coupling analysis. These spin waves survive fluctuation corrections as $k \to 0$, although the nature of the spectral density becomes different at larger k, as we will discuss shortly.

As was the case on the disordered side, we need an energy scale to characterize the ordered ground state, and its distance from the critical point. A convenient choice is to build an energy out of the **spin stiffness**, ρ_s. This quantity is a measure of how easy it is to make smooth changes in the order parameter orientation. Imagine, if instead of the uniform condensate $\langle \vec{n} \rangle = N_0(1, 0, 0, 0, \ldots)$ the system would choose on its own, we constrain the magnetization to precess smoothly in the $1-2$ plane (say)

$$\langle \vec{n} \rangle = N_0(\cos\varphi(x), \sin\varphi(x), 0, 0, \ldots) \tag{4.32}$$

where $\varphi(x)$ is a very slowly varying function of x. A constant $\varphi(x)$ cannot change the ground state energy of the constrained system, so the change in energy can depend only $\nabla\varphi(x)$. By inversion symmetry in x, the change cannot be linear in $\nabla\varphi$, and so the lowest order term in the change in energy has to be of the form

$$\delta E = \frac{\rho_s}{2} \int d^d x (\nabla\varphi)^2 \tag{4.33}$$

The coefficient appearing in the expression above is defined to be the spin stiffness ρ_s. We emphasize that the this stiffness is defined by changes in the ground state energy, and will always be assumed to be a $T = 0$ quantity, unless otherwise stated. The dimension of the stiffness under the scaling transformation of the $g = g_c$ point can now be easily deduced. The angle φ is a variable with period 2π, and must therefore necessary be dimensionless. By definition we have $\dim[\delta E] = z$, and therefore

$$\dim[\rho_s] = d + z - 2 \tag{4.34}$$

We can now construct the energy scale, which we denote Δ_-, which characterizes ground state for $g < g_c$. The requirement is that Δ_- should have scaling dimension z, and physical units of $(\text{time})^{-1}$. Such an object has to made out of powers of ρ_s, whose scaling dimension is above and whose physical units are $(\text{length})^{2-d}(\text{time})^{-1}$, and the velocity c, whose scaling dimension is 0 and physical units $(\text{length})(\text{time})^{-1}$; the unique combination is

$$\Delta_- \equiv (\rho_s/N)^{1/(d-1)} c^{(d-2)/(d-1)}. \tag{4.35}$$

The factor of N has been chosen for future convenience.

Knowledge of the spin stiffness allows us to make an exact statement on the form of the static transverse susceptibility $\chi_\perp(k, 0)$ in the limit $k \to 0$. This susceptibility is the response of the system to a very slowly varying static field, $h(x)$, which couples linearly to the 2 component (say). The system will respond to such an external field by a slowly varying shift in the angular orientation of the order parameter, and the net energy cost with then be

$$\delta E = \int d^d x \left[\frac{\rho_s}{2} (\nabla\varphi)^2 - h N_0 \sin\varphi \right] \tag{4.36}$$

Minimizing the energy cost with respect to variations in φ we get in Fourier space

$$\begin{aligned} \langle n_2(k) \rangle &\approx N_0 \varphi(k) \\ &= \frac{N_0^2}{\rho_s k^2} h(k). \end{aligned} \tag{4.37}$$

This gives us the exact result

$$\lim_{k \to 0} \chi_\perp(k, 0) = \frac{N_0^2}{\rho_s k^2} \tag{4.38}$$

Combining the $N = \infty$ results (4.29,4.31) with (4.38), we have

$$\rho_s = cN \left(\frac{1}{g} - \frac{1}{g_c} \right) \tag{4.39}$$

In general, from (4.34), ρ_s is expected to vanish $(g_c - g)^{(d-1)\nu}$, and the result (4.39) is consistent with the $N = \infty$ values of the exponents.

