

FINITE TEMPERATURE CORRELATIONS OF THE ISING CHAIN IN A TRANSVERSE FIELD

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Recent work on the finite temperature correlators of the Ising chain in a transverse field is reviewed. The crossover phase diagram in the coupling-constant and temperature plane is discussed, and various exact results and conjectures for static and dynamic correlators are presented.

1. Introduction

This paper will review recent results on the finite temperature (T) properties of the one-dimensional Ising chain in a transverse field. The discussion here is a synthesis of two recent papers^{1,2} by the author to which the reader is referred for further details. Further work on the one-dimensional Ising model appears in a recent paper³ and will not be reviewed here. We will also not consider properties of the Ising model in a transverse field in higher dimensions, and refer the reader to another recent paper.⁴

We will consider the Hamiltonian

$$H_I = -J \sum_i (g\sigma_x(i) + \sigma_z(i)\sigma_z(i+1)) \quad (1)$$

where $J > 0$ is an overall energy scale, $g > 0$ is a dimensionless coupling constant, $\sigma_x(i)$, $\sigma_z(i)$ are Pauli matrices on a chain of sites i . This model has a $T = 0$ quantum phase transition^{5,6} at $g = g_c = 1$ from a state with long-range-order with $\langle \sigma_z \rangle \neq 0$ ($g < g_c$), to a gapped quantum paramagnet ($g > g_c$). The dynamic critical exponent is $z = 1$ and the correlation length exponent is $\nu = 1$.

The simplest way to obtain the above well-known results is by noting that H_I can be written in a free-fermion form. One uses the Jordan-Wigner mapping to express the Pauli matrices in terms of creation and annihilation operators of spinless fermions and finds that H_I is a quadratic form in the fermion operators.⁵ Near the critical point at $g = g_c$, long-distance fluctuations are expected to dominate, and the continuum limit of this quadratic form can be written as a continuum quantum field theory (CQFT) of free Majorana fermions.⁶ Let the Majorana fermions have mass m and velocity c ; the energy difference between an excited state containing a single Majorana fermion at wavevector k , and the ground state is therefore

$(m^2 c^4 + \hbar^2 c^2 k^2)^{1/2}$. In terms of the couplings in H_I , the values of m and c are

$$m = \frac{\hbar^2(g_c - g)}{2Ja^2} \quad \text{and} \quad c = \frac{2Ja}{\hbar}, \quad (2)$$

where a is the lattice spacing. The Majorana fermion mass m can have either sign, and moving m through 0 tunes the system across the transition; we have chosen $m > 0$ to correspond to the ordered side. Scaling would suggest that $|m|c^2 \sim |g - g_c|^{z\nu}$, and the value of m for H_I agrees with $z\nu = 1$. The velocity, c , simply sets the relative distance and time scales and may be assumed to be constant across the transition. In the following we will use units of length, time and temperature in which $\hbar = c = k_B = 1$.

Our main focus here will be on the finite T properties of the CQFT of the Ising model. Alternatively stated, we will study finite T properties of H_I , under the scaling limit $a \rightarrow 0$, $g \rightarrow g_c$, at fixed m , c , and T . This CQFT is now characterized by just two energy scales, m and T , and we expect all observables to be completely universal functions of the dimensionless ratio m/T . On these general grounds, we can deduce the crossover phase diagram shown in Fig 1; the similarity of this phase diagram to that of the $d = 2$ $O(3)$ quantum rotor model is part of the reason for our interest in the present Ising model.¹ There are three different regimes, with smooth