We conclude this subsection by remarking on two features of the response functions of the ordered ground state which depend upon having a non-zero η, and are therefore absent in the $N = \infty$ theory: these features can be verified by explicit computation in a $1/N$ expansion. First, from (4.38), we deduce that the residue at the spin-wave pole (for $k \to 0$) is N_0^2/ρ_s; as g approaches g_c, this vanishes as $(g_c - g)^{\eta\nu}$, unlike the result (4.31) in which the spin-wave residue remains non-zero all the way upto g_c. Second, with energy scale Δ_- in hand, we can also define a corresponding length scale ξ_J

$$\xi_J = \frac{c}{\Delta_-}. \tag{4.40}$$

This is known as the Josephson length. The forms (4.38) and (4.31), which are characteristic long-wavelength transverse responses of a phase with spontanesouly broken continuous symmetry, remain valid at length scales larger than ξ_J, and times longer than Δ_-^{-1}. At shorter scales, the responses crossover to the isotropic response of the critical points like in (4.24).

D. Nonzero temperatures

The structure of the $N = \infty$ theory for $T > 0$ is especially simple, as the dynamic susceptibility retains the form (4.10) for all T, with the parameter σ determined by the solution of (4.9). Indeed, by Fourier transforming (4.10), we see that σ/c is the correlation length, which is a useful physical interpretation to keep in mind. The imaginary part of (4.10) also implies that there is a gap in the spectrum equal to σ. This feature is an artifact of the $N = \infty$ limit: the response of any interacting system has a non-zero spectral density at all frequencies (in certain cases, the response could vanish above some large ultraviolet cutoff $\sim c\Lambda$), as there are essentially no restrictions on the set of frequencies at which all the possible thermally excited states can absorb energy. The proper physical interpretation of the energy scale σ varies at different points in the g, T plane, and will be discussed further below.

As σ/c is a correlation length, it is quite easy to deduce its expected scaling form near the $g = g_c$ quantum critical point for $T > 0$. The scaling dimension of σ is $z = 1$, and from this, or by analogy with the result for the correlation length of the Ising chain in (3.40) we deduce [21]

$$\sigma = T F_{\pm}\left(\frac{\Delta_{\pm}}{T}\right) \tag{4.41}$$

We have written two separate scaling functions, F_+, F_-, which hold for $g \geq g_c$, $g \leq g_c$ respectively. We were forced into this somewhat awkward construction because the energy scales $\Delta_{\pm} \sim |g - g_c|^{z\nu}$, which characterize the ground states on either side of the $g = g_c$ quantum critical point, have quite different physical interpretations, and, in general, a rather different dependence on the bare coupling $g - g_c$. In the case of the Ising chain, we were able to use a single function because of some special features of that model: there was a gap in both ground states, the exponent $z\nu = 1$ implied that the gap $\sim |g - g_c|$, and we were able to define an energy scale $\Delta \sim g_c - g$ such that the gap $\Delta_+ = -\Delta$ for $g > g_c$, and the gap $\Delta_- = \Delta$ for $g < g_c$. In the present case, there is a gap only for $g > g_c$ and $z\nu \neq 1$. Nevertheless, the analyticity requirements on $T > 0$ physical properties, as a function of the bare coupling g, at the quantum critical coupling $g = g_c$ still hold. This means that given (say) F_+, and the dependencies of $\Delta_{\pm}$ on $g - g_c$, it is in principle possible to determine F_- by analytic continuation in g. It is perhaps worthwhile to reiterate that the analyticity holds *not* as a function of the physical, renormalized, energy scales $\Delta_{\pm}$, but as function of the bare coupling g at $g = g_c$. A new feature of models in $d > 1$, as we will see below, that for $T > 0$, there *can* be non-analyticities at couplings away from $g = g_c$. These non-analyticities correspond to finite T, thermal phase transitions.

We now determine the universal functions $F_{\pm}$, and will subsequently turn to a description of the physics in the various regions. The method used here introduces a number of tricks which repeatedly appear in the extraction of universal, cut-off independent crossover functions in other contexts [23].

We present first the calculation on the disordered side $g \geq g_c$. The first step is to subtract from (4.9) the corresponding equation (4.11) at the same coupling constants at $T = 0$; this gives us

$$\int^{\Lambda} \frac{d^d k}{(2\pi)^d} T \sum_{\omega_n} \frac{1}{c^2k^2 + \omega_n^2 + \sigma^2} - \frac{1}{c}\int^{\Lambda} \frac{d^{d+1}p}{(2\pi)^d} \frac{1}{p^2 + (\Delta_+/c)^2} = 0 \tag{4.42}$$

where Δ_+ is the gap at the current value of g. Next, subtract from the summation over frequencies of any quantity, the integration over frequences of precisely the same function; so we rewrite (4.42) as

$$\int^{\Lambda} \frac{d^d k}{(2\pi)^d}\left(T\sum_{\omega_n} \frac{1}{c^2k^2 + \omega_n^2 + \sigma^2} - \int \frac{d\omega}{2\pi}\frac{1}{c^2k^2 + \omega^2 + \sigma^2}\right)$$
$$+ \frac{1}{c}\int^{\Lambda} \frac{d^{d+1}p}{(2\pi)^d}\left(\frac{1}{p^2 + (\sigma/c)^2} - \frac{1}{p^2 + (\Delta_+/c)^2}\right) = 0 \tag{4.43}$$

Now we use the general relation

$$T\sum_{\omega_n} \frac{1}{\omega_n^2 + a^2} - \int \frac{d\omega}{2\pi}\frac{1}{\omega^2 + a^2} = \frac{1}{a}\frac{1}{e^{a/T} - 1} \tag{4.44}$$

valid for any positive a. Notice that the right-hand side falls off exponentially as a becomes large. This is a key property, and was the reason for considering the combination in (4.44).

Applying this identity to (4.43), we see that the first integration over k has an integrand which is exponentially small for large k, and hence is quite insensitive to Λ which can safely be sent to infinity. The integration over p in the second term is also ultraviolet convergent, again allowing Λ to be set to infinity. The resulting expression is then cutoff independent, and hence universal; we obtain

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{\sqrt{c^2 k^2 + \sigma^2}} \frac{1}{e^{\sqrt{c^2k^2+\sigma^2}/T} - 1} - \frac{X_{d+1}}{c^d} \left(\sigma^{d-1} - \Delta_+^{d-1} \right) = 0 \tag{4.45}$$

The solution of this equation is clearly of the form (4.41); after rescaling momenta by c/T in (4.45), we find that the function $F_+(s)$ is determined implicitly by solution of the equation

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{\sqrt{k^2 + F_+^2}} \frac{1}{e^{\sqrt{k^2+F_+^2}} - 1} - X_{d+1} \left(F_+^{d-1} - s^{d-1} \right) = 0 \tag{4.46}$$

We will discuss asymptotic features of the solution of this equation in the subsections below. We note here that precisely in $d = 2$, the equation (4.46) has a simple, explicit solution [21]

$$F_+(s) = 2 \sinh^{-1} \left(\frac{e^{s/2}}{2} \right) \qquad d = 2 \tag{4.47}$$

Now we turn to the ordered side, $g \leq g_c$. We assume that T is large enough that the magnetization is zero; the case of the magnetized state with $T \neq 0$ can be treated similarly, and will be referred to below. Subtract from (4.9), the value of ρ_s/N in (4.39), and insert the value of $1/g_c$ in (4.12). Evaluating the frequency summation as above we find

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{\sqrt{c^2 k^2 + \sigma^2}} \frac{1}{e^{\sqrt{c^2k^2+(\sigma/c)^2}/T} - 1} + \frac{1}{c} \int \frac{d^{d+1} p}{(2\pi)^d} \left(\frac{1}{p^2 + \sigma^2} - \frac{1}{p^2} \right) = \frac{\rho_s}{N c^2} \tag{4.48}$$

The solution of this is also in the form (4.41), and the function $F_-(s)$ is given by