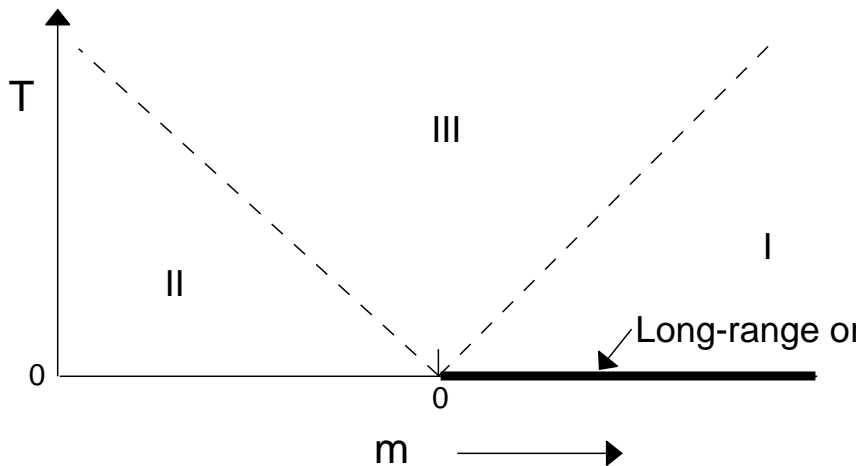


Fig. 1. Finite T phase diagram of the $d = 1$ quantum Ising model as a function of the fermion mass m and temperature T . Long-range order ($N_0 = \langle \sigma_z \rangle \neq 0$) is present only for $T = 0$ and $m > 0$.

crossovers between them:

(I) Low T limit ($T \ll m$), $m > 0$

The $T = 0$ state has long-range-order, but the order disappears at any non-zero T . The dominant configurations at low T consist of large ordered regions separated by thermally excited domain walls.

(II) Low T limit ($T \ll -m$), $m < 0$

The $T = 0$ state is a quantum paramagnet with short-range spin correlations. Finite T introduces a dilute concentration of quasi-particle excitations, which do not significantly modify the equal-time correlations.

(III) High T limit ($T \gg |m|$)

This novel state consists of strongly interacting thermal excitations from the critical continuum. At no length scale does the system display any characteristics of either the ordered or quantum-paramagnetic ground states.^{7,2}

A useful quantitative description of the crossovers in Fig 1 is contained in the two-point spin correlation function, C , as a function of space (x) and time (t). This must obey the scaling form⁷

$$C(x, t) \equiv \langle \sigma_z(x, t) \sigma_z(0, 0) \rangle = Z T^{1/4} \Phi_I \left(Tx, Tt, \frac{m}{T} \right), \quad (3)$$

where Z is a non-universal normalization constant, and Φ_I is a fully universal crossover function. The non-universality of Z is linked with the anomalous dimension of the σ_z operator, indicated by the $T^{1/4}$ prefactor (we will indicate steps deriving this pre-factor below), and requires a single normalization condition to set the overall scale of Φ_I : this will be specified in the next subsection and leads to the value $Z = 1/J^{1/4}$ for the specific lattice Hamiltonian H_I .

Finally, notice there are no phase transitions at finite T in Fig 1. We may therefore conclude that all properties of H_I are analytic functions of the bare coupling g at all non-zero T . Accidentally, because $z\nu = 1$, analyticity in g is equivalent to analyticity in m . So Φ_I is analytic as a function of m/T for all real, finite m/T . This is a powerful principle which shall be of considerable use to us.

We will now present known results for the function Φ_I , dividing our discussion into equal-time (Section 2) and unequal-time correlations (Section 3).

2. Statics

We expect equal-time correlations to decay exponentially at any non-zero T and so

$$\langle \sigma_z(|x| \rightarrow \infty, 0) \sigma_z(0, 0) \rangle \sim A \exp(-|x|/\xi), \quad (4)$$

where A is an amplitude, and ξ is the correlation length. The result (3) implies that these must obey the scaling forms

$$\begin{aligned} \xi^{-1} &= T f_I \left(\frac{m}{T} \right) \\ A &= Z T^{1/4} g_I \left(\frac{m}{T} \right), \end{aligned} \quad (5)$$

where $f_I(s)$ and $g_I(s)$ are completely universal functions of $-\infty < (s = m/T) < \infty$, and are analytic for all real, finite s .

We will now obtain the functions f_I and g_I in closed form. First, following Lieb, Schultz and Mattis,⁵ convert H_I into a free fermion Hamiltonian by the Jordan-Wigner transformation. Then, evaluate the equal-time spin correlator in terms of

the free-fermion correlators. This yields an expression for the correlator in terms of a Toeplitz determinant:^{5,8}

$$\langle \sigma_z(i) \sigma_z(i+n) \rangle = \begin{vmatrix} G_0 & G_{-1} & \cdots & G_{-n+1} \\ G_1 & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & G_0 & G_{-1} \\ G_{n-1} & \cdots & G_1 & G_0 \end{vmatrix} \quad (6)$$

where

$$G_p = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-ip\phi} \tilde{G}(\phi) \quad (7)$$

with

$$\tilde{G}(\phi) = \left(\frac{1 - ge^{i\phi}}{1 - ge^{-i\phi}} \right)^{1/2} \tanh \left[\frac{J}{T} ((1 - ge^{i\phi})(1 - ge^{-i\phi}))^{1/2} \right] \quad (8)$$

We can now take the large n limit of (6) by Szego's lemma⁹ provided $g < g_c = 1$. For $g \geq g_c$ a more complicated analysis is necessary to evaluate (6) directly;¹⁰ we shall not need this here as we shall obtain results for $g \geq g_c$ by our method of analytic continuation. By Szego's lemma we obtain

$$\lim_{n \rightarrow \infty} \langle \sigma_z(i) \sigma_z(i+n) \rangle \sim e^{n\lambda_0} \exp \left(\sum_{p=1}^{\infty} p \lambda_p \lambda_{-p} \right), \quad g < 1 \quad (9)$$

where

$$\ln \tilde{G}(\phi) = \sum_{p=-\infty}^{\infty} \lambda_p e^{ip\phi} \quad (10)$$

The correlation length is clearly $\xi = -a/\lambda_0$, and the above approach gives us the value of ξ for $g < g_c$. From (8-10) we can deduce that

$$\xi^{-1} = a \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \ln \coth \left[\frac{J}{T} (1 + g^2 - 2g \cos \phi)^{1/2} \right], \quad g < 1 \quad (11)$$

Now change variables of integration to $k = \phi/a$, re-express in terms of renormalized couplings by writing $J = \hbar c/2a$ and $g = 1 - mca/\hbar$, and take the scaling limit $a \rightarrow 0$. This yields, after reverting to units with $\hbar = c = 1$,

$$\xi^{-1} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \ln \coth \left[\frac{(m^2 + k^2)^{1/2}}{2T} \right], \quad m > 0 \quad (12)$$

Let the scaling function for the inverse correlation length, in (5), $f_I(s) = f_I^>(s)$ for $s > 0$. Then clearly

$$f_I^>(s) = \frac{1}{\pi} \int_0^{\infty} dy \ln \coth \frac{(y^2 + s^2)^{1/2}}{2}. \quad (13)$$

Notice that $f_I^>(s)$ is an even function of s . Later, it will be useful to have another form for $f_I^>(s)$: change variables of integration $y \rightarrow (y^2 - s^2)^{1/2}$, and integrate by parts to obtain

$$f_I^>(s) = \frac{1}{\pi} \int_{|s|}^{\infty} \frac{dy}{\sinh y} (y^2 - s^2)^{1/2} \quad (14)$$

We now need to get the scaling function of the inverse correlation length, $f_I(s)$ for $s \leq 0$. We could get this by returning to an analysis of (6), but as noted earlier, Szego's lemma does not directly apply and the analysis is not straightforward.¹⁰ Instead, we use the method of analytic continuation to obtain the answer quite rapidly. Unlike the needed $f_I(s)$, the function $f_I^>(s)$ is not analytic at $s = 0$. Let us examine the nature of the singularity in $f_I^>(s)$ near $s = 0$. Insert $1 = ((y^2 + s^2)/(y^2 + 1))^{1/2}((y^2 + 1)/(y^2 + s^2))^{1/2}$ in the logarithm in (13) to get

$$f_I^>(s) = \frac{1}{\pi} \int_0^{\infty} dy \ln \left[\left(\frac{y^2 + s^2}{y^2 + 1} \right)^{1/2} \coth \frac{(y^2 + s^2)^{1/2}}{2} \right] + \frac{1}{2\pi} \int_0^{\infty} dy \ln \left(\frac{y^2 + 1}{y^2 + s^2} \right) \quad (15)$$

The first integral has an integrand which is a smooth function of s^2 for all values of y ($x \coth x$ is regular at $x = 0$ and has a series expansion with only even powers of x); so for small s it can only yield a power series containing even, non-negative powers of s . The second integral can be performed exactly and we obtain

$$f_I^>(s \rightarrow 0) = -|s|/2 + \text{terms with even, non-negative powers of } s \quad (16)$$