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{\sqrt{k^2 + F_-^2}} \frac{1}{e^{\sqrt{k^2+F_-^2}} - 1} - X_{d+1} F_-^{d-1} - s^{d-1} = 0 \tag{4.49}$$

Again, there is a simple explicit solution in $d = 2$ [21]

$$F_-(s) = 2 \sinh^{-1} \left(\frac{e^{-2\pi s}}{2} \right) \qquad d = 2 \tag{4.50}$$

With expressions for the crossover functions $F_\pm$ in hand, let us discuss the physical properties of the system in different regimes of the g, T, plane

1. *Low T on the disordered side,* $g > g_c$, $T \ll \Delta_+$

Properties in this phase are essentially identical to those of the Ising model in Section III D 2, and so we will be quite brief. For the parameter σ we have

$$\sigma = \Delta_+ + \mathcal{O}(e^{-\Delta_+/T}) \tag{4.51}$$

There is a finite correlation length c/σ which has exponentially small corrections from its $T = 0$ value c/Δ_+. The gap in the spectrum is filled in at non-zero temperatures by classical relaxation in the dilute gas of thermally excited quasiparticles: this process is however not apparent in the $N = \infty$ susceptibility which has a gap even at non-zero T.

2. *High T,* $T \gg \Delta_+, \Delta_-$

Again properties are the same those of the Ising chain as discussed in Section III D 3. Now we have

$$\sigma = TF_+(0) = TF_-(0) \tag{4.52}$$

where $F_+(0)$, $F_-(0)$ are pure numbers. This represents a correlation length $\sim c/T$. Spin correlations decay through a 'quantum relaxation' process [21] at a rate $\sim T$, and the $N = \infty$ gap $\sim T$ is an artifact of the approximation.

3. *Low T on the ordered side,* $g < g_c$

Now there are some differences from the Ising chain for $2 < d < 3$.

Let us assume first that T is large enough so that $\langle \vec{n} \rangle = 0$ and so (4.49) can be used to determine F_-. For $1 < d \leq 2$, one finds that there is a solution of (4.49) for all T ($s \to \infty$). We find

$$\sigma = \begin{cases} \frac{T^{1/(2-d)}}{\Delta_-^{(d-1)/(2-d)} |X_d|^{1/(2-d)}} & 1 < d < 2 \\ T \exp(-2\pi\Delta_-/T) & d = 2 \end{cases} \tag{4.53}$$

So the correlation length $\sim c/\sigma$ diverges as $T \to 0$, but remains finite for all non-zero T. This was exactly the situation as in the Ising model, and the phase diagram for this model is therefore identical to Fig 7, except that the crossover boundaries do not come in linearly as $z\nu \neq 1$. There is a reduced scaling function which describes the classical physics of the system as $T \to 0$, as in the Ising model.

Now let us consider the case $2 < d < 3$. Although there is no physical dimension in this region, the results obtained below will apply in $d = 3$ with cutoff-dependent logarithmic corrections we do not want to discuss here. Further, the physics of the quantum Ising model in $d = 2$ is expected to be similar to that of the large N solution with $2 < d < 3$. The key observation in this case is that there is no solution of (4.49) for F_- below a critical value of $s = s_c$ given by