The $-|s|/2$ term is the only non-analyticity at $s = 0$. It is then evident that defining $f_I(s) = f_I^>(s) + |s|\theta(-s)$ gives the unique analytic function which equals $f_I^>(s)$ for $s > 0$ (here, θ is the unit step function). The amplitude scaling function $g_I(s)$ can be obtained by a related analysis of (9), and our final results¹ are

$$\begin{aligned} f_I(s) &= \int_0^{\infty} \frac{dy}{\pi} \ln \coth \frac{\sqrt{y^2 + s^2}}{2} + |s|\theta(-s) \\ \ln g_I(s) &= \int_s^1 \frac{dy}{y} \left[\left(\frac{df_I(y)}{dy} \right)^2 - \frac{1}{4} \right] + \int_1^{\infty} \frac{dy}{y} \left(\frac{df_I(y)}{dy} \right)^2. \end{aligned} \quad (17)$$

We have set the overall scale of g_I , and hence that of Φ_I , by the normalization condition $\lim_{s \rightarrow +\infty} g_I(s)/s^{1/4} = 1$.

It is useful to examine the values of ξ and the amplitude A in the regimes of Fig 1. Those for ξ can be easily obtained from (17)

$$\xi^{-1} = \begin{cases} \sqrt{2mT/\pi} e^{-m/T} & T \ll m, \text{ region I} \\ \pi T/4 & T \gg |m|, \text{ region III} \\ |m| + \sqrt{2|m|T/\pi} e^{-|m|/T} & T \ll -m, \text{ region II} \end{cases} \quad (18)$$

The corresponding asymptotic limits for A require a somewhat lengthier computation¹

$$A = \begin{cases} Zm^{1/4} & T \ll m, \text{ region I} \\ ZT^{1/4}g_I(0) & T \gg |m|, \text{ region III} \\ ZT|m|^{-3/4} & T \ll -m, \text{ region II} \end{cases} \quad (19)$$

Notice that as $T \rightarrow 0$ on the ordered side ($m > 0$) at fixed x , we have $C \rightarrow Zm^{1/4}$. This gives us the value of the spontaneous magnetization at $T = 0$: $N_0 \equiv \langle \sigma_z \rangle = Z^{1/2}m^{1/8}$.

Another useful $T = 0$ result can be obtained by combining the above with the conformal invariance of the theory at $m = 0$. At this critical coupling, the finite T form of $C(x, t)$ is completely determined by conformal invariance¹¹

$$C(x, 0) = Zg_I(0) \left(\frac{T}{2 \sinh(\pi T|x|)} \right)^{1/4} \quad m = 0, \quad (20)$$

where notice that the $x \rightarrow \infty$ limit of (20) agrees with (4), (18) and (19). The result (20) can also be evaluated at $T = 0$ where it gives $C(x, 0) = Zg_I(0)/(2\pi|x|)^{1/4}$.

We now use the fact that the $T = 0$ values of $\langle \sigma_z \rangle$ for $m > 0$, and of $C(x, 0)$ at $m = 0$, were obtained earlier by Pfeuty¹² by different methods. Demanding consistency between his and our results, we find that we must have

$$g_I(0) = \pi^{1/4} e^{1/4} 2^{1/12} A_G^{-3} \quad (21)$$

where A_G is Glaisher's constant⁹ (given explicitly by $\ln A_G = 1/12 - \zeta'(-1)$ where ζ is the Reimann zeta function), and recall that $g_I(0)$ is defined completely by (17). We have not been able to directly prove (21), but have checked it numerically to high accuracy. It appears that (21) is a new identity involving Glaisher's constant; a separate identity for A_G was also found by the author¹ while performing a related analysis for finite T crossovers in the dilute Bose gas.

3. Dynamics

Our knowledge of unequal time correlations is much less complete than that for the equal time case. Established results are limited to correlators in the high T limit ($T \gg |m|$, region III of Fig 1) which are discussed in Section 3.1. At low T above the ordered state (region I), there is a conjectured form based on physical arguments, and it is presented in Section 3.2.

3.1. *High T region, $T \gg |m|$*

First, consider correlations at $T = 0$ and at the critical coupling $m = 0$. In this case, we obtained above the equal-time correlator $C(x, 0) = Zg_I(0)/(2\pi|x|)^{1/4}$. The unequal time correlator now follows simply from the Lorentz invariance of the critical theory $C(x, -i\tau) = Zg_I(0)/(4\pi^2(x^2 + \tau^2))^{1/8}$. For our discussion, a physically more transparent form of this result is in terms of the dynamic susceptibility $\chi(k, \omega)$ which is the Fourier transform to momentum (k) and frequency (ω) space of the retarded σ_z, σ_z commutator

$$\chi(k, \omega) = Z(4\pi)^{3/4} g_I(0) \frac{\Gamma(7/8)}{\Gamma(1/8)} \frac{1}{(k^2 - \omega^2)^{7/8}} \quad T = 0, \quad m = 0 \quad (22)$$

We plot $\text{Im}\chi(k, \omega)/\omega$ in Fig 2. Notice that there are no delta functions in the spectral density, indicating the absence of any well-defined quasiparticles. Instead, we have a critical continuum of excitations.

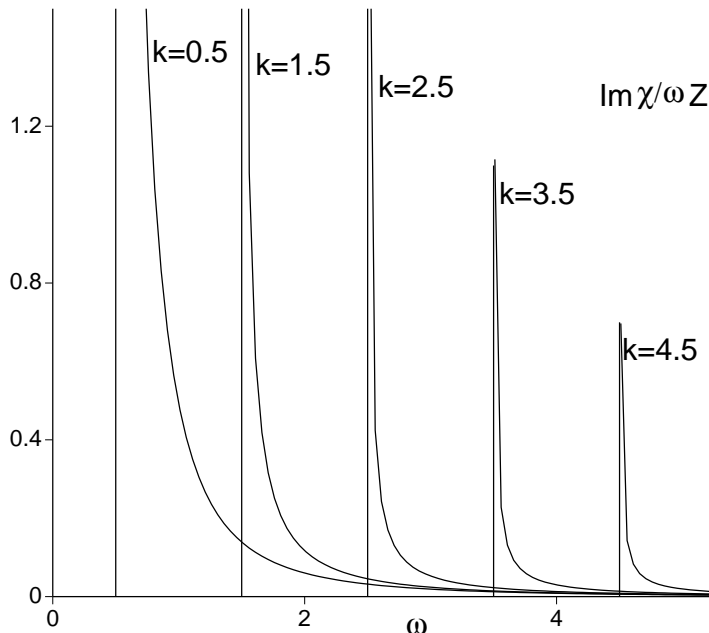


Fig. 2. Spectral density, $\text{Im}\chi(k, \omega)/\omega Z$, of the transverse field Ising model (1) at its critical point $g = g_c$ ($m = 0$) at $T = 0$, as a function of frequency ω , for a set of values of k .

Now let us turn to the non-zero T , $m = 0$ case. In general, the results here are the leading terms for $T \gg |m|$, which is region III of Fig 1. The conformal invariance argument leading to (20) applies for the unequal time case too, and leads to the result¹¹

$$C(x, -i\tau) = ZT^{1/4} \frac{2^{-1/8} g_I(0)}{(\cosh(2\pi T x) - \cos(2\pi T \tau))^{1/8}} \quad m = 0 \quad (23)$$

This result (apart from the prefactor) had been asserted earlier by an inspection of the partial differential equation satisfied by C .¹³ Although results like (23) have been known to conformal field theorists for some time, they usually interchange the roles of x and τ *i.e.* they consider models of finite spatial length L_x , and infinite temporal length, so that the system is in its ground state. For us, the spatial extent is infinite, and there is a finite length $1/T$ along the τ direction, with the correlator (23) periodic in τ with period $1/T$. Note that it doesn't really make sense to talk about the long imaginary time limit, $\tau \gg 1/T$. However, after analytic continuation to real time, the long time limit, $t \gg 1/T$ or $\omega \ll T$, is eminently sensible. As we will see shortly, the spin dynamics is relaxational in this

limit, and cannot be described by some effective classical model of dissipation, and so was dubbed⁷ “quantum relaxational”. The analytic continuation is a little more convenient in Fourier space: we Fourier transform (23) to obtain $\chi(k, i\omega_n)$ at the Matsubara frequencies ω_n and then analytically continue to real frequencies (there are some interesting subtleties in the Fourier transform to $\chi(k, i\omega_n)$ and its analytic structure in the complex ω plane, which are discussed elsewhere¹⁴). This gives us the leading result for $\chi(k, \omega)$ in region III.

$$\chi(k, \omega) = \frac{Z}{T^{7/4}} \frac{g_I(0)}{4\pi} \frac{\Gamma(7/8)}{\Gamma(1/8)} \frac{\Gamma\left(\frac{1}{16} + i\frac{\omega+k}{4\pi T}\right) \Gamma\left(\frac{1}{16} + i\frac{\omega-k}{4\pi T}\right)}{\Gamma\left(\frac{15}{16} + i\frac{\omega+k}{4\pi T}\right) \Gamma\left(\frac{15}{16} + i\frac{\omega-k}{4\pi T}\right)} \quad m=0. \quad (24)$$

We show a plot of $\text{Im}\chi/\omega$ in Fig 3. This result is the finite T version of Fig 2.