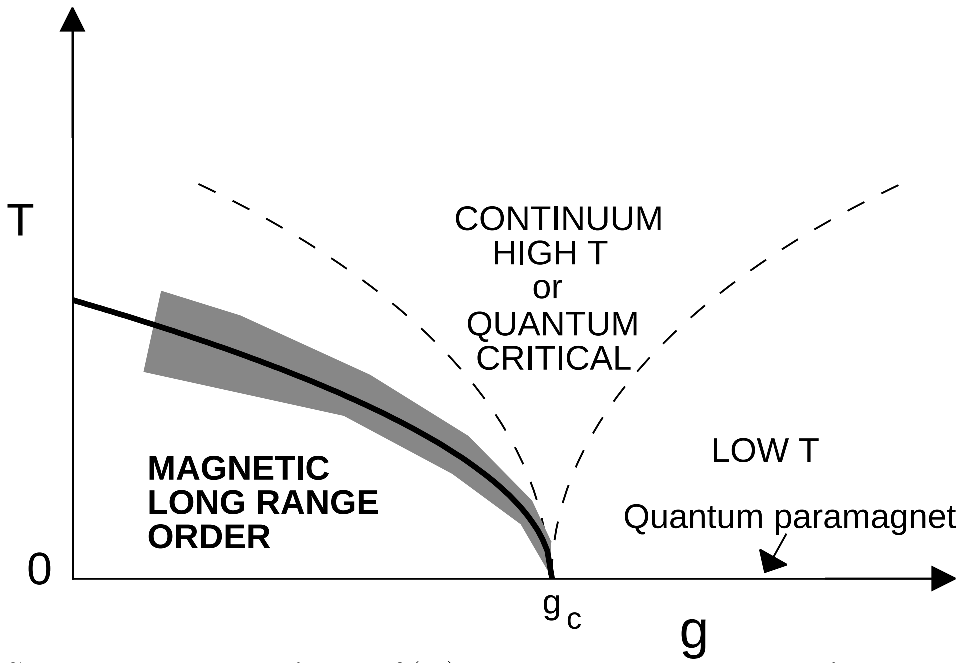

FIG. 11. Phase diagram for the $O(N)$ rotor model with $2 < d < 3$ for $N > 2$, or $2 \leq d < 3$ for $1 \leq N \leq 2$. The dashed lines are crossovers, while the full line is the locus of finite temperature phase transitions with T_c given by (4.55). The shaded region is where the reduced classical scaling functions apply.

$$s_c^{d-1} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k} \frac{1}{e^k - 1} = \frac{2\Gamma(d-1)\zeta(d-1)}{\Gamma(d/2)(4\pi)^{d/2}} \tag{4.54}$$

This defines a critical temperature T_c given precisely by

$$T_c \equiv \Delta_- / s_c \tag{4.55}$$

such that the system is in the paramagnetic phase only for $T > T_c$: the resulting phase diagram is shown in Fig 11 There is a finite temperature phase transition at $T = T_c$, and a magnetically ordered phase for $T < T_c$. As T approaches T_c, the conventional classical phase transition theory becomes applicable in the region $|T - T_c| \ll T_c$. The classical scaling functions of this transition emerge as reduced scaling functions of the quantum functions, in a manner very similar to the other cases that have been discussed above. One consequence of this behavior is that all the scale factors of the classical scaling functions, which are usually considered non-universal, are universally determined by the parameters Δ_-, c, and N_0 of the quantum crossover functions. We have already seen an example of this in (4.55), where T_c was universally determined by Δ_- [23].

Let us explicitly observe the collapse of the scaling function (4.41) in this classical region. As the primary quantum crossover function has only one argument, the reduced function would have no arguments i.e. it is a pure power law. Indeed, solution of (4.49) for s close to but above s_c gives us

$$\sigma = T_c \left[\left(\frac{T - T_c}{T_c} \right) \frac{(d-1)s_c^{d-1}}{X_d} \right]^{1/(d-2)} \tag{4.56}$$

The correlation length c/σ diverges with the classical exponent $\nu_c = 1/(d-2)$ with an amplitude that is universal. A similar collapse will occur in all other static and dynamic observables.

V. CONCLUSION

This review has described the physics of the vicinity of second-order quantum phase transition using three simple examples which had the virtue of capturing some essential physical phenomena. Among the physical issues we have not discussed are:
(i) Crossovers near quantum-critical points above their upper critical dimension: a simple understanding of this can be obtained from the large N solution of the $O(N)$ quantum rotor model with $d > 3$, or by a perturbative analysis [23].
(ii) Quantum phase transition in the presence of quenched randomness: this is a complicated subject still in its infancy, with only a few firm results.

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