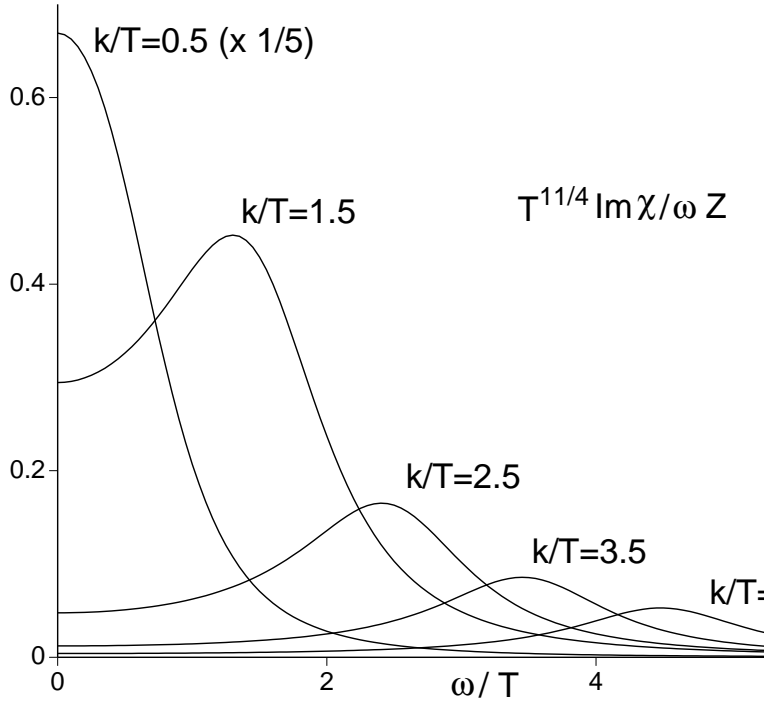


Fig. 3. The same observable as in Fig 2, $T^{11/4}\text{Im}\chi(k, \omega)/\omega Z$, but for $T \neq 0$. This is the leading result for $\text{Im}\chi$ for $T \gg |m|$ *i.e.* in region III of Fig 1. All quantities are scaled appropriately with powers of T , and the absolute numerical values of both axes are meaningful.

Notice that the sharp features of Fig 2 have been smoothed out on the scale T , and there is non-zero absorption at all frequencies. For $\omega, k \gg T$ there is a well-defined peak in $\text{Im}\chi/\omega$ (Fig 3) rather like the $T = 0$ critical behavior of Fig 2. However, for $\omega, k \ll T$ we cross-over to the quantum relaxational regime and the spectral

density $\text{Im}\chi/\omega$ is similar (but not identical) to a Lorentzian around $\omega = 0$. This relaxational behavior can be characterized by a relaxation rate Γ_R defined as

$$\Gamma_R^{-1} = -i \left. \frac{\partial \ln \chi(0, \omega)}{\partial \omega} \right|_{\omega=0} \quad (25)$$

(this is motivated by the phenomenological relaxational form $\chi(0, \omega) = \chi_0/(1 - i\omega/\Gamma_R + \mathcal{O}(\omega^2))$). From (24) we determine:

$$\Gamma_R = \left(2 \tan \frac{\pi}{16} \right) \frac{k_B T}{\hbar}, \quad (26)$$

where we have returned to physical units. At the scale of the characteristic rate Γ_R , the dynamics of the system involves intrinsic quantum effects (responsible for the non-Lorentzian lineshape) which cannot be neglected; description by an effective classical model would require that $\Gamma_R \ll k_B T/\hbar$, which is thus not satisfied in region III of Fig 1. The ease with which (26) was obtained belies its remarkable nature. Notice that we are working in a closed Hamiltonian system, evolving unitarily in time with the operator $e^{-iH_I t/\hbar}$, from an initial density matrix given by the Gibbs ensemble at a temperature T . Yet, we have obtained relaxational behavior at low frequencies, and determined an exact value for a dissipation constant. Such behavior is more typically obtained in phenomenological models which couple the system to an external heat bath and postulate an equation of motion of the Langevin type. Notice also that H_I in (1) is known to be integrable with an infinite number of conservation laws.⁵ However, the conservation laws are associated with a mapping to a free fermion model and are highly non-local in our σ_z degrees of freedom; they play no direct role in our considerations, and do not preclude relaxational behavior in the σ_z variables.

3.2. Low T above the ordered ground state

Let us now turn to region I of Fig 1. Recall that the ground state is ordered for $m > 0$, and for non-zero, but small, temperature it is almost so. The order is disrupted by a dilute concentration of domain walls, which in $d = 1$ is sufficient to destroy the long-range order. However, the spins are mostly in large ordered regions, and should display some collective dynamics. Further, as a large number of spins are flipping coherently together, one might expect that an effective *classical* model might be adequate.

We will now sharpen the above argument. Notice that in region I ($m \gg T$), the correlation length becomes (from (18)) exponentially large:

$$\xi_c \equiv \left(\frac{\pi}{2mT} \right)^{1/2} \exp \left(\frac{m}{T} \right). \quad (27)$$

We have labeled the correlation length ξ_c in this region, in anticipation of the classical model to follow. The spacing between the domain walls is of order ξ_c . Notice however that ξ_c is much larger than the de Broglie wavelength $\sim 1/(mT)^{1/2}$

of the quantum particle describing a domain wall. These are precisely the conditions under which a classical description of the domain wall dynamics should be adequate. The classical motion of these domain walls is expected to lead to a relaxation in the spin correlations at a time scale of order $\xi_c^{z_c}$, where z_c is the dynamic critical exponent of the effective classical model. Note that the quantum-critical point $T = 0$, $m = 0$ has a dynamic exponent $z = 1$, but there is no general scaling property which requires the dynamic exponents to be equal. Formally, the above arguments mean⁷ that in region I, the scaling function, Φ_I , of *three* arguments in (3) collapses into a reduced scaling function, ϕ_c , of *two* arguments:

$$\langle \sigma_z(x, t) \sigma_z(0, 0) \rangle = N_0^2 \phi_c \left(\frac{x}{\xi_c}, \gamma \frac{t}{m^{z_c-1} \xi_c^{z_c}} \right) \quad m \gg T, |x| \gg \frac{1}{(mT)^{1/2}}, |t| \gg \frac{1}{T}, \quad (28)$$

where $N_0 = Z^{1/2} m^{1/8} = \langle \sigma_z \rangle_{T=0}$ is the ground state magnetization, and ϕ_c is a scaling function of the effective classical model of spin dynamics. The limits on x and t for classical behavior to hold were somewhat different in earlier work,¹ and we now believe (28) contains the correct conditions. The dimensionless constant γ determines the prefactor of ξ_c in the relaxation rate of the classical model; on dimensional grounds we expect

$$\gamma = \mathcal{R} \left(\frac{T}{m} \right)^\rho \quad (29)$$

where \mathcal{R} is a universal number, and ρ an unknown exponent.

To make further progress, we need to specify the classical model. In earlier work, it was conjectured¹ that the Glauber model¹⁵ of Ising spin dynamics would reproduce the correlations of the quantum model at low T above the ordered state, and thus allow computation of ϕ_c . However, in very recent recent work with A.P. Young, it has been realized that the consequences of global energy and momentum conservation are particularly strong in one dimension, and that these quantities must be conserved in the classical theory; the Glauber model describes Ising spins coupled to a heat bath, and conserve neither the total energy, nor the total momentum, of the Ising spin system. Further details of this on-going work will appear shortly.

4. Conclusions

The major gap in our knowledge is the form of Ising spin correlators for $m \neq 0$. It would be of considerable interest to obtain explicit results for these to test the conjectures of Section 3.2. Some steps in this direction have been taken recently³ by the derivation of new non-linear partial differential equations obeyed by the correlators, and the prospect for additional developments is bright.

Acknowledgements

This research was supported by NSF Grant DMR-96-23181.

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